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FEKETE TYPE POINTS FOR RIDGE FUNCTION INTERPOLATION AND HYPERBOLIC POTENTIAL THEORY

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Abstract. We apply hyperbolic potential theory to the study of the asymptotics of Fekete type points for univariate ridge function interpolation.

1. Introduction

Suppose that $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C}^d \to \mathbb{C}$ is defined by $g(x) = f(t \cdot x)$ where $t \in \mathbb{C}^d$ is fixed and $t \cdot x = t_1 x_1 + \cdots + t_d x_d$. In the case of $t = i\omega$ with $\omega \in \mathbb{R}^d$ and $f(z) = e^z$, $g(x) = e^{i\omega \cdot x}$ and hence we refer to t as a (generalized) "frequency". Such an g is called a *ridge function* (or sometimes a planar wave). If we have n such frequencies $t_1, \ldots, t_n \in \mathbb{C}^d$ then the span of the associated ridge functions

$$V_n := \mathrm{span}\big(\{f(t_1 \cdot x), f(t_2 \cdot x), \ldots, f(t_n \cdot x)\}\big)$$

form a linear "frequency space" and may be used as the basis of a multivariate interpolation scheme for data in $\mathbb{C}^d$ in the following way. Suppose that the "sites" $s_i \in K \subset \mathbb{C}^d$, $1 \leqslant i \leqslant n$, where K is compact, are given together with values $z_i \in \mathbb{C}$, $1 \leqslant i \leqslant n$. We look for an interpolant of the form

$$p(x) = \sum_{j=1}^{n} a_j f(t_j \cdot x),$$

i.e., a $p \in V_n$ such that

$$p(s_i) = z_i, \quad 1 \leqslant i \leqslant n. \tag{1.1}$$

Applying the conditions (1.1) to the equation for p results in a linear system with coefficient matrix $M_n(s,t) := [f(t_j \cdot s_i)]_{1 \leqslant i,j \leqslant n}$.

If the frequencies t_j or the sites s_i, or both, may freely be adjusted within K, then it is reasonable to ask for those points which produce "best" or at least "good" interpolants. Of course, the numerical conditioning of the matrix M_n will play an important role in the answer to such questions and hence it would be

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We dedicate this paper to Prof. Giuseppe Mastroianni on the occasion of his retirement.

useful to know which frequencies t_j and/or points s_i produce the best conditioned matrix M_n. Unfortunately, this is likely a forbiddingly difficult problem, and hence, as a first step, it is reasonable to ask for those frequencies t_j and/or points s_i in K for which which $\det(M_n)$ is as large as possible. M_n is an analogue of the classical Vandermonde matrix and so in analogy with this case, we refer to such optimal points as *ridge* Fekete points. In [**4**], specifying to $\mathbb{R}^d$ and two particular classes of ridge functions, we proved the following theorem.

THEOREM 1.1. *Suppose that $f(x) = \exp(\alpha x)$ or $f(x) = \exp(-\beta x^2)$ for some $\alpha, \beta > 0$. Suppose further that $\hat{s}_1 < \hat{s}_2 < \cdots < \hat{s}_n \in [a, b]$ are points which maximize either*

(1) *$\det(M_n(s, t))$, $s \in [a, b]^n$, where $t \in [a, b]^n$ are fixed but distinct*
(2) *$\det(M_n(s, s))$, $s \in [a, b]^n$.*

Then the discrete measures $\mu_{\hat{s}}^{(n)} := \frac{1}{n}\sum_{i=1}^n \delta_{\hat{s}_i}$ tend weak- to the arcsine measure μ^* given by*

$$d\mu^* = \frac{1}{\pi}\frac{1}{\sqrt{(b-x)(x-a)}}dx.$$

See [**10**] for error estimates of such interpolants. We also remark that, in contrast, for radial basis interpolation by basis functions of the form $g(|x|)$ with $g'(0) \neq 0$, the optimal points are asymptotically *uniformly* distributed; see [**5**] or [**3**].

As is well known, the arcsine measure μ^* is also the so-called equilibrium measure of complex potential theory, a theory fundamental for the study of the asymptotics of good points for univariate *polynomial* interpolation; see, for example [**1**] or [**2**]. This theorem may be paraphrased to say that for the exponential basis functions optimal points for ridge function interpolation behave (asymptotically) exactly like those for polynomial interpolation. In this paper we show that, depending on the basis function $f(x)$, this is not always the case. Indeed, for a different family of functions, it is *hyperbolic* potential theory that plays a central role. In particular, this shows that the asymptotic distribution of ridge Fekete points, in general, depends on the function f.

2. A first example and hyperbolic potential theory

Consider the function $f(z) := 1/(1-z)$ and the corresponding ridge function $g(x) = f(tx)$ where $d = 1$ so that $t, x \in \mathbb{C}$. Then, for sites s_i, $1 \leqslant i \leqslant n$ and frequencies t_i, $1 \leqslant i \leqslant n$ distinct, the matrix $M_n(s,t) = [1/(1-s_i t_j)]_{1 \leqslant i,j \leqslant n}$ is a variant of the so-called Cauchy matrix (see [**6**, p. 268]) and its determinant may be explicitly calculated.

PROPOSITION 2.1. *We have*

$$\det(M_n(s,t)) = \frac{V(s)V(t)}{\prod_{i,j=1}^n (1 - s_i t_j)}$$

where $V(x) := \prod_{i>j}(x_i - x_j)$ is the classical Vandermonde determinant.

PROOF. We may write

$$\frac{1}{1 - s_i t_j} = \frac{1}{s_i} \frac{1}{a_i + b_j}$$

where $a_i := 1/s_i$ and $b_j := -t_j$. Hence

$$\det(M_n(s,t)) = \left(\prod_{i=1}^{n} \frac{1}{s_i} \right)^n \det\big([1/(a_i + b_j)]\big).$$

This latter determinant is the Cauchy determinant for which Davis [**6**, p. 268] gives the formula

$$\det\big([1/(a_i + b_j)]\big) = \frac{V(a)V(b)}{\prod_{i,j=1}^{n}(a_i + b_j)}.$$

Elementary algebra then gives us the result. □

Take now $t = s \in \mathbb{R}^n$. Then

$$\det(M_n(s,s)) = \frac{(V(s))^2}{\prod_{i,j=1}^{n}(1 - s_i s_j)} = \frac{\big(\prod_{i>j}(s_i - s_j)\big)^2}{\prod_{i \neq j}(1 - s_i s_j)} \frac{1}{\prod_{i=1}^{n}(1 - s_i^2)}$$

$$= \left(\frac{\prod_{i>j}(s_i - s_j)}{\prod_{i>j}(1 - s_i s_j)} \right)^2 \frac{1}{\prod_{i=1}^{n}(1 - s_i^2)} = \left(\prod_{i>j} [s_i, s_j]_h \right)^2 \frac{1}{\prod_{i=1}^{n}(1 - s_i^2)} \tag{2.1}$$

where

$$[\alpha, \beta]_h := \left| \frac{\alpha - \beta}{1 - \overline{\alpha}\beta} \right|$$

is the *pseudohyperbolic distance* between $\alpha, \beta \in \mathbb{C}$.

REMARK 2.1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Equipped with the hyperbolic distance

$$\{\alpha, \beta\}_h := \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1 - |z|^2}, \quad \alpha, \beta \in \mathbb{D}$$

where the inf is taken over all rectifiable curves in $\mathbb{D}$ connecting α to β, $\mathbb{D}$ becomes the *hyperbolic plane*. It can be shown that $[\alpha, \beta]_h = \tanh(\{\alpha, \beta\}_h)$.

We then define the hyperbolic Vandermonde determinant to be

$$H(s) := \prod_{i>j} [s_i, s_j]_h.$$

Suppose that $K \subset \mathbb{D}$ is compact. A set of points $t^{(n)} = \{t_1^{(n)}, \ldots, t_n^{(n)}\} \subset K$ that maximize $H(s)$ for $s \in K^n$ form a hyperbolic analogue of classical Fekete points. As they were first studied by Tsuji they are often referred to as *Tsuji points*. Hyperbolic potential theory, as introduced in Tsuji [**9**, p. 94], may be thought of as classical complex potential theory with the euclidean distance $|\alpha - \beta|$ replaced by the pseudohyperbolic distance; see also the survey by Kirsch [**8**, § 6.2]. In particular, for a probability measure μ with support in K, its *energy* is

$$I(\mu) := \int_K \int_K \log\left(\frac{1}{[\alpha, \beta]_h} \right) d\mu(\alpha)\, d\mu(\beta)$$

and its *hyperbolic conductor potential* is

$$U_\mu^h(\alpha) := \int_K \log\left(\frac{1}{[\alpha,\beta]_h}\right) d\mu(\beta).$$

Let $V_h(K) := \inf_\mu I(\mu)$. It is known that

$$\lim_{n\to\infty} H(t^{(n)})^{1/\binom{n}{2}} = \exp(-V_h(K)) =: \mathrm{cap}_h(K),$$

the *hyperbolic capacity* of K. If $\mathrm{cap}_h(K) > 0$ then there exists a unique minimizing measure, μ_K^h, called the *hyperbolic equilibrium measure.* For $\mu = \mu_K^h$, the potential function U_μ is harmonic in $\mathbb{D}\setminus K$ and has the properties that $U_\mu(\alpha) = 0$ for $\alpha \in \partial\mathbb{D}$, $U_\mu(\alpha) \leqslant V_h(K)$ on $\mathbb{D}$ and $U_\mu(\alpha) = V_h(K)$ q.e. on K; i.e., for $\alpha \in K \setminus P$ where P is a (possibly empty) *polar* set (a set P is polar if there exists a subharmonic function $u \not\equiv -\infty$ with $P \subset \{u = -\infty\}$). Points of $K \setminus P$ are called *regular points* of K.

If we define the discrete measures supported on the Tsuji points,

$$\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{t_i^{(n)}},$$

then $\mu_n \to \mu_K^h$, weak$-*$. More generally we have (cf. the proof of Thm. 1.5 in [**1**])

Theorem 2.1. *Suppose that $s^{(n)} \in K^n$ is a sequence of sets of points such that*

$$\lim_{n\to\infty} H(s^{(n)})^{1/\binom{n}{2}} = \mathrm{cap}_h(K).$$

Then for the discrete measures $\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{s_i^{(n)}}$, we have $\lim_{n\to\infty} \mu_n = \mu_K^h$, weak-.*

From this we may conclude

Theorem 2.2. *Suppose that for $K \subset \mathbb{D}$ compact, $s^{(n)} \in K^n$ is such that*

$$\left|\det\left(M_n\left(s^{(n)}, s^{(n)}\right)\right)\right| = \max_{s\in K^n} |\det(M_n(s,s))|, \quad n = 1,2,\ldots,$$

i.e., $s^{(n)}$ is a set of ridge Fekete points, and $\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{s_i^{(n)}}$. Then

$$\lim_{n\to\infty} \mu_n = \mu_K^h, \quad \textit{weak-}* .$$

Proof. By Theorem 2.1 it is sufficient to prove that

$$\lim_{n\to\infty} H(s^{(n)})^{1/\binom{n}{2}} = \mathrm{cap}_h(K).$$

First note that since $K \subset \mathbb{D}$, there exists a constant $\delta > 0$ such that $0 < \delta \leqslant |1 - s_i^2| \leqslant 2$, for all $s \in K$. If we let, as before, $t^{(n)} \in K^n$ denote the Tsuji points for K, we have immediately that $H(s^{(n)}) \leqslant H(t^{(n)})$. Further, by the definition of $s^{(n)}$, $\left|\det\left(M_n\left(t^{(n)}, t^{(n)}\right)\right)\right| \leqslant \left|\det\left(M_n\left(s^{(n)}, s^{(n)}\right)\right)\right|$ so that from (2.1) applied to

$s = t^{(n)}$ and to $s = s^{(n)}$ we have

$$H^2(t^{(n)})\frac{1}{\prod_{i=1}^n |1-(t_i^{(n)})^2|} = \left|\det\left(M_n\left(t^{(n)}, t^{(n)}\right)\right)\right|$$
$$\leqslant \left|\det\left(M_n\left(s^{(n)}, s^{(n)}\right)\right)\right| = H^2(s^{(n)})\frac{1}{\prod_{i=1}^n |1-(s_i^{(n)})^2|}.$$

It follows that

$$H^2(t^{(n)})\left(\prod_{i=1}^n \frac{|1-(s_i^{(n)})^2|}{|1-(t_i^{(n)})^2|}\right) \leqslant H^2(s^{(n)}) \leqslant H^2(t^{(n)})$$

and hence $(\delta/2)^n H^2(t^{(n)}) \leqslant H^2(s^{(n)}) \leqslant H^2(t^{(n)})$. Clearly then

$$\lim_{n\to\infty} H(s^{(n)})^{1/\binom{n}{2}} = \lim_{n\to\infty} H(t^{(n)})^{1/\binom{n}{2}} = \mathrm{cap}_h(K)$$

and we are done. □

Now let us return to the case when K is a real interval. For simplicity let us take $K = [-a, a] \subset \mathbb{D}$. We first note that μ_K^h is *not* the same as the classical equilibrium measure

$$\mu^* = \frac{1}{\pi}\frac{1}{\sqrt{a^2-x^2}}dx.$$

For if $\mu_K^h = \mu^*$ then it would have to be the case that

$$\int_{-a}^{a} \log\left(\frac{1}{|\alpha-\beta|}\right)d\mu^*(\beta) - \int_{-a}^{a} \log\left(\frac{1}{[\alpha,\beta]_h}\right)d\mu^*(\beta) \tag{2.2}$$

is constant q.e. for $\alpha \in [-a, a]$ as, from the classical theory, the first term in (2.2) is also constant on $[-a, a]$. Hence we would have that

$$\int_{-a}^{a} \log\left(\frac{[\alpha,\beta]_h}{|\alpha-\beta|}\right)d\mu^*(\beta) = -\frac{1}{\pi}\int_{-a}^{a} \log(1-\alpha\beta)\frac{d\beta}{\sqrt{a^2-\beta^2}}$$

is constant q.e. on $[-a, a]$. However, a direct calculation shows that

$$\frac{1}{\pi}\int_{-a}^{a} \log(1-\alpha\beta)\frac{d\beta}{\sqrt{a^2-\beta^2}} = \log\left(\frac{1+\sqrt{1-a^2\alpha^2}}{2}\right)$$

which is clearly *not* constant in α, a contradiction.

Alternatively, we may note that in this case

$$U_\mu^h(\alpha) = \int_{-a}^{a} \log\left|\frac{1-\overline{\alpha}\beta}{\alpha-\beta}\right| d\mu(\beta)$$

and for $|\alpha| = 1$,

$$\left|\frac{1-\overline{\alpha}\beta}{\alpha-\beta}\right| = \left|\frac{\overline{\alpha}(\alpha-\beta)}{\alpha-\beta}\right| = |\overline{\alpha}| = 1$$

so that $U_\mu^h(\alpha) = 0$, $|\alpha| = 1$. Then, from the fact that, for $\mu = \mu_K^h$, $U_K^h(\alpha) \equiv V_h(K)$ on $[-a, a]$, it follows that U_K^h is a multiple of the *relative extremal function*

$$\omega(\alpha, K, \mathbb{D}) := \sup\{u(\alpha) : u \text{ shm in } \mathbb{D},\ \ u < 0 \text{ on } \mathbb{D},\ \ u \leqslant -1 \text{ on } K\},$$

so that $\mu_K^h = c\,\Delta\omega(\alpha, K, \mathbb{D})$ for some constant c. In particular,

$$\mu_K^h \neq \frac{1}{\pi}\frac{d\beta}{\sqrt{a^2-\beta^2}},$$

since the right-hand side is the classical equilibrium measure of $[-a, a]$, which is a multiple of the laplacian of the *global extremal function*

$$\sup\big\{u(\alpha) : u \text{ shm in } \mathbb{C},\ u(z) - \log|z| = 0(1)\ (|z| \to \infty),\ u \leqslant 0 \text{ on } [-a, a]\big\}$$
$$= \log\big|\alpha/a - \sqrt{(\alpha/a)^2 - 1}\big|.$$

3. A generalized family of functions

Consider now the family of functions $f_c(z) := c^2/(c^2 - z)$, $c \geqslant 1$ with $g_c(x) = f_c(tx)$. These functions are analytic in the disks $\mathbb{D}_c := \{z \in \mathbb{C} : |z| < c^2\} \supset \mathbb{D}$. The matrices $M_n(s, t)$ for $s, t \in \mathbb{R}^n$ then become $M_n(s, t) = [c^2/(c^2 - s_i t_j)] \in \mathbb{R}^{n\times n}$. It is easy to verify, using Proposition 2.1, that

Proposition 3.1. *We have*

$$\det(M_n(s,t)) = \frac{V(s')V(t')}{\prod_{i,j=1}^n (1 - s_i' t_j')}$$

where $s' := s/c$, $t' = t/c$ *and* $V(x) := \prod_{i>j}(x_i - x_j)$ *is the classical Vandermonde determinant.*

Suppose that $K \subset \mathbb{D}$. It follows that the points that maximize the determinant of $M_n(s, s)$, $s \subset K$, have a limiting measure given by that of the dilation by c of that for K/c. Specifically, if we denote this measure by $d\mu_c^*$ it is such that

$$\int_K f(\beta)\, d\mu_c^*(\beta) = \int_{K/c} f(c\beta)\, d\mu_{K/c}^h(\beta).$$

In particular

$$\int_K \log\Big|\frac{1-\overline{\alpha}\beta}{\alpha-\beta}\Big|\, d\mu_c^*(\beta) = \int_{K/c} \log\Big|\frac{1-\overline{\alpha}c\beta}{\alpha - c\beta}\Big|\, d\mu_{K/c}^h(\beta).$$

Now, we claim that μ_c^* cannot (in general) be equal to the hyperbolic equilibrium measure $d\mu_K^h$. For suppose that they were equal and suppose that $0 \in K$ and that $\alpha = 0 \in K \cap (K/c)$ is a regular point. It would follow that, evaluating at $\alpha = 0$,

$$V_h(K) = \int_K \log\Big|\frac{1}{\beta}\Big|\, d\mu_c^*(\beta) = \int_{K/c} \log\Big|\frac{1}{c\beta}\Big|\, d\mu_{K/c}^h(\beta) = V_h(K/c) - \log(c)$$

so that $\mathrm{cap}_h(K) = c\ \mathrm{cap}_h(K/c)$. However, cap_h does not in general have this scaling property. In fact, Kirsch [**8**, p. 278], reports that

$$\mathrm{cap}_h([0,r]) = \exp\bigg\{-\frac{\pi}{2}\frac{K(\sqrt{1-r^2})}{K(r)}\bigg\}$$

where

$$K(r) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}$$

is the complete elliptic integral of the first kind.

In summary, the limiting measure seems to depend on the domain of analyticity of the basis function. To illustrate this further, we consider in the next section a family of functions with the same domain of analyticity.

4. The family of functions $f^c(z) := (1-z)^{-c}$, $c \geqslant 1$

We again take $K = [-a, a] \subset \mathbb{D}$ with $0 < a < 1$. For $s \in K^n$, the matrix $M_n(s,s) = [(1-s_i s_j)^{-c}] \in \mathbb{R}^{n\times n}$. For $c = 1$ we recover the matrix of the first section. Gross and Richards [**7**, (3.21)] give the remarkable formula

$$\det(M_n(s,s)) = c_n (V(s))^2 \int_{U(n)} \det(I - susu^{-1})^{-(c+n-1)} du \tag{4.1}$$

where $U(n)$ is the group of $n \times n$ complex unitary matrices. We remark that on the right-hand side, we may take $s \in \mathbb{R}^{n\times n}$ the diagonal *matrix* with diagonal the vector s of the left-hand side. The measure is Haar measure on $U(n)$. The constant c_n depends on the parameter c but its exact value will not play a role for us.

In particular this formula allows them to conclude that the matrices $M_n(s,s)$ are positive definite and hence have positive determinant.

First note that $\|susu^{-1}\|_2 \leqslant \|s\|_2^2 = (\max_{1\leqslant i\leqslant n}|s_i|)^2 \leqslant a^2$ so that the spectral radius $\rho(susu^{-1}) \leqslant a^2$. It follows that $1 - a^2 \leqslant |\lambda| \leqslant 1 + a^2$ for any eigenvalue λ of $I - susu^{-1}$, and hence

$$(1-a^2)^n \leqslant \det(I - susu^{-1}) \leqslant (1+a^2)^n.$$

Now consider the formula (4.1). We have

$$\begin{aligned}
&\det([(1-s_i s_j)^{-c}]) \\
&= c_n (V(s))^2 \int_{U(n)} \det(I - susu^{-1})^{-(c+n-1)} du \\
&= c_n (V(s))^2 \int_{U(n)} \frac{\det(I - susu^{-1})^{-(c+n-1)}}{\det(I - susu^{-1})^{-(1+n-1)}} \det(I - susu^{-1})^{-(1+n-1)} du \\
&= c_n (V(s))^2 \int_{U(n)} \det(I - susu^{-1})^{-(c-1)} \det(I - susu^{-1})^{-n} du.
\end{aligned}$$

Consequently, since by assumption $c \geqslant 1$,

$$\begin{aligned}
\det([(1-s_i s_j)^{-c}]) &\leqslant (1-a^2)^{-n(c-1)} c_n (V(s))^2 \int_{U(n)} \det(I - susu^{-1})^{-n} du \\
&= (1-a^2)^{-n(c-1)} \det([(1-s_i s_j)^{-1}]) \qquad (4.2)
\end{aligned}$$

and similarly

$$\begin{aligned}
\det([(1-s_i s_j)^{-c}]) &\geqslant (1+a^2)^{-n(c-1)} c_n (V(s))^2 \int_{U(n)} \det(I - susu^{-1})^{-n} du \\
&= (1+a^2)^{-n(c-1)} \det([(1-s_i s_j)^{-1}]). \qquad (4.3)
\end{aligned}$$

Let now $s^\star$ denote the points in K^n which maximize $\det([(1-s_is_j)^{-c}])$ and $t^\star$ those points in K^n which maximize $\det([(1-s_is_j)^{-1}])$. By the definition of $t^\star$ we have directly that

$$\det([(1-s_i^\star s_j^\star)^{-1}]) \leqslant \det([(1-t_i^\star t_j^\star)^{-1}]).$$

Furthermore,

$$\begin{aligned}\det([(1-s_i^\star s_j^\star)^{-1}]) &\geqslant (1-a^2)^{n(c-1)}\det([(1-s_i^\star s_j^\star)^{-c}]) \quad \text{by (4.2)}\\ &\geqslant (1-a^2)^{n(c-1)}\det([(1-t_i^\star t_j^\star)^{-c}])\\ &\geqslant (1-a^2)^{n(c-1)}(1+a^2)^{-n(c-1)}\det([(1-t_i^\star t_j^\star)^{-1}]) \quad \text{by (4.3)}\\ &= \Big(\frac{1-a^2}{1+a^2}\Big)^{n(c-1)}\det([(1-t_i^\star t_j^\star)^{-1}]).\end{aligned}$$

We conclude that

$$\lim_{n\to\infty} H(s^\star)^{1/\binom{n}{2}} = \lim_{n\to\infty} H(t^\star)^{1/\binom{n}{2}}$$

and hence, by Theorem 2.2, that the optimal points for f^c also are asymptotically distributed according to the hyperbolic equilibrium measure, μ_K^h.

References

1. T. Bloom, L. Bos, C. Christensen, N. Levenberg, *Polynomial interpolation of holomorphic functions in $\mathbb{C}$ and $\mathbb{C}^n$*, Rocky Mountain J. Math. **22**(2) (1992), 441–470.
2. T. Bloom, L. Bos, J.-P. Calvi, N. Levenberg, *Polynomial interpolation and approximation in $\mathbb{C}^d$*, Ann. Polon. Math. **106** (2012), 53–81.
3. L. Bos, S. De Marchi, *Univariate radial basis functions with compact support cardinal functions*, East J. Approx. **14**(1) (2008), 69–80.
4. ———, *On optimal points for interpolation by univariate exponential functions*, Dolomites Res. Notes Approx. DRNA **4** (2011), 8–12.
5. L. Bos, U. Maier, *On the asymptotics of Fekete-type points for univariate radial basis functions*, J. Approx. Theory **119**(2) (2002), 252–270.
6. P. Davis, *Interpolation and Approximation*, Dover, New York, 1975.
7. K. Gross, D. St. P. Richards, *Total positivity, spherical series, and hypergeometric functions of matrix argument*, J. Approx. Theory **59**(2) (1989), 224–246.
8. S. Kirsch, *Transfinite Diameter, Chebyshev Constant and Capacity*, Chapter 6 of *Handbook of Complex Analysis, Volume 2: Geometric Function Theory*, North Holland, 2005.
9. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1958.
10. D. Yarotsky, *Univariate interpolation by exponential functions and Gaussian RBFs for generic sets of nodes*, J. Approx. Theory **166** (2013), 163–175.

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