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NEW INTEGRAL REPRESENTATIONS IN THE LINEAR THEORY OF VISCOELASTIC MATERIALS WITH VOIDS

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Abstract. We investigate the two basic internal BVPs related to the linear theory of viscoelasticity for Kelvin–Voigt materials with voids by means of the potential theory. By using an indirect boundary integral method, we represent the solution of the first (second) BVP of steady vibrations in terms of a simple (double) layer elastopotential. The representations achieved are different from the previously known ones. Our approach hinges on the theory of reducible operators and on the theory of differential forms.

1. Introduction

The theory of viscoelasticity is involved in different branches of applied sciences like, for instance, civil engineering, geotechnical engineering, technology and biomechanics (see [**13**, **20**] and the references therein).

Recently, Svanadze [**20**] has studied some properties related to the linear theory of viscoelasticity for Kelvin–Voigt materials with voids. In particular, the existence and uniqueness theorems for classical solutions of the internal and external two basic boundary value problems (BVPs) of steady vibrations are proved by means of boundary integral method.

The purpose of this work is to obtain integral representations for the solution different from those given in [**20**]. The method we use has been introduced for the first time in [**1**] for the n-dimensional laplacian and it leads to the solution of the Dirichlet problem by means of a simple layer potential. The double layer potential ansatz for the Neumann problem can be treated in a similar way as shown in [**5**]. This approach does not require neither the knowledge of pseudodifferential operators nor the use hypersingular integral, but it hinges on the theory of singular integral operators and the theory of differential forms. The method has been applied to several PDEs in simply and multiply connected domains (see [**3**–**8**, **15**, **16**]).

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Dedicated to Professor Giuseppe Mastroianni on the occasion of his retirement.

The paper is organized as follows. After giving some definitions and preliminary results in Sections 2 and 3, in Section 4 we deal with some properties of simple and double layer elastopotential. Section 5 concerns the study of the first BVP of steady vibrations. We show how to obtain the solution by means of a simple layer elastopotentials. In particular, we construct a left reduction for the related singular integral system. We prove that this singular integral system is equivalent to the Fredholm system obtained through the reduction. Section 6 is devoted to the second BVP of steady vibrations. It turns out that the solution does exist in the form of double layer elastopotential.

We mention that results of this kind are of interest also in numerical applications, inasmuch they allow to apply Boundary Element Method to BVPs of the linear theory of viscoelasticity for Kelvin–Voigt materials with voids.

2. Definitions and preliminary results

Let us consider Ω as a bounded domain of $\mathbb{R}^3$ such that its boundary $\partial\Omega$ is a Lyapunov surface, i.e., $\Sigma \in C^{1,\beta}$, $\beta \in (0,1]$, and such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected; $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward unit normal vector at the point $x = (x_1, x_2, x_3) \in \Sigma$ and $D = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. If $v = (v_1, v_2, v_3, v_4)$, $w = (w_1, w_2, w_3, w_4)$ are two vectors, then $v \cdot w = \sum_{j=1}^{4} v_j \overline{w}_j$, where $\overline{w}_j$ is the conjugate of w_j.

In the sequel p indicates a real number such that $p \in\,]1, +\infty[$. We denote by $L^p(\Sigma)$ the space of all complex-valued measurable functions u such that $|u|^p$ is integrable over Σ and by $W^{1,p}(\Sigma)$ the space of all complex-valued measurable functions $u \in L^p(\Sigma)$ such that $Du \in L^p(\Sigma)$.

The symbol $C^h_k(\Sigma)$ (resp. $L^p_k(\Sigma)$) stands for the space of the differential forms of degree k ($k = 0, 1, 2, 3$) whose components are continuously differentiable up to the order h (resp. belong to $L^p(\Sigma)$) in a coordinate system of class C^{h+1} (resp. C^1) (and then in every coordinate system of class C^{h+1} (resp. C^1)).

We recall that if v is a k-form on Σ, the symbol dv denotes the differential of v and $*v$ denotes the adjoint of u ($*$ stands for the star Hodge operator). In the sequel we shall use the symbol $\underset{\Sigma}{*}$; it means that, if w is a 2-form on Σ and $w = w_0 d\sigma$, then $\underset{\Sigma}{*} w = w_0$. For more details about differential forms we refer the reader to [**10**, **11**].

We mention that if B and $\widetilde{B}$ are two Banach spaces and $S : B \to \widetilde{B}$ is a continuous linear operator, we say that S can be reduced on the left if there exists a continuous linear operator $S' : \widetilde{B} \to B$ such that $S'S = I + T$, where I stands for the identity operator on B and $T : B \to B$ is compact. One of the main properties of such operators is that the equation $S\alpha = \beta$ has a solution if and only if $\langle \gamma, \beta \rangle = 0$ for any γ such that $S^*\gamma = 0$, S^* being the adjoint of S (see [**9**, **17**]). A left reduction is said to be equivalent if $\mathrm{N}(S') = \{0\}$, where $\mathrm{N}(S')$ denotes the kernel of S' (see [**17**, pp. 19–20]).

In what follows we shall make use of the theory of singular integral operators, for which we refer to [**9**, **14**, **17**].

3. The system of the linear theory viscoelasticity for Kelvin–Voigt material with voids

In this section we follow Svanadze [**20**]. Assume that the region Ω is occupied by an isotropic homogeneous viscoelatic Kelvin–Voigt material with voids. The system of homogeneous equations of motion in the linear theory of viscoelasticity for such materials is

$$\begin{aligned} &\mu\Delta u' + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u' + b\,\mathrm{grad}\,\varphi' - \rho\ddot{u}' \\ &\qquad + \mu^*\Delta\dot{u}' + (\lambda^*+\mu^*)\,\mathrm{grad}\,\mathrm{div}\,\dot{u}' + b^*\,\mathrm{grad}\,\dot{\varphi}' = 0, \\ &(\alpha\Delta-\xi)\varphi' - b\,\mathrm{div}\,u' - \rho_0\ddot{\varphi}' + (\alpha^*\Delta-\xi^*)\dot{\varphi}' - \nu^*\,\mathrm{div}\,\dot{u}' = 0, \end{aligned}$$

where $u' = (u_1', u_2', u_3')$ is the displacement vector, φ' is the volume fraction field, ρ is the reference mass density ($\rho > 0$), $\rho_0 = \rho\kappa$, κ is the equilibrated inertia ($\kappa > 0$); λ, μ, b, α, ξ, λ^*, μ^*, b^*, α^*, ν^*, ξ^* are (real) constitutive coefficients, and a superposed dot denotes differentiation with respect to t. In particular λ and μ are the Lamé constants and λ^* and μ^* are the dynamic viscosity constants.

We are interested in the case where u' and φ' have a harmonic time variation, that is

$$u'(x,t) = \mathrm{Re}[u(x)e^{-i\omega t}], \qquad \varphi'(x,t) = \mathrm{Re}[\varphi(x)e^{-i\omega t}],$$

with $u(x) = (u_1(x), u_2(x), u_3(x))$ a complex time-independent vector function and $\varphi(x)$ a complex time-independent function. Then we obtain the following system of homogeneous equations of steady vibrations

$$\tag{3.1} \begin{aligned} \mu_1\Delta u + (\lambda_1+\mu_1)\,\mathrm{grad}\,\mathrm{div}\,u + b_1\,\mathrm{grad}\,\varphi + \rho\omega^2 u = 0, \\ (\alpha_1\Delta+\xi_2)\varphi - \nu_1\,\mathrm{div}\,u = 0, \end{aligned}$$

where ω is the oscillation frequency ($\omega > 0$),

$$\begin{gathered} \lambda_1 = \lambda - i\omega\lambda^*, \quad \mu_1 = \mu - i\omega\mu^*, \quad b_1 = b - i\omega b^*, \quad \alpha_1 = \alpha - i\omega\alpha^*, \\ \nu_1 = b - i\omega\nu^*, \quad \xi_1 = \xi - i\omega\xi^*, \quad \xi_2 = \rho_0\omega^2 - \xi_1. \end{gathered}$$

We observe that (3.1) is a system of partial differential equations with complex coefficients. It is convenient to write it in the following matrix form

$$\tag{3.2} A(D_x)U(x) = 0, \qquad x \in \Omega,$$

where $U = (u, \varphi)$ and $A(D_x) = (A_{pq}(D_x))_{4\times 4}$ denotes the matrix whose entries are

$$\begin{gathered} A_{lj}(D_x) = (\mu_1\Delta + \rho\omega^2)\delta_{lj} + (\lambda_1+\mu_1)\frac{\partial^2}{\partial x_l\partial x_j}, \;\; A_{l4}(D_x) = b_1\frac{\partial}{\partial x_l}, \\ A_{4l}(D_x) = -\nu_1\frac{\partial}{\partial x_l}, \;\; A_{44}(D_x) = \alpha_1\Delta + \xi_2, \quad l,j = 1,2,3 \end{gathered}$$

(δ_{lj} being the Kronecker delta).

Let us introduce the matrix of differential operators

$$L(D_x) = (L_{pq}(D_x))_{4\times 4},$$

$$L_{lj}(D_x) = \frac{1}{\mu_1}(\Delta+\tau_1^2)(\Delta+\tau_2^2)\delta_{lj} - \frac{1}{\alpha_1\mu_1\mu_2}[(\lambda_1+\mu_1)(\alpha_1\Delta+\xi_2) + b_1\nu_1]\frac{\partial^2}{\partial x_l\partial x_j},$$

$$L_{l4}(D_x) = -\frac{b_1}{\alpha_1\mu_2}\frac{\partial}{\partial x_l}, \qquad L_{4l}(D_x) = \frac{\nu_1}{\alpha_1\mu_1\mu_2}(\mu_1\Delta + \rho\omega^2)\frac{\partial}{\partial x_l},$$
$$L_{44}(D_x) = \frac{1}{\alpha_1\mu_2}(\mu_2\Delta + \rho\omega^2), \qquad l, j = 1, 2, 3,$$

where τ_1^2 and τ_2^2 are the roots of the equation (with respect to τ)

$$(\mu_2\tau - \rho\omega^2)(\alpha_1\tau - \xi_2) - b_1\nu_1\tau = 0$$

and $\mu_2 = \lambda_1 + 2\mu_1$. Further set $\tau_3^2 = \rho\omega^2/\mu_1$. From now on we assume that $\tau_1^2 \neq \tau_2^2 \neq \tau_3^2 \neq \tau_1^2$.

The matrix of fundamental solution of homogeneous system (3.1) is

$$\Gamma = (\Gamma_{pq})_{4\times 4}, \tag{3.3}$$

defined by $\Gamma(x) = L(D_x)Y(x)$, where

$$Y(x) = (Y_{pq}(x))_{4\times 4}, \quad Y_{ll}(x) = \sum_{j=1}^{3} c_{1j}\gamma_j(x), \qquad l = 1, 2, 3,$$

$$Y_{44}(x) = \sum_{j=1}^{2} c_{2j}\gamma_j(x), \quad Y_{pq}(x) = 0, \qquad p, q = 1, 2, 3, 4, \quad p \neq q,$$

with $\gamma_j(x) = s(x)e^{i\tau_j|x|}$,

$$s(x) = -\frac{1}{4\pi|x|} \tag{3.4}$$

being the fundamental solution of the Laplace equation, and

$$c_{1j} = \prod_{\substack{l=1\\ l\neq j}}^{3} \frac{1}{\tau_l^2 - \tau_j^2}, \qquad c_{21} = -c_{22} = \frac{1}{\tau_2^2 - \tau_1^2}, \qquad j = 1, 2, 3.$$

If $\alpha_1\mu_1\mu_2 \neq 0$, then each column of the matrix $\Gamma(x)$ satisfies system (3.2) at every point $x \in \mathbb{R}^3$ except the origin [**20**, Corollary 4.1]. Moreover (see [**20**, Corollary 4.2]), the fundamental solution of the system

$$\mu_1\Delta u(x) + (\lambda_1 + \mu_1)\,\mathrm{grad}\,\mathrm{div}\, u(x) = 0,$$
$$\alpha_1\Delta\varphi(x) = 0$$

is the matrix $\Psi = (\Psi_{pq})_{4\times 4}$, whose entries are

$$\begin{aligned} &\Psi_{lj}(x) = -\frac{1}{8\pi}\left(\frac{1}{\mu_1}\Delta\delta_{lj} - \frac{\lambda_1+\mu_1}{\mu_1\mu_2}\frac{\partial^2}{\partial x_l\partial x_j}\right)|x|,\\ &\Psi_{44}(x) = \frac{1}{\alpha_1}s(x), \quad \Psi_{l4}(x) = \Psi_{4j}(x) = 0, \qquad l, j = 1, 2, 3, \end{aligned} \tag{3.5}$$

with $s(x)$ given by (3.4).

In the sequel the following result will be useful (see [**20**, Theorem 4.2]).

LEMMA 3.1. *If $\alpha_1\mu_1\mu_2 \neq 0$, then the relations*

$$\Psi_{pq}(x) = \mathcal{O}(1/|x|), \qquad \Gamma_{pq}(x) - \Psi_{pq}(x) = \mathcal{O}(1 + |x|),$$
$$\frac{\partial^m}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} [\Gamma_{pq}(x) - \Psi_{pq}(x)] = \mathcal{O}(|x|^{1-m})$$

hold in a neighborhood of the origin, where $m = m_1 + m_2 + m_3$, $m \geqslant 1$, $m_l \geqslant 0$, $l = 1, 2, 3$ and $p, q = 1, 2, 3, 4$.

Thus, $\Psi(x)$ is the singular part of the matrix $\Gamma(x)$ in the neighborhood of the origin.

Denote by $P(D_x, n)$ the matrix of differential operators whose entries are

$$\begin{aligned} &P_{ij}(D_x, n) = T_{ij}(D_x, n), \quad T_{ij}(D_x, n) = \mu_1 \delta_{ij} \frac{\partial}{\partial n} + \mu_1 n_j \frac{\partial}{\partial x_i} + \lambda_1 n_i \frac{\partial}{\partial x_j}, \\ &P_{i4}(D_x, n) = b_1 n_i, \quad P_{4j}(D_x, n) = 0, \quad P_{44}(D_x, n) = \alpha_1 \frac{\partial}{\partial n}, \quad i, j = 1, 2, 3. \end{aligned} \tag{3.6}$$

Let us introduce the following notation:

$$\widetilde{A}(D_x) = A^T(-D_x)$$

(the superscript T denotes the transposition),

$$\begin{aligned} &\widetilde{P}(D_x, n) = (\widetilde{P}_{pq}(D_x, n))_{4\times 4}, \quad \widetilde{P}_{pj}(D_x, n) = P_{pj}(D_x, n), \\ &\widetilde{P}_{j4}(D_x, n) = \nu_1 n_j, \quad \widetilde{P}_{44}(D_x, n) = P_{44}(D_x, n), \quad j = 1, 2, 3, \ p = 1, 2, 3, 4, \end{aligned} \tag{3.7}$$

$$\widetilde{\Gamma}(x) = \Gamma^T(-x). \tag{3.8}$$

$\widetilde{\Gamma}$ is the fundamental solution of $\widetilde{A}(D_x)U = 0$.

The basic internal BVPs of steady vibrations in the theory of viscoelastic materials with voids consist in finding a solution of system (3.2) satisfying the boundary condition (with F assigned complex-valued vector function)

$$\lim_{\Omega \ni x \to z \in \Sigma} U(x) = [U(z)]^+ = F(z) \quad \text{in the first problem,}$$
$$[P(D_z, n)U(z)]^+ = F(z) \quad \text{in the second problem.}$$

4. Some properties of simple and double layer elastopotentials

Throughout this article the symbol $\mathcal{S}^p$ stands for the class of simple layer elastopotentials

$$U[g](x) = \int_\Sigma \Gamma(x - y) g(y)\, d\sigma_y, \qquad x \in \Omega, \tag{4.1}$$

with density belonging to $[L^p(\Sigma)]^4$; by $\mathcal{D}^p$ we mean the class of double layer elastopotentials

$$W[g](x) = \int_\Sigma [\widetilde{P}(D_y, n)\Gamma^T(x - y)]^T g(y)\, d\sigma_y, \qquad x \in \Omega, \tag{4.2}$$

with density in $[W^{1,p}(\Sigma)]^4$. For simplicity of notation, we omit to specify the density when it is not necessary.

We begin to note that, from Lemma 3.1 it follows that

$$\Gamma = \Psi + H, \tag{4.3}$$

where Ψ is the matrix defined by (3.5) and H is a 4×4 complex matrix whose entries are $H_{pq}(x) = \mathcal{O}(1 + |x|)$. Then, on account of [**14**, Theorem 7.2, p. 317] and Lemma 3.1, we have the following properties of the simple layer elastopotential.

THEOREM 4.1. *If* $G \in [C^{0,\beta'}(\Sigma)]^4$, $0 < \beta' < \beta \leqslant 1$, *then*

(i) $A(D_x)U[G] = 0$ *in* Ω; (ii) $U[G] \in [C^{1,\beta'}(\overline{\Omega}) \cap C^{\infty}(\Omega)]^4$.

Now consider the double layer elastopotential W with density a complex-valued function G:

$$W(x) = \int_{\Sigma} [\widetilde{P}(D_y, n)\Gamma^T(x - y)]^T G(y)\, d\sigma_y, \qquad x \in \Omega.$$

Then, from (4.3),

$$\widetilde{P}(D_y, n)\Gamma^T(x - y) = \widetilde{P}(D_y, n)\Psi^T(x - y) + \widetilde{P}(D_y, n)\, H^T(x - y)$$

and, consequently, we can rewrite

$$\begin{aligned} W(x) = &\int_{\Sigma} [\widetilde{P}(D_y, n)\Psi^T(x - y)]^T G(y)\, d\sigma_y \\ &+ \int_{\Sigma} [\widetilde{P}(D_y, n)H^T(x - y)]^T G(y)\, d\sigma_y = W^{\Psi}(x) + W^{H}(x). \end{aligned} \tag{4.4}$$

A direct calculation shows that $W^{\Psi} = (w^{\Psi}, \varphi^{\Psi})$ with

$$\begin{aligned} w^{\Psi}(x) &= \int_{\Sigma} [\widetilde{P}(D_y, n)\Psi^T(x - y)]^T g(y)\, d\sigma_y, \\ \varphi^{\Psi}(x) &= \int_{\Sigma} \left[\frac{\nu_1}{\alpha_1} s(x - y) n_y \cdot g(y) + \frac{\partial}{\partial n_y} s(x - y) g_4(y) \right] d\sigma_y, \end{aligned} \tag{4.5}$$

where $G = (g_1, g_2, g_3, g_4) = (g, g_4)$.

Set now $\mathcal{H}(x - y) = [\widetilde{P}(D_y, n)H^T(x - y)]^T$, $\mathcal{H} = (\mathcal{H}_{pq})_{4\times 4}$, where

$$\mathcal{H}_{pq}(x - y) = \sum_{j=1}^{4} \widetilde{P}_{qj}(D_y, n) H_{pj}(x - y). \tag{4.6}$$

In particular, if $p = 1, 2, 3, 4$ and $q = 1, 2, 3$, from (4.6) and (3.7) we get

$$\mathcal{H}_{pq}(x - y) = \sum_{j=1}^{3} T_{qj}(D_y, n) H_{pj}(x - y) + \nu_1 n_q(y) H_{p4}(x - y), \tag{4.7}$$

$$\mathcal{H}_{44}(x - y) = \alpha_1 \frac{\partial}{\partial n_y} H_{44}(x - y) \tag{4.8}$$

and $\mathcal{H}_{q4}(x - y) = 0$.

This yields that $W^H = (w^H, \varphi^H)$ with

$$
\begin{aligned}
w_p^H(x) &= \int_\Sigma \mathcal{H}_{pq}(x-y) g_q(y)\, d\sigma_y, \quad p = 1,2,3, \\
\varphi^H(x) &= \int_\Sigma \mathcal{H}_{44}(x-y) g_4(y)\, d\sigma_y.
\end{aligned} \tag{4.9}
$$

Note that, if $p = 1,2,3,4$ and $q = 1,2,3$, from (4.7) and Lemma 3.1 it follows that

$$
\mathcal{H}_{pq}(x-y) = \mathcal{O}(1 + |x-y|), \tag{4.10}
$$

since

$$
\begin{aligned}
T_{qj}(D_y, n) H_{pj}(x-y) = \mu_1 \delta_{qj} \frac{\partial}{\partial n_y} H_{pj}(x-y) + \mu_1 n_j(y) \frac{\partial}{\partial y_q} H_{pj}(x-y) \\
+ \lambda_1 n_q(y) \frac{\partial}{\partial y_j} H_{pj}(x-y) = \mathcal{O}(1)
\end{aligned}
$$

and

$$
H_{p4}(x-y) = \mathcal{O}(1 + |x-y|). \tag{4.11}
$$

Moreover, again from Lemma 3.1 and (4.8), we get

$$
\mathcal{H}_{44}(x-y) = \mathcal{O}(1). \tag{4.12}
$$

We end this section with the following result.

THEOREM 4.2. *If* $G \in [C^{1,\beta'}(\Sigma)]^4$, $0 < \beta' < \beta \leqslant 1$, *then*
(i) $A(D_x) W[G] = 0$ *in* Ω; (ii) $W[G] \in [C^{1,\beta'}(\overline{\Omega}) \cap C^\infty(\Omega)]^4$.

PROOF. Statement (i) is obvious. In order to obtain (ii), keeping in mind (4.4), (4.5) and (4.9), it is sufficient to apply [**14**, Theorem 6.2, p. 315]. □

5. First problem

In this section we look for the solution of the first BVP in the form of a simple layer elastopotential. Namely, we consider the BVP

$$
\begin{aligned}
&U \in \mathcal{S}^p, \\
&A(D_x) U = 0 \quad \text{in } \Omega, \\
&U = F \quad \text{on } \Sigma, \quad F \in [W^{1,p}(\Sigma)]^4.
\end{aligned} \tag{5.1}
$$

Imposing the boundary condition we get the integral system of the first kind

$$
\int_\Sigma \Gamma(x-y) \phi(y)\, d\sigma_y = F(x) \tag{5.2}
$$

on Σ. Following the approach introduced in [**1**], we take the differential of both sides of (5.2), obtaining the following singular integral system

$$
\int_\Sigma d_x[\Gamma(x-y)] \phi(y)\, d\sigma_y = dF(x). \tag{5.3}
$$

In (5.3) the unknown is the vector $(\phi_1, \dots, \phi_4)$ whose components are scalar functions, while the data is the vector $(dF_1, \dots, dF_4)$ whose components are differential forms of degree 1. We shall see that (5.3) is solvable and we shall obtain the solution of (5.1). Moreover system (5.3) is shown to be equivalent to a Fredholm one.

The following result was proved in [**1**, Theorem I, p. 186].

Lemma 5.1. *The singular integral operator*

$$J : L^p(\Sigma) \longrightarrow L^p_1(\Sigma)$$

$$J\phi(x) = \int_\Sigma \phi(y)\, d_x s(x-y)\, d\sigma_y\,,$$

where s *is given by* (3.4)*, can be reduced on the left. A reducing operator is*

$$J' : L^p_1(\Sigma) \longrightarrow L^p(\Sigma)$$

$$J'\psi(z) = \underset{\Sigma}{*} \int_\Sigma \psi(x) \wedge d_z[s_1(z-x)],$$

where $s_1(z-x)$ *is the double 1-form introduced by Hodge in [**12**]:*

$$s_1(z-x) = \sum_{j=1}^{3} s(z-x)\, dz^j dx^j.$$

As in [**5**, Theorem 4, p. 38], one can show

Lemma 5.2. *The singular integral operator*

$$R : [L^p(\Sigma)]^3 \longrightarrow [L^p_1(\Sigma)]^3$$

$$R_j\phi(x) = \int_\Sigma \phi_k(y)\, d_x[\Psi_{jk}(x-y)] d\sigma_y \qquad (j=1,2,3),$$

Ψ_{jk}*, being defined by* (3.5)*, can be reduced on the left. A reducing operator is*

$$R' : [L^p_1(\Sigma)]^3 \longrightarrow [L^p(\Sigma)]^3$$

$$R'_i\psi = \frac{(\lambda_1+\mu_1)(\lambda_1+2\mu_1)}{\lambda_1+3\mu_1}\mathcal{K}_{jj}(\psi)n_i + 2\mu_1\frac{(\lambda_1+2\mu_1)}{(\lambda_1+3\mu_1)}\mathcal{K}_{ij}(\psi)n_j + \mu_1\frac{(\lambda_1+\mu_1)}{(\lambda_1+3\mu_1)}\delta^{ij}_{sp} n_j \mathcal{K}_{ps}(\psi).$$

Here $\mathcal{K}_{js}$ *are the operators defined by*

$$\mathcal{K}_{js}(\psi)(x) = *\int_\Sigma d_x[s_1(x-y)]\wedge\psi_j(y)\wedge dx^s - \delta^{123}_{ihp}\int_\Sigma \frac{\partial}{\partial x_s}[K_{ij}(x-y)]\wedge\psi_h(y)\wedge dy^p,$$

where

$$K_{ij}(x-y) = \frac{1}{4\pi}\left[\frac{(\lambda_1+\mu_1)}{(\lambda_1+3\mu_1)}\frac{\partial|x-y|}{\partial y_j}\frac{\partial|x-y|}{\partial y_i}\right]\frac{1}{|x-y|}$$

and δ^{123}_{ihp} *is the Levi-Civita symbol.*

Lemma 5.3. *The singular integral operator*

$$S_0 : [L^p(\Sigma)]^4 \longrightarrow [L^p_1(\Sigma)]^4$$

$$S_0(\phi)(x) = \int_\Sigma d_x[\Psi(x-y)]\phi(y)\, d\sigma_y,$$

where the matrix Ψ *is given by* (3.5), *can be reduced on the left. A reducing operator is*

$$S' : [L_1^p(\Sigma)]^4 \longrightarrow [L^p(\Sigma)]^4$$
(5.4)
$$S'_k(\psi) = (1-\delta_{k4})R'_k(\psi^{(1)}) + \delta_{k4}\alpha_1 J'(\psi_4),$$

where $\psi = \binom{\psi^{(1)}}{\psi_4}$, *and* $\psi^{(1)}$ *is a three-component column vector.*

Proof. We remark that

$$[S_0(\phi)]_k = (1-\delta_{k4})R_k(\phi^{(1)}) + \delta_{k4}\frac{1}{\alpha_1}J(\phi_4), \qquad k=1,2,3,4.$$

We have

$$[S'S_0(\phi)]_k = (1-\delta_{k4})R'_k([S_0(\phi)]^{(1)}) + \delta_{k4}\alpha_1 J'([S_0(\phi)]_4) = \\ = (1-\delta_{k4})R'_k R_k(\phi^{(1)}) + \delta_{k4}J'J(\phi_4).$$

Lemmas 5.1 and 5.2 complete the proof. □

We are now in a position to find the reducing operator for S and to obtain an existence theorem for the equation $S\phi = \omega$.

Proposition 5.1. *The singular integral operator*

$$S : [L^p(\Sigma)]^4 \longrightarrow [L_1^p(\Sigma)]^4$$
(5.5)
$$S\phi(x) = \int_\Sigma d_x[\Gamma(x-y)]\phi(y)\, d\sigma_y,$$

Γ *being the matrix* (3.3), *can be reduced on the left by* S' *(see* (5.4)*).*

Proof. We can write $S = (S-S_0)+S_0$. Since Lemma 3.1 implies that $S-S_0$ is compact, by the previous lemma we have that $S'S = S'(S-S_0) + S'S_0$ is a Fredholm operator. □

Theorem 5.1. *If*

(5.6) $$\mu^* > 0, \quad 3\lambda^*+2\mu^* > 0, \quad \alpha^* > 0, \quad (3\lambda^*+2\mu^*)\xi^* > \tfrac{3}{4}(b^*+\nu^*)^2$$

are satisfied, then, given $\omega \in [L_1^p(\Sigma)]^4$, *there exists a solution* $\phi \in [L^p(\Sigma)]^4$ *of the singular integral system*

(5.7) $$S\phi = \omega \qquad a.e.\ x \in \Sigma,$$

where S *is given by* (5.5), *if and only if*

(5.8) $$\int_\Sigma \gamma_i \wedge \overline{\omega}_i = 0, \qquad i=1,2,3,4$$

for every $\gamma \in [L_1^q(\Sigma)]^4$, $q = \frac{p}{p-1}$, *such that* γ_i $(i=1,2,3,4)$ *is a weakly closed* 1*-form, i.e.,*

$$\int_\Sigma \gamma_i \wedge dg = 0, \qquad \forall\, g \in C^\infty(\mathbb{R}^3)$$

($g : \mathbb{R}^3 \to \mathbb{C}$*).*

PROOF. Proposition 5.1 implies that the range of S is closed in $[L_1^p(\Sigma)]^4$. Then integral system (5.7) has a solution $\phi \in [L^p(\Sigma)]^4$ if and only if compatibility conditions (5.8) hold for every $\gamma \in [L_1^q(\Sigma)]^4$ solution of the homogeneous adjoint system

$$S_j^*\gamma(x) = \int_\Sigma \gamma_i(y) \wedge d_y[\Gamma_{ij}(y-x)] = 0 \qquad \text{a.e. } x \in \Sigma, \quad j = 1,2,3,4.$$

On the other hand $S^*\gamma = 0$ if and only if γ_i is a weakly closed 1-form. Indeed, if γ is such that $S^*\gamma = 0$, that is

$$\int_\Sigma \gamma_i(y) \wedge d_y\,[\Gamma_{ij}(y-x)] = 0 \qquad \text{a.e. } x \in \Sigma, \tag{5.9}$$

we have

$$\begin{aligned} 0 &= \int_\Sigma p_j(x)\,d\sigma_x \int_\Sigma \gamma_i(y) \wedge d_y\,[\Gamma_{ij}(y-x)] \\ &= \int_\Sigma \gamma_i(y) \wedge d_y \int_\Sigma p_j(x)\Gamma_{ij}(y-x)d\sigma_x \qquad \forall\ p_i \in C^\lambda(\Sigma). \end{aligned}$$

We can represent every smooth solution of (3.2) by means of a simple layer elastopotential (see Theorem 4.1) $U_i(y) = \int_\Sigma p_j(x)\Gamma_{ij}(y-x)\,d\sigma_x$, and then $\int_\Sigma \gamma_j \wedge dU_j = 0$ for any $U \in [C^{1,\beta'}(\overline{\Omega}) \cap C^\infty(\Omega)]^4$ such that $A(D_x)U = 0$. Therefore we have

$$\int_\Sigma \gamma_i(y) \wedge d_y[\Gamma_{ij}(y-x)] = 0 \qquad \forall x \in \mathbb{R}^3 \setminus \overline{\Omega}. \tag{5.10}$$

Let us denote by $w_j(x)$, $j = 1,2,3,4$, the left-hand side of (5.10). By (3.8) it follows that $w_j(x) = \int_\Sigma \gamma_i(y) \wedge d_y[\widetilde{\Gamma}_{ji}(x-y)]$.

If $v \in [C^\infty(\mathbb{R}^3)]^4$ and $\eta \in [C^1(\overline{\Omega})]^4 \cap [C^2(\Omega)]^4$ are such that $A(D_x)\eta = A(D_x)v$ in Ω and $\eta = 0$ on Σ, we have

$$\begin{aligned} \int_\Omega w_j(A(D_x)v)_j dx &= \int_\Omega w_j(A(D_x)\eta)_j dx \\ &= \int_\Omega (A(D_x)\eta)_j(x)dx \int_\Sigma \gamma_i(y) \wedge d_y[\widetilde{\Gamma}_{ji}(x-y)] \\ &= \int_\Sigma \gamma_i(y) \wedge d_y \int_\Omega (A(D_x)\eta)_j(x)\widetilde{\Gamma}_{ji}(x-y)\,dx. \end{aligned}$$

From [**20**, Theorem 7.3], it follows that

$$\eta_i(y) = -\int_\Sigma \widetilde{\Gamma}_{ji}(x-y)(P(D_x,n)\eta)_j(x)\,d\sigma_x + \int_\Omega \widetilde{\Gamma}_{ji}(x-y)(A(D_x)\eta)_j(x)\,dx, \quad y \in \Omega. \tag{5.11}$$

Letting $y \to \Sigma$, (5.11) gives

$$\int_\Sigma \widetilde{\Gamma}_{ji}(x-y)(P(D_x,n)\eta)_j(x)\,d\sigma_x = \int_\Omega \widetilde{\Gamma}_{ji}(x-y)(A(D_x)\eta)_j(x)\,dx, \quad y \in \Sigma.$$

Then, by (5.9) we have

$$\int_\Omega w_j(A(D_x)v)_j dx = \int_\Sigma \gamma_i(y) \wedge d_y \int_\Sigma \widetilde{\Gamma}_{ji}(x-y)(P(D_x,n)\eta)_j(x)\,d\sigma_x \tag{5.12}$$

$$= \int_\Sigma (P(D_x,n)\eta)_j(x)\,d\sigma_x \int_\Sigma \gamma_i(y)\wedge d_y[\widetilde{\Gamma}_{ji}(x-y)] = 0.$$

Formulas (5.10) and (5.12) lead to

$$0 = \int_{\mathbb{R}^3} w_j (A(D_x)\psi)_j dx = \int_{\mathbb{R}^3} (A(D_x)\psi)_j(x) dx \int_\Sigma \gamma_i(y)\wedge d_y[\widetilde{\Gamma}_{ji}(x-y)]$$
$$= \int_\Sigma \gamma_i(y)\wedge d_y \int_{\mathbb{R}^3} (A(D_x)\psi)_j(x)\widetilde{\Gamma}_{ji}(x-y)\,dx = \int_\Sigma \gamma_i \wedge d\psi_i,$$

for any $\psi \in [\overset{\circ}{C}{}^\infty(\mathbb{R}^3)]^4$. The last equality follows from [**20**, Theorem 7.3]. This shows that γ_i is a weakly closed 1-form and the theorem is proved. □

Lemma 5.4. *If* (5.6) *hold, given* $F \in [W^{1,p}(\Sigma)]^4$, $1 < p < \infty$, *the boundary value problem*

$$\begin{aligned} &U \in \mathcal{S}^p, \\ &A(D_x)U = 0 \quad \text{in } \Omega, \\ &dU = dF \quad \text{on } \Sigma \end{aligned} \tag{5.13}$$

is solvable. A solution is given by a simple layer elastopotential (4.1) *where its density g solves the singular integral system*

$$Sg = dF, \tag{5.14}$$

S being operator (5.5).

Proof. There exists a solution of (5.13) if and only if, there exists a solution $g \in [L^p(\Sigma)]^4$ of singular integral system (5.14). Such a system is always solvable, by Theorem 5.1. □

Lemma 5.5. *If* (5.6) *are satisfied, then the solution of the boundary value problem*

$$\begin{aligned} A(D_x)V = 0 \quad &\text{in } \Omega, \\ V = C \quad &\text{on } \Sigma, \end{aligned} \tag{5.15}$$

where $C = (c_1,\dots,c_4) \in \mathbb{C}^4$, *can be represented by a simple layer elastopotential with density* $g \in [C^{1,\beta'}(\Sigma)]^4$, $0 < \beta' < \beta \leqslant 1$.

Proof. From Theorem 4.2(i) and [**20**, Theorem 9.1], it follows that the solution of (5.15) can be represented by a double layer elastopotential

$$V(x) = \int_\Sigma [\widetilde{P}(D_y,n)\Gamma^T(x-y)]^T g(y)\,d\sigma_y, \quad g \in [C^{1,\beta'}(\Sigma)]^4, \quad (0<\beta'<\beta\leqslant 1),$$

where g is a solution of the singular integral equation

$$\frac{1}{2}\,g(z) + \int_\Sigma [\widetilde{P}(D_y,n)\Gamma^T(x-y)]^T g(y)\,d\sigma_y = C,$$

which is always solvable. Then $V \in [C^{1,\beta'}(\overline{\Omega})]^4$ (see Theorem 4.2(ii)).

Let us consider now the boundary value problem

$$(5.16)\qquad \begin{aligned} A(D_x)U = 0 \quad &\text{in } \Omega, \\ P(D_x,n)U = P(D_x,n)V \quad &\text{on } \Sigma. \end{aligned}$$

Since the solution U of problem (5.16) exists, is unique and can be represented by a simple layer elastopotential ([**20**, Theorem 9.3] and Theorem 4.1), we obtain $U = V$, which proves the theorem. $\square$

Now we can solve the first BVP in the class $\mathcal{S}^p$.

Theorem 5.2. *Assuming conditions* (5.6)*, the first BVP* (5.1) *admits a unique solution U. In particular, the density ϕ of U can be written as $\phi = \phi_0 + \psi_0$, where ϕ_0 solves the singular integral system*

$$\int_\Sigma d_x\,[\Gamma_{ij}(y-x)]\,\phi_{0j}(y)\,d\sigma_y = dF_i(x), \quad i = 1,\ldots,4, \quad a.e.\ x \in \Sigma$$

and ψ_0 is the density of a simple layer elastopotential which is constant on Σ.

Proof. Let W be a solution of (5.13). Since $dW = dF$ on Σ and Σ is connected, we have $W = F - C$ on Σ, with $C \in \mathbb{C}^4$. Then $U = W + V$, V being a solution of (5.15), solves (5.1).

In order to show the uniqueness, suppose that the simple layer elastopotential U defined by (4.1) solves (5.1) with $F = 0$. From Proposition 5.1 it follows that the condition $U = 0$ on Σ implies that

$$(5.17)\qquad \phi + K\phi = 0,$$

where K is a suitable compact operator from $[L^p(\Sigma)]^4$ into itself such that $S'S = I + K$ (S and S' being given by (5.5) and (5.4) resp.). By bootstrap techniques (5.17) implies that $\phi \in [C^{1,\beta'}(\Sigma)]^4$, $0 < \beta' < \beta \leqslant 1$. Then $U \in [C^{1,\beta'}(\overline{\Omega}) \cap C^2(\Omega)]^4$ thanks to Theorem 4.1, the uniqueness proved in [**20**, Theorem 6.1] completes the proof. $\square$

When we solve the first BVP (5.1) by means of a simple layer elastopotential, we need to study singular integral system (5.14). We known that this system can be reduced to a Fredholm one by the operator S'. Note that this reduction is not an equivalent one because $\mathrm{N}(S') \neq \{0\}$. Nevertheless, we still have a kind of equivalence, as we prove in the next theorem.

Theorem 5.3. *The singular integral system* (5.14) *is equivalent to the Fredholm system $S'Sg = S'(dF)$, where $g \in [L^p(\Sigma)]^4$, $F \in [W^{1,p}(\Sigma)]^4$.*

Proof. As in [**2**, pp. 253–254], one can show that $N(S'S) = N(S)$. This implies that, if G is such that there exists a solution g of the system $Sg = G$, this system is satisfied if and only if $S'Sg = SG$. Since we know that the system $Sg = dF$ is solvable, we have that $Sg = dF$ if, and only if, $S'Sg = S'(dF)$. $\square$

6. Second problem

In this section we achieve the representability of the solution of the second BVP by means of a double layer elastopotential, i.e.,

$$
\begin{aligned}
&W \in \mathcal{D}^p, \\
&A(D_x)W = 0 \qquad \text{in } \Omega, \\
&[P(D_x,n)W]^+ = F \quad \text{on } \Sigma, \quad F \in [L^p(\Sigma)]^4.
\end{aligned} \tag{6.1}
$$

In order to prove the claim of this section we need the following lemma.

LEMMA 6.1. *We have that*

$$
[P(D_x,n)W[U]]^+ = -\frac{1}{4}\,g + V^2 g, \qquad g \in [L^p(\Sigma)]^4, \tag{6.2}
$$

where $P(D_x,n)$ *are the matrix of differential operators defined by* (3.6), U *is a simple layer elastopotential* (4.1) *with density* g, W *is the double layer elastopotential* (4.2) *with density* U *and*

$$
Vg(x) = \int_\Sigma P(D_x,n)\Gamma(x-y)g(y)\,d\sigma_y. \tag{6.3}
$$

PROOF. By [**20**, Theorem 7.3], we have

$$
W[U](x) = U(x) + \int_\Sigma \Gamma(x-y)P(D_y,n)U(y)\,d\sigma_y, \quad x \in \Omega.
$$

It is also known that (see [**20**, formula (8.1)])

$$
\big[P(D_x,n)U[g](x)\big]^+ = -\frac{1}{2}\,g(x) + P(D_x,n)U[g](x), \quad x \in \Sigma. \tag{6.4}
$$

Then we have

$$
\begin{aligned}
[P(D_x,n)W[U](x)]^+ &= \Big[P(D_x,n)\Big(U(x) + \int_\Sigma \Gamma(x-y)P(D_y,n)U(y)\,d\sigma_y\Big)\Big]^+ \\
&= \Big[P(D_x,n)U(x) + P(D_x,n)\int_\Sigma \Gamma(x-y)P(D_y,n)U(y)\,d\sigma_y\Big]^+ \\
&= \Big(1-\frac{1}{2}\Big)P(D_x,)U(x) + P(D_x,n)\int_\Sigma \Gamma(x-y)P(D_y,n)U(y)\,d\sigma_y, \quad x \in \Sigma.
\end{aligned}
$$

Keeping in mind (4.1) and (6.4), we get

$$
\begin{aligned}
[P(D_x,n)W[U](x)]^+ &= \frac{1}{2}\,P(D_x,n)\Big(\int_\Sigma \Gamma(x-y)g(y)\,d\sigma_y\Big) \\
&\quad + P(D_x,n)\int_\Sigma \Gamma(x-y)P(D_y,n)\Big(\int_\Sigma \Gamma(y-z)g(z)\,d\sigma_z\Big)d\sigma_y \\
&= -\frac{1}{4}g(x) + P(D_x,n)\int_\Sigma \Gamma(x-y)P(D_y,n)\int_\Sigma \Gamma(y-z)g(z)\,d\sigma_z d\sigma_y \\
&= -\frac{1}{4}g(x) + V^2 g(x), \quad x \in \Sigma,
\end{aligned}
$$

V being the operator (6.3). □

Lemma 6.2. *Let T be the following linear integral operator*

$$T(G)(x) = \int_\Sigma K(x-y)G(y)\,d\sigma_y, \quad x \in \Omega,$$

where K is a 4×4 matrix whose entries are $K_{pq}(x-y) = \mathcal{O}(|x-y|^{-2})$. Then T is a continuous operator from $[L^2(\Sigma)]^4$ into $[L^2(\Omega)]^4$, i.e., there exists C such that

$$\|T(G)\|_{[L^2(\Omega)]^4} \leqslant C\|G\|_{[L^2(\Sigma)]^4}, \qquad \forall\, G \in [L^2(\Sigma)]^4.$$

Proof. Observe that for every $q = 1,2,3,4$

$$\begin{aligned}\int_\Omega dx \int_\Sigma \frac{|G_q(y)|}{|x-y|^2}\,d\sigma_y \int_\Sigma \frac{|G_q(w)|}{|x-w|^2}\,d\sigma_w \\ = \int_\Sigma |G_q(y)|\,d\sigma_y \int_\Sigma |G_q(w)|\,d\sigma_w \int_\Omega \frac{1}{|x-y|^2|x-w|^2}dx \\ \leqslant C_1 \int_\Sigma |G_q(y)|\,d\sigma_y \int_\Sigma \frac{|G_q(w)|}{|y-w|}\,d\sigma_w,\end{aligned}$$

the last inequality being true thanks to [**19**, p. 806] or [**18**, p. 45]. Then

$$\begin{aligned}&\int_\Omega dx \int_\Sigma \frac{|G_q(y)|}{|x-y|^2}\,d\sigma_y \int_\Sigma \frac{|G_q(w)|}{|x-w|^2}\,d\sigma_w \\ &\leqslant C_1\Big(\int_\Sigma |G_q(y)|^2\,d\sigma_y\Big)^{1/2}\Big(\int_\Sigma\Big(\int_\Sigma \frac{|G_q(w)|}{|y-w|}\,d\sigma_w\Big)^2 d\sigma_y\Big)^{1/2} \\ &\leqslant C_1\Big(\int_\Sigma |G_q(y)|^2\,d\sigma_y\Big)^{1/2}\Big(\int_\Sigma\int_\Sigma \frac{|G_q(\tau)|^2}{|y-\tau|}\,d\sigma_\tau \int_\Sigma \frac{d\sigma_w}{|y-w|}\,d\sigma_y\Big)^{1/2} \\ &\leqslant C_2\Big(\int_\Sigma |G_q(y)|^2\,d\sigma_y\Big)^{1/2}\Big(\int_\Sigma |G_q(\tau)|^2\,d\sigma_\tau \int_\Sigma \frac{d\sigma_y}{|y-\tau|}\Big)^{1/2} \leqslant C_3\|G_q\|^2_{L^2(\Sigma)},\end{aligned}$$

and hence the claim. □

Lemma 6.3. *Let $W = (w,\varphi) \in \mathcal{D}^2$ be a double layer elastopotential with density $G = (g, g_4) \in [W^{1,2}(\Sigma)]^4$ and set*

$$\begin{aligned}\mathcal{E}(w,\lambda_1,\mu_1) &= \frac{1}{3}(3\lambda_1+2\mu_1)|\operatorname{div} w|^2 \\ &\quad + \mu_1\Big[\frac{1}{2}\sum_{l,j=1,l\neq j}^{3}\Big|\frac{\partial w_j}{\partial x_l}+\frac{\partial w_l}{\partial x_j}\Big|^2 + \frac{1}{3}\sum_{l,j=1}^{3}\Big|\frac{\partial w_l}{\partial x_l}-\frac{\partial w_j}{\partial x_j}\Big|^2\Big],\end{aligned}$$

$$B(D_x) = (B_{lj}(D_x))_{3\times 3}, \quad B_{lj}(D_x) = \mu_1\delta_{lj} + (\lambda_1+\mu_1)\frac{\partial^2}{\partial x_l \partial x_j}, \quad l,j = 1,2,3,4.$$

Then

$$\int_\Omega [B(D_x)w\cdot w + \mathcal{E}(w,\lambda_1,\mu_1)]\,dx = \int_\Sigma Tw\cdot w\,d\sigma, \tag{6.5}$$

$$\int_\Omega [\Delta\varphi\overline{\varphi} + |\operatorname{grad}\varphi|^2]\,dx = \int_\Sigma \frac{\partial\varphi}{\partial n}\overline{\varphi}\,d\sigma. \tag{6.6}$$

PROOF. Let $(G_k)_{k\geqslant 1}$ be a sequence of functions in $[C^{1,\beta'}(\Sigma)]^4$ $(0<\beta'<\beta)$ such that $G_k\to G$ in $[W^{1,2}(\Sigma)]^4$, that is $g_k\to g$ in $[W^{1,2}(\Sigma)]^3$ and $g_{4_k}\to g_4$ in $W^{1,2}(\Sigma)$. Setting

$$W_k(x)=\int_\Sigma[\widetilde{P}(D_y,n)\Gamma^T(x-y)]^T G_k(y)\,d\sigma_y,\quad x\in\Omega,$$

on account of Theorem 4.2 we get $W_k\in[C^{1,\beta'}(\overline{\Omega})\cap C^2(\Omega)]^4$ and then, from Green's formulas, identities (6.5) and (6.6) hold for $W_k=(w_k,\varphi_k)$:

$$\int_\Omega[B(D_x)w_k\cdot w_k+\mathcal{E}(w_k,\lambda_1,\mu_1)]\,dx=\int_\Sigma Tw_k\cdot w_k\,d\sigma, \tag{6.7}$$

$$\int_\Omega[\Delta\varphi_k\overline{\varphi}_k+|\operatorname{grad}\varphi_k|^2]\,dx=\int_\Sigma\frac{\partial\varphi_k}{\partial n}\overline{\varphi}_k\,d\sigma. \tag{6.8}$$

Observe that $g_k\to g$ in $[L^2(\Sigma)]^3$ and $g_{4_k}\to g_4$ in $L^2(\Sigma)$ imply $w_k\to w$ in $[L^2(\Sigma)]^3$ and $\varphi_k\to\varphi$ in $L^2(\Sigma)$, respectively. Indeed, taking (4.4)–(4.9) into account, $w_k=w_k^\Psi+w_k^H$ and $\varphi_k=\varphi_k^\Psi+\varphi_k^H$. Hence $w_k^\Psi\to w^\Psi$ in $[L^2(\Sigma)]^3$ and $\varphi_k^\Psi\to\varphi^\Psi$ in $L^2(\Sigma)$ because of the well-known properties of singular integral operators; thanks to (4.10)–(4.12) we get $w_k^H\to w^H$ in $[L^2(\Sigma)]^3$ and $\varphi_k^H\to\varphi^H$ in $L^2(\Sigma)$.

Arguing as in [**6**, Lemma 6.1], one can show that $Tw_k^\Psi\to Tw^\Psi$ in $[L^2(\Sigma)]^3$. In view of (4.6)–(4.11), we have that $Tw_k^H\to Tw^H$ in $[L^2(\Sigma)]^3$. Therefore $Tw_k\to Tw$ in $[L^2(\Sigma)]^3$. Analogously we can obtain $\partial\varphi_k/\partial n\to\partial\varphi/\partial n$ in $L^2(\Sigma)$ (see [**7**, Lemma 5.1 and Remark 1] and (4.12)).

Further, according to Lemma 6.2, from $G_k\to G$ in $[L^2(\Sigma)]^4$ it follows that $W_k\to W$ in $[L^2(\Omega)]^4$.

We proceed to show that $\operatorname{grad}w_k\to\operatorname{grad}w$ in $[L^2(\Omega)]^3$. Indeed, the same argument as in [**6**, Lemma 6.1] applies to show that $\operatorname{grad}w_k^\Psi\to\operatorname{grad}w^\Psi$ in $[L^2(\Omega)]^3$. The kernel of

$$\frac{\partial}{\partial x_p}(w_p^H)_k(x)=\int_\Sigma\frac{\partial}{\partial x_p}\mathcal{H}_{pq}(x-y)(g_q)_k(y)\,d\sigma_y,\quad p=1,2,3,$$

being $\mathcal{O}(|x-y|^{-1})$, $\operatorname{grad}w_k^H\to\operatorname{grad}w^H$ in $[L^2(\Omega)]^3$.

In the same way we get $\operatorname{grad}\varphi_k\to\operatorname{grad}\varphi$ in $L^2(\Omega)$ (see [**7**, Lemma 5.1 and Remark 1] and (4.12)).

Finally, since $B(D_x)w_k^\Psi=0$ and $\Delta\varphi_k^\Psi=0$, we have

$$B(D_x)w_k=B(D_x)w_k^H\to B(D_x)w^H=B(D_x)w\qquad\text{in }[L^2(\Omega)]^3,$$
$$\Delta\varphi_k=\Delta\varphi_k^H\to\Delta\varphi^H=\Delta\varphi,\qquad\text{in }L^2(\Omega).$$

This is because the integral operators have weakly singular kernels (see Lemma 3.1).

We get the claim letting $k\to+\infty$ in (6.7) and (6.8). □

THEOREM 6.1. *Assume that conditions* (5.6) *are satisfied. Then, there exists a unique solution of the second BVP* (6.1). *In particular, the density of the double layer elastopotential W* (4.2) *is given by a simple layer elastopotential $U[g]$, (see* (4.1)*), $g\in[L^p(\Sigma)]^4$ being a solution of the singular integral system*

$$-\tfrac14\,g(x)+V^2g(x)=F(x),\quad x\in\Sigma, \tag{6.9}$$

where V *is defined by* (6.3).

PROOF. Let $W \in \mathcal{D}^p$. From (6.2) the boundary condition $[P(D_x, n)W]^+ = F$ turns into the system (6.9) which can be rewritten in the following way

$$\Big(-\frac{I}{2} + V\Big)\Big(\frac{1}{2}g + Vg\Big) = F.$$

It is known (see [**20**, Theorem 9.3]) that there exists $h \in [L^p(\Sigma)]^4$ such that

$$-\frac{1}{2}\,h + Vh = F$$

and from [**20**, Theorem 9.2] there exists $g \in [L^p(\Sigma)]^4$ solution of the system

$$\frac{1}{2}\,g + Vg = h.$$

Consequently, system (6.9) is solvable and the second BVP (6.1) admits a solution.

Finally, for the proof of the uniqueness we can proceed as in the proof of [**20**, Theorem 6.1], keeping in mind the identities (6.5) and (6.6). □

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