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LAGRANGE-TYPE OPERATORS ASSOCIATED WITH U_n^ϱ

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Abstract. We consider a class of positive linear operators which, among others, constitute a link between the classical Bernstein operators and the genuine Bernstein–Durrmeyer mappings. The focus is on their relation to certain Lagrange-type interpolators associated to them, a well known feature in the theory of Bernstein operators. Considerations concerning iterated Boolean sums and the derivatives of the operator images are included. Our main tool is the eigenstructure of the members of the class.

1. Introduction

The present note continues the authors' research on a class U_n^ϱ of Bernstein-type operators which constitute a link between the classical Bernstein operators $B_n(\varrho \to \infty)$ and the so-called genuine Bernstein–Durrmeyer operators $U_n(\varrho = 1)$. This class of operators was introduced by Păltănea in [**12**] and has since then been the subject of several papers dealing with various aspects of the matter. A detailed description is given in the next section. Here we present their relationship to the Lagrange interpolation using the eigenstructure of the U_n^ϱ, thus extending in a natural way results known for B_n. The eigenstructure is also useful to describe the convergence behavior of iterated Boolean sums based on a single mapping U_n^ϱ, ϱ and n fixed. In the final Section 4 a relationship between certain divided differences used in Section 2 and the representation of the derivatives $(U_n^\varrho)^{(j)}$ is established.

2. Lagrange type operators associated with U_n^ϱ

2.1. A first description of L_n^ϱ. Let $\varrho > 0$ and $n \geqslant 1$ be fixed. Consider the functionals $F_{n,k}^\varrho : C[0,1] \to \mathbb{R}$, $k = 0, 1, \dots, n$, defined by

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Dedicated to Professor Giuseppe Mastroianni on the occasion of his 75th birthday.

$$F_{n,0}^{\varrho}(f) = f(0), F_{n,n-1}^{\varrho}(f) = f(1),$$
$$F_{n,k}^{\varrho}(f) = \int_0^1 \frac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho,(n-k)\varrho)} f(t)\,dt, \quad k = 1,\dots,n-1.$$

The operator $U_n^{\varrho} : C[0,1] \to \Pi_n$ (the space of algebraic polynomials of degree no greater than n) is given by $U_n^{\varrho}(f;x) := \sum_{k=0}^{n} F_{n,k}^{\varrho}(f) p_{n,k}(x)$, $f \in C[0,1]$, where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$. With a slight abuse of notation consider also the operator $U_n^{\varrho} : \Pi_n \to \Pi_n$. Its eigenvalues $\lambda_{n,k}^{(n)}$ and eigenfunctions $p_{n,k}^{(n)}$, $k = 0,1,\dots,n$, are described in [**4**]; in particular,

$$1 = \lambda_{\varrho,0}^{(n)} = \lambda_{\varrho,1}^{(n)} > \lambda_{\varrho,2}^{(n)} > \lambda_{\varrho,3}^{(n)} > \cdots > \lambda_{\varrho,n}^{(n)} > 0,$$

which means that $U_n^{\varrho} : \Pi_n \to \Pi_n$ is invertible. Consider the inverse operator $(U_n^{\varrho})^{-1} : \Pi_n \to \Pi_n$ (note the domain of definition here!) and define $L_n^{\varrho} : C[0,1] \to \Pi_n$ by

$$L_n^{\varrho} = (U_n^{\varrho})^{-1} \circ U_n^{\varrho}. \tag{2.1}$$

Then $U_n^{\varrho}(L_n^{\varrho} f) = U_n^{\varrho}(f), f \in C[0,1]$, which leads to

$$F_{n,k}^{\varrho}(L_n^{\varrho} f) = F_{n,k}^{\varrho}(f), \quad f \in C[0,1], \quad k = 0,1,\dots,n. \tag{2.2}$$

Relation (2.2) expresses an interpolatory property with respect to the functionals $F_{n,0}^{\varrho},\dots,F_{n,n}^{\varrho}$; more precisely, given $f \in C[0,1], L_n^{\varrho} f$ is the unique polynomial in Π_n satisfying (2.2). In particular, $L_n p = p$, for all $p \in \Pi_n$. It is known [**3**] that

$$\lim_{\varrho\to\infty} F_{n,k}^{\varrho}(f) = f(k/n), \quad f \in C[0,1], \quad k = 0,1,\dots,n. \tag{2.3}$$

This entails

$$\lim_{\varrho\to\infty} U_n^{\varrho}(f) = B_n f, \quad \text{uniformly on } [0,1], \tag{2.4}$$

for all $f \in C[0,1]$; here B_n denotes the classical Bernstein operator on $C[0,1]$. Let L_n be the Lagrange operator on $C[0,1]$ based on the nodes $0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1$. It is easy to see that

$$L_n = B_n^{-1} \circ B_n, \tag{2.5}$$

where $C[0,1] \xrightarrow{B_n} \Pi_n \xrightarrow{(B_n)^{-1}} \Pi_n$. Similar to the above, the symbol $(B_n)^{-1}$ means the inverse of $B_n : \Pi_n \to \Pi_n$. We will see that

$$\lim_{\varrho\to\infty} L_n^{\varrho}(f) = L_n f, \quad \text{uniformly on } [0,1], \tag{2.6}$$

for all $f \in C[0,1]$. If we interpret (2.4) by saying that $U_n^{\infty} = B_n$, then (2.6) can be interpreted as $L_n^{\infty} = L_n$. On the other hand, one has (see [**4**, (3.3)])

$$U_n^{\varrho} f = \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad f \in C[0,1], \tag{2.7}$$

where $\mu_{\varrho,k}^{(n)}$ are the dual functionals of $p_{\varrho,k}^{(n)}$. This leads to

$$L_n^\varrho(f) = (U_n^\varrho)^{-1}(U_n^\varrho f) = \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} \frac{1}{\lambda_{\varrho,k}^{(n)}} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad \text{i.e.,}$$

$$L_n^\varrho(f) = \sum_{k=0}^{n} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad f \in C[0,1]. \tag{2.8}$$

So the relationship between U_n^ϱ and L_n^ϱ, expressed by (2.7) and (2.8), is similar to the relationship between $B_n = U_n^\infty$ and $L_n = L_n^\infty$, described in [**1**, Sect. 6].

To conclude this section let us recall that $U_n^\varrho = B_n \circ \overline{\mathbb{B}}_{n\varrho}$ where

$$\overline{\mathbb{B}}_r(f;x) = \begin{cases} f(0), & x = 0; \\ \dfrac{1}{B(rx, r-rx)} \displaystyle\int_0^1 t^{rx-1}(1-t)^{r-rx-1} f(t)\,dt, & 0 < x < 1; \\ f(1), & x = 1. \end{cases}$$

is the Lupaş–Mühlbach Beta operator (see [**7**, p. 63], [**11**]). From (2.1) and (2.5) it follows

$$L_n^\varrho = (\overline{\mathbb{B}}_{n\varrho})^{-1} \circ L_n \circ \overline{\mathbb{B}}_{n\varrho}, \ \varrho > 0, \tag{2.9}$$

i.e., the operators L_n^ϱ and L_n are similar.

2.2. A concrete approach to L_n^ϱ. In order to obtain other representations of the operators L_n^ϱ we shall use a classical method described, for example, in [**13**, Sect. 1.2], [**2**], [**8**, Sect. 1.3]. Let $n \geqslant 1, \varrho > 0$ and $f \in C[0,1]$ be fixed. Then $L_n^\varrho f \in \Pi_n$ has the form $L_n^\varrho f = c_0 e_0 + c_1 e_1 + \cdots + c_n e_n$, where $e_j(x) = x^j$, $x \in [0,1]$, $j \geqslant 0$, and $c_j \in \mathbb{R}$. According to (2.2), the coefficients $c_0, \ldots, c_n$ satisfy the system of equations

$$\begin{aligned} L_n^\varrho f &= c_0 e_0 + c_1 e_1 + \cdots + c_n e_n \\ F_{n,0}^\varrho(f) &= c_0 F_{n,0}^\varrho(e_0) + c_1 F_{n,0}^\varrho(e_1) + \cdots + c_n F_{n,0}^\varrho(e_n) \\ &\cdots \\ F_{n,n}^\varrho(f) &= c_0 F_{n,n}^\varrho(e_0) + c_1 F_{n,n}^\varrho(e_1) + \cdots + c_n F_{n,n}^\varrho(e_n). \end{aligned}$$

By eliminating $c_0, \ldots, c_n$, we get

$$\begin{vmatrix} L_n^\varrho f & e_0 & e_1 & \ldots & e_n \\ F_{n,0}^\varrho(f) & F_{n,0}^\varrho(e_0) & F_{n,0}^\varrho(e_1) & \ldots & F_{n,0}^\varrho(e_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ F_{n,n}^\varrho(f) & F_{n,n}^\varrho(e_0) & F_{n,n}^\varrho(e_1) & \ldots & F_{n,n}^\varrho(e_n) \end{vmatrix} = 0 \tag{2.10}$$

Since $F_{n,j}^\varrho(e_m) = (i\varrho)^{\overline{m}}/(n\varrho)^{\overline{m}}$, where by $x^{\overline{k}} = x(x+1) \cdot \cdots \cdot (x+k-1)$ we have denoted the rising factorial, from (2.10) we get after elementary computations

(2.11)
$$L_n^\varrho f = -V\Big(0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\Big)^{-1} \begin{vmatrix} 0 & e_0 & \frac{(n\varrho)^{\bar{1}}}{(n\varrho)} e_1 & \dots & \frac{(n\varrho)^{\bar{n}}}{(n\varrho)^n} e_n \\ F_{n,0}^\varrho(f) & 1 & \frac{(0\varrho)^{\bar{1}}}{(n\varrho)} & \dots & \frac{(0\varrho)^{\bar{n}}}{(n\varrho)^n} \\ F_{n,1}^\varrho(f) & 1 & \frac{(1\varrho)^{\bar{1}}}{(n\varrho)} & \dots & \frac{(1\varrho)^{\bar{n}}}{(n\varrho)^n} \\ \dots & \dots & \dots & \dots & \dots \\ F_{n,n}^\varrho(f) & 1 & \frac{(n\varrho)^{\bar{1}}}{(n\varrho)} & \dots & \frac{(n\varrho)^{\bar{n}}}{(n\varrho)^n} \end{vmatrix}$$

where V is the Vandermonde determinant. Now we are in the position to prove (2.6).

THEOREM 2.1. *For each* $f \in C[0,1]$ *we have*

$$\lim_{\varrho\to\infty} L_n^\varrho f = L_n f, \text{ uniformly on } [0,1].$$

PROOF. Let us remark that

$$\lim_{\varrho\to\infty} \frac{(j\varrho)^{\bar{k}}}{(n\varrho)^k} = \Big(\frac{j}{n}\Big)^k. \tag{2.12}$$

From (2.3), (2.11), and (2.12) we deduce
(2.13)
$$\lim_{\varrho\to\infty} L_n^\varrho f = -V\Big(0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\Big)^{-1} \begin{vmatrix} 0 & e_0 & e_1 & \dots & e_n \\ f(0) & 1 & 0 & \dots & 0 \\ f(\frac{1}{n}) & 1 & \frac{1}{n} & \dots & (\frac{1}{n})^n \\ \dots & \dots & \dots & \dots & \dots \\ f(\frac{n-1}{n}) & 1 & \frac{n-1}{n} & \dots & (\frac{n-1}{n})^n \\ f(1) & 1 & 1 & \dots & 1 \end{vmatrix}.$$

Since the right-hand side of (2.13) is $L_n f$ (see, e.g., [**15**, Sect. 3.1], [**8**, Sect. 1.3]), the proof is complete. □

2.3. The associated divided difference. The coefficient of e_n in the expression of $L_n f$ is the divided difference of f at the nodes $0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1$, and is given by (see e.g. [**15**, Sect. 2.6]):

$$\Big[0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1; f\Big] = V\Big(0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\Big)^{-1} \times \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & f(0) \\ 1 & \frac{1}{n} & (\frac{1}{n})^2 & \dots & (\frac{1}{n})^{n-1} & f(\frac{1}{n}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{n-1}{n} & (\frac{n-1}{n})^2 & \dots & (\frac{n-1}{n})^{n-1} & f(\frac{n-1}{n}) \\ 1 & 1 & 1 & \dots & 1 & f(1) \end{vmatrix}. \tag{2.14}$$

Let us denote by $[F_{n,0}^\varrho, F_{n,1}^\varrho, \dots, F_{n,n}^\varrho; f]$ the coefficient of e_n in $L_n^\varrho f$.

THEOREM 2.2. *For each $f \in C[0,1]$ we have*

$$[F^\varrho_{n,0}, F^\varrho_{n,1}, \dots, F^\varrho_{n,n}; f] = \frac{(n\varrho)^{\bar n}}{(n\varrho)^n} V\Big(0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\Big)^{-1} \times \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & F^\varrho_{n,0}(f) \\ 1 & \frac{1}{n} & (\frac{1}{n})^2 & \dots & (\frac{1}{n})^{n-1} & F^\varrho_{n,1}(f) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{n-1}{n} & (\frac{n-1}{n})^2 & \dots & (\frac{n-1}{n})^{n-1} & F^\varrho_{n,n-1}(f) \\ 1 & 1 & 1 & \dots & 1 & F^\varrho_{n,n}(f) \end{vmatrix}. \tag{2.15}$$

PROOF. From (2.11) we get immediately

$$\begin{aligned}
&[F^\varrho_{n,0}, F^\varrho_{n,1}, \dots, F^\varrho_{n,n}; f] \\
&= \frac{(n\varrho)^{\bar n}}{(n\varrho)^n} V\left(0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right)^{-1} \begin{vmatrix} 1 & \dfrac{(0\varrho)^{\bar 1}}{n\varrho} & \dots & \dfrac{(0\varrho)^{\overline{n-1}}}{(n\varrho)^{n-1}} & F^\varrho_{n,0}(f) \\ 1 & \dfrac{(1\varrho)^{\bar 1}}{n\varrho} & \dots & \dfrac{(1\varrho)^{\overline{n-1}}}{(n\varrho)^{n-1}} & F^\varrho_{n,1}(f) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \dfrac{(n\varrho)^{\bar 1}}{n\varrho} & \dots & \dfrac{(n\varrho)^{\overline{n-1}}}{(n\varrho)^{n-1}} & F^\varrho_{n,n}(f) \end{vmatrix} \\
&= \frac{(n\varrho)^{\bar n}/(n\varrho)^n}{(n\varrho)^{\frac{n(n-1)}{2}} V\left(0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right)} \begin{vmatrix} 1 & (0\varrho)^{\bar 1} & \dots & (0\varrho)^{\overline{n-1}} & F^\varrho_{n,0}(f) \\ 1 & (1\varrho)^{\bar 1} & \dots & (1\varrho)^{\overline{n-1}} & F^\varrho_{n,1}(f) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & (n\varrho)^{\bar 1} & \dots & (n\varrho)^{\overline{n-1}} & F^\varrho_{n,n}(f) \end{vmatrix} \\
&= \frac{(n\varrho)^{\bar n}/(n\varrho)^n}{(n\varrho)^{\frac{n(n-1)}{2}} V\left(0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right)} \begin{vmatrix} 1 & 0 & \dots & 0 & F^\varrho_{n,0}(f) \\ 1 & \varrho & \dots & \varrho^{n-1} & F^\varrho_{n,1}(f) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n\varrho & \dots & (n\varrho)^{n-1} & F^\varrho_{n,n}(f) \end{vmatrix}
\end{aligned}$$

and this leads to (2.15). □

REMARK 2.1. From (2.3), (2.14) and (2.15) we derive

$$\lim_{\varrho\to\infty} [F^\varrho_{n,0}, F^\varrho_{n,1}, \dots, F^\varrho_{n,n}; f] = \Big[0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1; f\Big]$$

for all $f \in C[0,1]$. Moreover, let $f \in C[0,1]$ and Φ_n a (Lagrange type) polynomial with $\Phi_n \in \Pi_n$, $\Phi_n(\frac{j}{n}) = F^\varrho_{n,j}(f)$, $j = 0, \dots, n$. From (2.14) and (2.15) it is easy to deduce

$$[F^\varrho_{n,0}, F^\varrho_{n,1}, \dots, F^\varrho_{n,n}; f] = \frac{(n\varrho)^{\bar n}}{(n\varrho)^n} \Big[0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1; \Phi_n\Big].$$

The last (classical) divided difference can be computed by recurrence; see [**15**].

REMARK 2.2. Using (2.8), we see that

$$\mu^{(n)}_{\varrho,n} = [F^\varrho_{n,0}, F^\varrho_{n,1}, \dots, F^\varrho_{n,n}; \cdot]. \tag{2.16}$$

For the Bernstein operator (i.e., for $\varrho \to \infty$), (2.16) can be found in [**1**, p. 164] .

REMARK 2.3. $u^\varrho_{n+1} := e_{n+1} - L^\varrho_n e_{n+1}$ is the unique monic polynomial in Π_{n+1} such that $L^\varrho_n u^\varrho_{n+1} = 0$. For example, $u^\infty_{n+1} = x(x-1)(x-\frac{1}{n})\cdot\cdots\cdot(x-\frac{n-1}{n})$. Moreover, $u^1_{n+1}(x) = x(x-1)J_{n-1}(x)$, where $J_0(x), J_1(x), \ldots$ are the monic Jacobi polynomials, orthogonal on $[0,1]$ with respect to the weight function $x(1-x)$. Indeed, $F^1_{n,0}(u^1_{n+1}) = F^1_{n,n}(u^1_{n+1}) = 0$, and

$$\int_0^1 t^{k-1}(1-t)^{n-k-1}u^1_{n+1}(t)\,dt = \int_0^1 t^{k-1}(1-t)^{n-k-1}t(t-1)J_{n-1}(t)\,dt = 0$$

(since for all $k = 1,\ldots,n-1$, $t^{k-1}(1-t)^{n-k-1}$ is a polynomial of degree $n-2$). This implies $F^1_{n,k}(u^1_{n+1}) = 0, k = 1,\ldots,n-1$, and so $L^1_n(u^1_{n+1}) = 0$. For general values of ϱ we have not found such compact representations.

Now we shall prove a general result.

THEOREM 2.3. *The polynomial u^ϱ_{n+1} has $n+1$ distinct roots in $[0,1]$.*

PROOF. By using Remark 2.3 and (2.9) we get $(\overline{\mathbb{B}}^{-1}_{n\varrho}\circ L_n\circ\overline{\mathbb{B}}_{n\varrho})(u^\varrho_{n+1}) = 0$, which entails $L_n(\overline{\mathbb{B}}_{n\varrho}u^\varrho_{n+1}) = 0$. Now the same Remark 2.3 yields

$$\overline{\mathbb{B}}_{n\varrho}u^\varrho_{n+1} = \frac{(n\varrho)^{n+1}}{(n\varrho)^{\overline{n+1}}}u^\infty_{n+1}.$$

So $\overline{\mathbb{B}}_{n\varrho}u^\varrho_{n+1}$ has $n+1$ distinct roots in $[0,1]$. According to [**6**], u^ϱ_{n+1} has at least $n+1$ distinct roots in $[0,1]$; to finish the proof, it suffices to remark that u^ϱ_{n+1} is a polynomial of degree $n+1$. □

Now let us recall the representation of L_n in terms of the fundamental Lagrange polynomials:

$$L_nf(x) = \sum_{k=0}^n l_{n,k}(x)f\Big(\frac{k}{n}\Big), \quad f\in C[0,1], \quad x\in[0,1].$$

Using (2.9) we infer that L^ϱ_n has a similar representation, namely

$$L^\varrho_nf(x) = \sum_{k=0}^n l^\varrho_{n,k}(x)F^\varrho_{n,k}(f),$$

where

$$l^\varrho_{n,k} := \overline{\mathbb{B}}^{-1}_{n\varrho}(l_{n,k}), \quad k = 0,1,\ldots,n. \tag{2.17}$$

THEOREM 2.4. *For each $k = 0,1,\ldots,n$, the polynomial $l^\varrho_{n,k}$ has n distinct roots in $[0,1]$.*

PROOF. Since, according to (2.17), $\overline{\mathbb{B}}_{n\varrho}(l^\varrho_{n,k}) = l_{n,k}$, the proof is similar to that of Theorem 2.3 and we omit it. □

In what follows we shall establish mean value theorems for the generalized divided difference and for the remainder $R^\varrho_nf := f - L^\varrho_n$.

Theorem 2.5. *Let $n \geqslant 1, \varrho > 0$ and $f \in C[0,1]$ be given. Then there exist $0 = t_0 < t_1 < \cdots < t_n = 1$ such that*

$$R_n^\varrho f(t_i) = 0, \quad i = 0, 1, \ldots, n. \tag{2.18}$$

Proof. According to (2.2), $F_{n,k}^\varrho(R_n^\varrho f) = 0, k = 0, 1, \ldots, n$, i.e.

$$R_n^\varrho f(0) = R_n^\varrho f(1) = 0, \tag{2.19}$$

$$\int_0^1 t^{k\varrho-1}(1-t)^{(n-k)\varrho-1} R_n^\varrho f(t)\,dt = 0, \quad k = 1, \ldots, n-1. \tag{2.20}$$

Set $x := \big(\frac{t}{1-t}\big)^\varrho$, $j := k-1$, and $h(x) := R_n^\varrho f\big(\frac{x^{1/\varrho}}{1+x^{1/\varrho}}\big)$, $x \geqslant 0$. Then (2.20) becomes

$$\int_0^\infty (1+x^{1/\varrho})^{-n\varrho} x^j h(x)\,dx = 0, \quad j = 0, 1, \ldots, n-2. \tag{2.21}$$

Suppose that the number of the roots of h in $(0, +\infty)$ is at most $n-2$, i.e.,

$$\{x \in (0, +\infty) : h(x) = 0\} = \{x_1, \ldots, x_r\}, r \leqslant n-2.$$

Then there exists a polynomial $p \in \Pi_{n-2}$ such that $\{x \in (0, +\infty) : p(x) = 0\} \subset \{x_1, \ldots, x_r\}$ and, moreover,

$$\int_0^\infty (1+x^{1/\varrho})^{-n\varrho} p(x) h(x)\,dx > 0. \tag{2.22}$$

Obviously (2.22) contradicts (2.21), which means that h has at least $n-1$ roots in $(0, +\infty)$. It follows that $R_n^\varrho f$ has at least $n-1$ roots in $(0,1)$. Together with (2.19), this proves the theorem. $\square$

Corollary 2.1. *Let $n \geqslant 1, \varrho > 0$ and $f \in C^n[0,1]$ be given. Then there exists $\xi \in (0,1)$ such that*

$$[F_{n,0}^\varrho, F_{n,1}^\varrho, \ldots, F_{n,n}^\varrho; f] = \frac{f^{(n)}(\xi)}{n!}.$$

Proof. According to Theorem 2.5, $R_n^\varrho f$ has at least $n+1$ roots in $[0,1]$. It follows that $(R_n^\varrho f)^{(n)}$ has at least a root $\xi \in (0,1)$. Thus

$$0 = (R_n^\varrho f)^{(n)}(\xi) = f^{(n)}(\xi) - n![F_{n,0}^\varrho, F_{n,1}^\varrho, \ldots, F_{n,n}^\varrho; f],$$

and the proof is finished. $\square$

Let now $n \geqslant 1, \varrho > 0$ and $f \in C^{n+1}[0,1]$ be given. Consider the points $t_0, t_1, \ldots, t_n$ satisfying (2.18), and let $\omega(t) = (t-t_0)\cdots(t-t_n)$.

Corollary 2.2. *Let $x \in [0,1] \setminus \{t_0, t_1, \ldots, t_n\}$. Under the above assumption there exists $\eta_x \in (0,1)$ such that*

$$R_n^\varrho f(x) = \omega(x) \frac{f^{(n+1)}(\eta_x)}{(n+1)!}.$$

Proof. Consider the function $w(t) = \omega(x) R_n^\varrho f(t) - \omega(t) R_n^\varrho f(x)$, $t \in [0,1]$. Then $x, t_0, \ldots, t_n$ are roots of w, which means that there exists $\eta_x \in (0,1)$ such that $w^{(n+1)}(\eta_x) = 0$. Now it suffices to remark that $w^{(n+1)}(t) = \omega(x) f^{(n+1)}(t) - (n+1)! R_n^\varrho f(x)$. $\square$

Corollaries 2.1 and 2.2 generalize the mean value theorems for the divided difference and the remainder in the classical Lagrange interpolation; see [**15**, Sect. 3.1], [**8**, Sect. 1.4]..

3. Iterated Boolean sums of the operators U_n^ϱ

For $M \geqslant 1$, let $\bigoplus^M U_n^\varrho = I - (I - U_n^\varrho)^M$ be the iterated Boolean sum of U_n^ϱ; here I stands for the identity operator on $C[0,1]$. Iterated Boolean sums of the classical Bernstein operator and modifications thereof were investigated by numerous authors in the past, among them Mastroiani and Occorsio [**9, 10**]. Some historical information on this method which may be traced to Natanson can be found in [**5**]. From a general result of Wenz [**16**, Theorem 2] it follows that $\lim_{M\to\infty} \bigoplus^M U_n^\varrho f = L_n^\varrho f$, $f \in C[0,1]$, $n \geqslant 1$. With the notation from the preceding sections, we can say more, namely

THEOREM 3.1. *Let $n \geqslant 2$ and $f \in C[0,1]$ be given. Then*

$$\lim_{M\to\infty} (1-\lambda_{\varrho,n}^{(n)})^{-M}\Big(\bigoplus^M U_n^\varrho f - L_n^\varrho f\Big) = -[F_{n,0}^\varrho, F_{n,1}^\varrho, \dots, F_{n,n}^\varrho; f]\, p_{\varrho,n}^{(n)}, \tag{3.1}$$

uniformly on $[0,1]$.

PROOF. We have, according to (2.7)

$$\begin{aligned}
\bigoplus^M U_n^\varrho f &= \big(I - (I - U_n^\varrho)^M\big) f = \sum_{i=1}^M (-1)^{i+1}\binom{M}{i}(U_n^\varrho)^i f\\
&= \sum_{i=1}^M (-1)^{i+1}\binom{M}{i}\sum_{k=0}^n \big(\lambda_{\varrho,k}^{(n)}\big)^i p_{\varrho,k}^{(n)}\mu_{\varrho,k}^{(n)}(f)\\
&= \sum_{k=0}^n p_{\varrho,k}^{(n)}\mu_{\varrho,k}^{(n)}(f)\sum_{i=1}^M (-1)^{i+1}\binom{M}{i}\big(\lambda_{\varrho,k}^{(n)}\big)^i\\
&= \sum_{k=0}^n p_{\varrho,k}^{(n)}\mu_{\varrho,k}^{(n)}(f)\big(1 - \big(1 - \big(\lambda_{\varrho,k}^{(n)}\big)^M\big)\big).
\end{aligned}$$

Combined with (2.8) this yields

$$\bigoplus^M U_n^\varrho f - L_n^\varrho f = -\sum_{k=0}^n p_{\varrho,k}^{(n)}\mu_{\varrho,k}^{(n)}(f)\big(1-\lambda_{\varrho,k}^{(n)}\big)^M, \quad \text{i.e.,}$$

$$\big(1-\lambda_{\varrho,n}^{(n)}\big)^{-M}\Big(\bigoplus^M U_n^\varrho f - L_n^\varrho f\Big) = -p_{\varrho,n}^{(n)}\mu_{\varrho,n}^{(n)}(f) - \sum_{k=0}^{n-1}\mu_{\varrho,k}^{(n)}(f)\mu_{\varrho,k}^{(n)}(f)\bigg(\frac{1-\lambda_{\varrho,k}^{(n)}}{1-\lambda_{\varrho,n}^{(n)}}\bigg)^M.$$

Since $0 < \big(1-\lambda_{\varrho,k}^{(n)}\big)/\big(1-\lambda_{\varrho,n}^{(n)}\big) < 1$, $k = 0,\dots,n-1$, we get

$$\lim_{M\to\infty}\big(1-\lambda_{\varrho,n}^{(n)}\big)^{-M}\Big(\bigoplus^M U_n^\varrho f - L_n^\varrho f\Big) = -\mu_{\varrho,n}^{(n)}(f)p_{\varrho,n}^{(n)}.$$

To conclude the proof it suffices to use (2.16). □

REMARK 3.1. For $\varrho \to \infty$, (3.1) was obtained in [**14**, Th. 26.7].

4. The derivatives of U_n^ϱ

In this section we show that there is a natural relationship between the derivatives of the operator images and the divided differences $[\dots;\Phi_n]$ which we introduced in Remark 2.1.

THEOREM 4.1. *With the usual notation the following relationships hold*

$$\text{(i)}\qquad (U_n^\varrho(f;x))' = n\sum_{k=0}^{n-1} p_{n-1,k}(x)\Delta^1 F_{n,k}^\varrho(f) = \sum_{k=0}^{n-1} p_{n-1,k}(x)\Big[\frac{k}{n},\frac{k+1}{n};\Phi_n\Big];$$

$$\begin{aligned}\text{(ii)}\ (U_n^\varrho(f;x))^{(j)} &= n(n-1)\cdots(n-j+1)\sum_{k=0}^{n-j} p_{n-j,k}(x)\Delta^j F_{n,k}^\varrho(f)\\ &= n(n-1)\cdots(n-j+1)\sum_{k=0}^{n-j} p_{n-j,k}(x)\frac{j!}{n^j}\Big[\frac{k}{n},\dots,\frac{k+j}{n};\Phi_n\Big];\end{aligned}$$

$$\text{(iii)}\qquad U_n^\varrho(f;x) = \sum_{k=0}^{n}\binom{n}{k}\Delta^k F_{n,0}^\varrho(f)e_k(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{n^k}\Big[0,\frac{1}{n},\dots,\frac{k}{n};\Phi_n\Big]e_k(x);$$

where as before $\Phi_n\left(\frac{k}{n}\right) = F_{n,k}^\varrho(f)$.

PROOF. (i) The forward difference was defined in [**3**, p. 792] by

$$\Delta^j F_{n,k}^\varrho(f) = \sum_{i=0}^{j}\binom{j}{i}(-1)^{i+j}F_{n,k+i}^\varrho(f).$$

Thus we have

$$\Big[\frac{k}{n},\frac{k+1}{n};\Phi_n\Big] = \frac{\Phi_n(\frac{k+1}{n})-\Phi_n(\frac{k}{n})}{\frac{k+1}{n}-\frac{k}{n}} = n\big[F_{n,k+1}^\varrho(f)-F_{n,k}^\varrho(f)\big] = n\Delta^1 F_{n,k}^\varrho(f);$$

(ii) The first equality can be found in [**3**, p. 792]. It remains to show that

$$\Delta^j F_{n,k}^\varrho(f) = \frac{j!}{n^j}\Big[\frac{k}{n},\dots,\frac{k+j}{n};\Phi_n\Big].$$

We have that $\Delta^{j+1}F_{n,k}^\varrho(f) = \Delta(\Delta^j F_{n,k}^\varrho(f)) = \Delta^j F_{n,k+1}^\varrho(f) - \Delta^j F_{n,k}^\varrho(f)$. By using the recurrence formula for divided differences (see e.g. [**15**, p.104]) we get

$$\begin{aligned}\Delta^j F_{n,k+1}^\varrho(f) - \Delta^j F_{n,k}^\varrho(f) &= \frac{j!}{n^j}\cdot\frac{j+1}{n}\cdot\frac{\big[\frac{k+1}{n},\dots,\frac{k+j+1}{n};\Phi_n\big]-\big[\frac{k}{n},\dots,\frac{k+j}{n};\Phi_n\big]}{\frac{k+j+1}{n}-\frac{k}{n}}\\ &= \frac{(j+1)!}{n^{j+1}}\Big[\frac{k}{n},\dots,\frac{k+j+1}{n};\Phi_n\Big] = \Delta^{j+1}F_{n,k}^\varrho(f),\\ U_n^\varrho(f;x) &= \sum_{j=0}^{n}\frac{(U_n^\varrho f)^{(j)}(0)}{j!}x^k\end{aligned}$$

and show that $(U_n^\varrho(f;x))^{(j)} = n(n-1)\cdots(n-j+1)\Delta^j F_{n,0}^\varrho(f)$.

To this end we take $x=0$ in (ii); because $p_{n-j,0}(0)=1$ and for all $k\geqslant 1$, $p_{n-j,k}(0)=0$, from $\sum_{k=0}^{n-j}$ only the first term remains, which concludes the proof. $\square$

REMARK 4.1. In the case $\varrho \to \infty$ we can find the analogues of the above relationships in [**15**, pp. 300–302].

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