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NEW MODULI OF SMOOTHNESS

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Abstract. We discuss various properties of the new modulus of smoothness

$$\omega^{\varphi}_{k,r}(f^{(r)},t)_p := \sup_{0<h\leqslant t} \|\mathcal{W}^r_{kh}(\cdot)\Delta^k_{h\varphi(\cdot)}(f^{(r)},\cdot)\|_{\mathbb{L}_p[-1,1]},$$

where $\varphi(x) := \sqrt{1-x^2}$ and $\mathcal{W}_\delta(x) = \big((1-x-\delta\varphi(x)/2)(1+x-\delta\varphi(x)/2)\big)^{1/2}$. Related moduli with more general weights are also considered.

1. Introduction

1.1. Trigonometric approximation. Let $\widetilde{\mathbb{L}}_p$, $1 \leqslant p \leqslant \infty$, denote the space of 2π-periodic measurable functions for which the norm $\|f\|_{\widetilde{\mathbb{L}}_p} := \big(\int_{-\pi}^{\pi} |f(x)|^p dx\big)^{1/p}$ is finite. Here, by $\widetilde{\mathbb{L}}_\infty$ we mean the space of continuous 2π-periodic functions $\widetilde{\mathbb{C}}$ equipped with the uniform norm, i.e., $\|f\|_{\widetilde{\mathbb{C}}} := \max_{x\in[-\pi,\pi]} |f(x)|$.

Let $\mathcal{T}_n$, $n \in \mathbb{N}$, be the space of $(n-1)$st degree trigonometric polynomials

$$T_n(x) = \sum_{j=0}^{n-1}(a_j \cos jx + b_j \sin jx).$$

For $f \in \widetilde{\mathbb{L}}_p$, denote by

$$\Delta^k_h(f,x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f(x+(i-k/2)h) \tag{1.1}$$

the kth symmetric difference of the function f, and by

$$\omega_k(f,t)_p := \inf_{h\in[0,t]} \|\Delta^k_h(f,\cdot)\|_{\widetilde{\mathbb{L}}_p}$$

its kth modulus of smoothness. Finally, let $\widetilde{E}_n(f)_p := \inf_{T_n\in\mathcal{T}_n} \|f-T_n\|_{\widetilde{\mathbb{L}}_p}$ denote the degree of approximation of f by trigonometric polynomials from $\mathcal{T}_n$.

In 1908, de la Vallée Poussin (see [**20**, Section 7], for example) posed a problem on a connection between the rate of polynomial approximation of functions and

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their differential properties. To quote de la Vallée Poussin [**20**, p. 119], “It is the memoir by D. Jackson [**9**] which answers most completely the direct question, and that of S. Bernstein [**3**] which answers most completely the inverse problem”. These results were generalized by de la Vallée Poussin in [**21**], though as he writes in [**20**, p. 119], “I combined the results obtained by the two authors above named, and filled them out in many points; I changed or simplified the proofs; but I contributed little in the way of new materials to the construction”.

In 1911, Jackson [**9**, Theorem VIII] (see also [**8**, p. 428]) proved the following inequality (which is now commonly known as one of “Jackson’s inequalities”):

$$\widetilde{E}_n(f)_\infty \leqslant c\omega_1(f, n^{-1})_\infty, \quad n \geqslant 1.$$

This result was later extended by Zygmund [**22**, Theorems 8 and 8′], Bernstein [**2**], Akhiezer [**1**, Section 89], and Stechkin [**15**, Theorem 1] as follows.

THEOREM $\widetilde{\mathbf{D}}_\mathbf{0}$ (**D**irect theorem, $r = 0$). *Let $k \in \mathbb{N}$. If $f \in \widetilde{\mathbb{L}}_p$, $1 \leqslant p \leqslant \infty$, then $\widetilde{E}_n(f)_p \leqslant c(k)\omega_k(f, n^{-1})_p$, $n \geqslant 1$.*

We note that “$r = 0$” and the subscript “0” in “$\widetilde{\mathrm{D}}_0$” will become clear once one compares this result with Theorem $\widetilde{\mathrm{D}}_\mathrm{r}$ below.

Matching inverse theorems are due to Bernstein [**3**], de la Vallée Poussin [**21**, Section 39], Quade [**19**, Theorem 1], Salem [**14**, Chapter V], Zygmund [**22**, Theorems 8, 8′, 9, and 9′], the Timan brothers [**18**], and Stechkin [**15**, Theorem 8].

THEOREM 1.1. *Let $k \in \mathbb{N}$ and $f \in \widetilde{\mathbb{L}}_p$, $1 \leqslant p \leqslant \infty$. Then*

$$\omega_k(f, n^{-1})_p \leqslant \frac{c(k)}{n^k} \sum_{\nu=1}^{n} \nu^{k-1} \widetilde{E}_\nu(f)_p, \quad n \geqslant 1.$$

This theorem can be restated in the following form.

THEOREM $\widetilde{\mathbf{I}}_\mathbf{0}$ (**I**nverse theorem, $r = 0$). *Let $k \in \mathbb{N}$ and let $\phi : [0, 1] \to [0, \infty)$ be a nondecreasing function such that $\phi(0+) = 0$. If a function $f \in \widetilde{\mathbb{L}}_p$, $1 \leqslant p \leqslant \infty$, is such that $\widetilde{E}_n(f)_p \leqslant \phi\left(n^{-1}\right)$, $n \geqslant 1$, then*

$$\omega_k(f, t)_p \leqslant c(k)t^k \int_t^1 \frac{\phi(u)}{u^{k+1}} du, \quad 0 < t \leqslant 1/2.$$

These direct and inverse theorems yield a constructive characterization of the class $\widetilde{\mathrm{Lip}}(\alpha, p) = \left\{f \in \widetilde{\mathbb{L}}_p \mid \omega_{\lfloor\alpha\rfloor+1}(f, t)_p \leqslant ct^\alpha\right\}$.

THEOREM $\widetilde{\mathbb{C}}_\mathbf{0}$ (**C**onstructive characterization, $r = 0$). *Let $f \in \widetilde{\mathbb{L}}_p$, $1 \leqslant p \leqslant \infty$, and $\alpha > 0$. If $\omega_k(f, t)_p \leqslant t^\alpha$, then $\widetilde{E}_n(f)_p \leqslant c(k)n^{-\alpha}$, $n \geqslant 1$.*

Conversely, if $0 < \alpha < k$ and $\widetilde{E}_n(f)_p \leqslant n^{-\alpha}$, $n \geqslant 1$, then $\omega_k(f, t)_p \leqslant c(k, \alpha)t^\alpha$.

Jackson’s inequalities of the second type involve differentiable functions.

Let $\widetilde{\mathbb{W}}_p^r$, $r \in \mathbb{N}$, be the space of 2π-periodic functions f such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in \widetilde{\mathbb{L}}_p$, where by $\widetilde{\mathbb{W}}_\infty^r$ we mean $\widetilde{\mathbb{C}}^r$.

The following result is an immediate consequence of Theorem $\widetilde{\mathrm{D}}_0$ and the well known property $\omega_{k+r}(f, t)_p \leqslant t^r \omega_k(f^{(r)}, t)_p$, $t > 0$.

THEOREM $\widetilde{\mathbf{D}}_{\mathbf{r}}$ (Direct theorem, $r \in \mathbb{N}$). *Let $k \in \mathbb{N}$ and $r \in \mathbb{N}$. If $f \in \widetilde{\mathbb{W}}_p^r$, then*

$$\widetilde{E}_n(f)_p \leqslant c(k,r)n^{-r}\omega_k(f^{(r)}, n^{-1})_p, \quad n \geqslant 1.$$

The following inverse theorems are due to Bernstein [**3**], de la Vallée Poussin [**21**, Section 39], Quade [**19**, Theorem 1], Zygmund [**22**, Theorems 8, 8′, 9 and 9′], Stechkin [**15**, Theorem 11], and A. Timan [**17**], [**16**, Theorem 6.1.3].

THEOREM 1.2. *Let $r \in \mathbb{N}$ and $f \in \widetilde{\mathbb{L}}_p$, $1 \leqslant p \leqslant \infty$. If $\sum_{\nu=1}^{\infty} \nu^{r-1}\widetilde{E}_\nu(f)_p < \infty$, then f is a.e. identical with a function from $\widetilde{\mathbb{W}}_p^r$. In addition, for any $k \in \mathbb{N}$,*

$$\omega_k(f^{(r)}, n^{-1})_p \leqslant \frac{c(k,r)}{n^k}\sum_{\nu=1}^{n}\nu^{k+r-1}\widetilde{E}_\nu(f)_p + c(k,r)\sum_{\nu=n+1}^{\infty}\nu^{r-1}\widetilde{E}_\nu(f)_p, \quad n \geqslant 1.$$

This theorem can be restated as follows.

THEOREM $\widetilde{\mathbf{I}}_{\mathbf{r}}$ 1 (Inverse theorem, $r \in \mathbb{N}$). *Let $k \in \mathbb{N}$, $r \in \mathbb{N}$ and $\phi : [0,1] \to [0,\infty)$ be a nondecreasing function such that $\phi(0+) = 0$ and*

$$\int_0^1 \frac{\phi(t)}{t^{r+1}}dt < \infty.$$

If $f \in \widetilde{\mathbb{L}}_p$ be such that $\widetilde{E}_n(f)_p \leqslant \phi\left(n^{-1}\right)$, $n \geqslant 1$, then f is a.e. identical with a function from $\widetilde{\mathbb{W}}_p^r$, and

$$\omega_k(f^{(r)}, t)_p \leqslant c(k,r)\left(\int_0^t \frac{\phi(u)}{u^{r+1}}du + t^k\int_t^1 \frac{\phi(u)}{u^{k+r+1}}du\right), \quad 0 < t \leqslant 1/2.$$

Finally, we have a constructive characterization of functions $f \in \widetilde{\mathbb{W}}^r$ such that $f^{(r)} \in \widetilde{\mathrm{Lip}}(\alpha - r, p)$.

THEOREM $\widetilde{\mathbb{C}}_{\mathbf{r}}$ (Constructive characterization, $r \in \mathbb{N}$). *Let $r \in \mathbb{N}$, $\alpha > r$, $f \in \widetilde{\mathbb{W}}_p^r$, $1 \leqslant p \leqslant \infty$, and $\omega_k(f^{(r)}, t)_p \leqslant t^{\alpha-r}$. Then $\widetilde{E}_n(f)_p \leqslant c(k,r)n^{-\alpha}$, $n \geqslant 1$.*

Conversely, if $f \in \widetilde{\mathbb{L}}_p$, $1 \leqslant p \leqslant \infty$, $r < \alpha < k + r$, and $\widetilde{E}_n(f)_p \leqslant n^{-\alpha}$, $n \geqslant 1$, then f is a.e. identical with a function from $\widetilde{\mathbb{W}}_p^r$ and $\omega_k(f^{(r)}, t)_p \leqslant c(k,r,\alpha)t^{\alpha-r}$.

1.2. Algebraic approximation. Let $\mathbb{L}_p[-1,1]$, $1 \leqslant p \leqslant \infty$ denote the usual $\mathbb{L}_p$ space equipped with the norm $\|f\|_p := \left(\int_{-1}^1 |f(x)|^p dx\right)^{1/p}$, where by $L_\infty[-1,1]$ we mean $C[-1,1]$ equipped with the uniform norm.

Let $\mathcal{P}_n$ denote the space of algebraic polynomials of degree $< n$ and set

$$E_n(f)_p := \inf_{P_n \in \mathcal{P}_n} \|f - P_n\|_p$$

the degree of best approximation of f by algebraic polynomials in $\mathbb{L}_p$.

Define

$$\Delta_h^k(f, x; [-1,1]) := \begin{cases} \Delta_h^k(f,x), & x \pm kh/2 \in [-1,1], \\ 0, & \text{otherwise} \end{cases},$$

where $\Delta_h^k(f,x)$ was defined in (1.1).

Finally, define the Ditzian–Totik (DT) moduli of smoothness [**6**], by

$$\omega_k^{\varphi}(f,t)_p := \sup_{0<h\leqslant t} \|\Delta_{h\varphi(\cdot)}^k(f,\cdot;[-1,1])\|_p \tag{1.2}$$

where $\varphi(x) := (1-x^2)^{1/2}$.

It is well known that the DT moduli of smoothness yield results which are completely analogous to Theorems $\widetilde{\mathrm{D}}_0$, $\widetilde{\mathrm{I}}_0$ and $\widetilde{\mathbb{C}}_0$. Namely, we have the following results (see [**6**]).

THEOREM $\mathbf{D_0}$. *Let $k\in\mathbb{N}$. If $f\in\mathbb{L}_p[-1,1]$, $1\leqslant p\leqslant\infty$, then*

$$E_n(f)_p \leqslant c(k)\omega_k^{\varphi}(f,n^{-1})_p, \quad n\geqslant k.$$

THEOREM $\mathbf{I_0}$. *Let $k\in\mathbb{N}$ and $\phi:[0,1]\to[0,\infty)$ be a nondecreasing function such that $\phi(0+)=0$. If a function $f\in\mathbb{L}_p[-1,1]$, $1\leqslant p\leqslant\infty$, is such that $E_n(f)_p\leqslant\phi(n^{-1}$, $n\geqslant k$, then*

$$\omega_k^{\varphi}(f,t)_p \leqslant c(k)t^k\int_t^1 \frac{\phi(u)}{u^{k+1}}du, \quad t\in[0,1/2].$$

THEOREM $\mathbf{C_0}$. *Let $\alpha>0$ and $f\in\mathbb{L}_p[-1,1]$, $1\leqslant p\leqslant\infty$. If $\omega_k^{\varphi}(f,t)_p\leqslant t^{\alpha}$, then $E_n(f)_p\leqslant c(k)n^{-\alpha}$, $n\geqslant k$.*

Conversely, if $0<\alpha<k$ and $E_n(f)_p\leqslant n^{-\alpha}$, $n\geqslant k$, then $\omega_k^{\varphi}(f,t)_p\leqslant c(k,\alpha)t^{\alpha}$.

The purpose of this paper is to discuss our new moduli of smoothness (introduced in [**10**]) that allow to obtain the analogs of Theorems $\widetilde{\mathrm{D}}_{\mathrm{r}}$, $\widetilde{\mathrm{I}}_{\mathrm{r}}$ and $\widetilde{\mathbb{C}}_{\mathrm{r}}$.

2. New moduli of smoothness

2.1. Definitions. For $1\leqslant p<\infty$ and $r\in\mathbb{N}$, denote

$$\mathbb{B}_p^r := \{f : f^{(r-1)}\in \mathrm{AC}_{\mathrm{loc}}(-1,1) \quad \text{and} \quad \|f^{(r)}\varphi^r\|_p<+\infty\}.$$

If $p=\infty$, then

$$\mathbb{B}_\infty^r := \{f : f\in C^r(-1,1) \quad \text{and} \quad \lim_{x\to\pm1} f^{(r)}(x)\varphi^r(x)=0\}.$$

Finally, if $r=0$, then $\mathbb{B}_p^0 := \mathbb{L}_p[-1,1]$, $1\leqslant p<\infty$ and $\mathbb{B}_\infty^0 := \mathbb{C}[-1,1]$.

For $f\in\mathbb{B}_p^r$, define

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_p := \sup_{0<h\leqslant t}\|\mathcal{W}_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)},\cdot)\|_p,$$

where

$$\mathcal{W}_\delta(x) := \begin{cases} \big((1-x-\delta\varphi(x)/2)(1+x-\delta\varphi(x)/2)\big)^{1/2}, \\ \qquad\qquad\qquad\qquad 1\pm x-\delta\varphi(x)/2\in[-1,1], \\ 0, \qquad\qquad\qquad\quad \text{otherwise}. \end{cases}$$

Note that, if $r=0$, then $\omega_{k,0}^{\varphi}(f,t)_p=\omega_k^{\varphi}(f,t)_p$ are the usual DT moduli defined in (1.2).

It turns out (see [**10**, Lemma 3.2]) that if $f\in\mathbb{B}_p^r$, then $\lim_{t\to0+}\omega_{k,r}^{\varphi}(f^{(r)},t)_p=0$.

2.2. Weighted DT moduli of smoothness. Let

$$\overrightarrow{\Delta}_h^k f(x) := \begin{cases} \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} f(x+ih), & \text{if } x, x+kh \in [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\overleftarrow{\Delta}_h^k f(x) := \begin{cases} \sum_{i=0}^k \binom{k}{i}(-1)^{i} f(x-ih), & \text{if } x-kh, x \in [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

be the forward and backward kth differences, respectively. Note that

$$\overrightarrow{\Delta}_h^k f(x) := \Delta_h^k(f, x+kh/2) \quad \text{and} \quad \overleftarrow{\Delta}_h^k f(x) := \Delta_h^k(f, x-kh/2).$$

Let

$$w(x) := w_{\alpha,\beta}(x) := (1-x)^\alpha (1+x)^\beta, \quad \alpha, \beta \geqslant 0, \tag{2.1}$$

and denote $\mathbb{L}_p(w) := \mathbb{L}_p(w_{\alpha,\beta}) := \{f : [-1,1] \to \mathbb{R} \mid \|w_{\alpha,\beta} f\|_p < \infty\}$. For $f \in \mathbb{L}_p(w)$, the weighted DT moduli of smoothness were defined (see [**6**, (8.2.10) and Appendix B]) by

$$\begin{aligned} \omega_k^\varphi(f,t)_{w,p} := \sup_{0<h\leqslant t} \|w\Delta_{h\varphi}^k f\|_{\mathbb{L}_p[-1+2k^2h^2, 1-2k^2h^2]} \\ + \sup_{0<h\leqslant 2k^2t^2} \|w\overrightarrow{\Delta}_h^k f\|_{\mathbb{L}_p[-1,-1+2k^2t^2]} \\ + \sup_{0<h\leqslant 2k^2t^2} \|w\overleftarrow{\Delta}_h^k f\|_{\mathbb{L}_p[1-2k^2t^2,1]}. \end{aligned} \tag{2.2}$$

The first term on the right-hand side of (2.2) is the main part modulus which is denoted by $\Omega_k^\varphi(f,t)_{w,p}$ (see [**6**, (8.1.2)]) and is further discussed in Section 5.

It was shown in [**6**, Theorem 6.1.1] that $\omega_k^\varphi(f,t)_{w,p}$ is equivalent to the following weighted K-functional $K_{k,\varphi}(f,t^k)_{w,p}$ (with $0 < t \leqslant t_0$):

$$K_{k,\varphi}(f,t^k)_{w,p} := \inf_{g^{(k-1)} \in \mathrm{AC_{loc}}} \big(\|(f-g)w\|_p + t^k \|w\varphi^k g^{(k)}\|_p\big).$$

2.3. Properties of the new moduli. For $r \geqslant 0$ and $f \in \mathbb{B}_p^r$, we denote

$$K_{k,r}^\varphi(f^{(r)}, t^k)_p := \inf_{g \in \mathbb{B}_p^{k+r}} (\|(f^{(r)} - g^{(r)})\varphi^r\|_p + t^k \|g^{(k+r)}\varphi^{k+r}\|_p).$$

Then, we have the following equivalence results (see [**10**, Theorem 2.7]).

THEOREM 2.1. *If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$ and $f \in \mathbb{B}_p^r$, then, for all $0 < t \leqslant 2/k$,*

$$cK_{k,r}^\varphi(f^{(r)}, t^k)_p \leqslant \omega_{k,r}^\varphi(f^{(r)}, t)_p \leqslant cK_{k,r}^\varphi(f^{(r)}, t^k)_p,$$

where constants $c > 0$ and may depend only on k, r and p.

COROLLARY 2.1. *If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$ and $f \in \mathbb{B}_p^r$, then, for all $0 < t \leqslant 2/k$,*

$$cK_{k,\varphi}(f^{(r)}, t^k)_{\varphi^r,p} \leqslant \omega_{k,r}^\varphi(f^{(r)}, t)_p \leqslant cK_{k,\varphi}(f^{(r)}, t^k)_{\varphi^r,p}.$$

Also, the following was proved in [**10**, Theorem 7.1].

THEOREM 2.2. *If $f \in \mathbb{B}_p^{r+1}$, $1 \leqslant p \leqslant \infty$, $r \in \mathbb{N}_0$ and $k \geqslant 2$, then*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant ct\omega_{k-1,r+1}^{\varphi}(f^{(r+1)}, t)_p.$$

The following sharp Marchaud inequality was proved in [**4**].

THEOREM 2.3. [**4**, Theorem 7.5] *For $\alpha > -1/p$, $\beta > -1/p$, $1 < p < \infty$, $m \in \mathbb{N}$ and a weight w defined in* (2.1), *we have*

$$K_{m,\varphi}(f, t^m)_{w,p} \leqslant Ct^m \left(\int_t^1 \frac{K_{m+1,\varphi}(f, u^{m+1})_{w,p}^q}{u^{mq+1}} \, du + E_m(f)_{w,p}^q \right)^{1/q}$$

$$K_{m,\varphi}(f, t^m)_{w,p} \leqslant Ct^m \left(\sum_{n<1/t} n^{qm-1} E_n(f)_{w,p}^q \right)^{1/q},$$

where $q = \min(2, p)$ and $E_n(f)_{w,p}$ is the degree of best weighted approximation of f by polynomials from $\mathcal{P}_n$, i.e., $E_n(f)_{w,p} := \inf\{\|(f - P_n)w\|_p \mid P_n \in \mathcal{P}_n\}$.

COROLLARY 2.2. *For $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $f \in \mathbb{B}_p^r$, we have*

$$\omega_{m,r}^{\varphi}(f^{(r)}, t)_p \leqslant Ct^m \left(\int_t^1 \frac{\omega_{m+1,r}^{\varphi}(f^{(r)}, u)_p^q}{u^{mq+1}} \, du + E_m(f^{(r)})_{\varphi^r,p}^q \right)^{1/q},$$

$$\omega_{m,r}^{\varphi}(f^{(r)}, t)_p \leqslant Ct^m \left(\sum_{n<1/t} n^{qm-1} E_n(f^{(r)})_{\varphi^r,p}^q \right)^{1/q},$$

where $q = \min(2, p)$.

The following sharp Jackson inequality was proved in [**5**].

THEOREM 2.4. [**5**, Theorem 6.2] *For $\alpha > -1/p$, $\beta > -1/p$, $1 < p < \infty$, $m \in \mathbb{N}$ and a weight w defined in* (2.1), *we have*

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs} E_{2^j}(f)_{w,p}^s \right)^{1/s} \leqslant CK_{m,\varphi}(f, 2^{-nm})_{w,p},$$

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs} K_{m+1,\varphi}(f, 2^{-j(m+1)})_{w,p}^s \right)^{1/s} \leqslant CK_{m,\varphi}(f, 2^{-nm})_{w,p},$$

where $2^{j_0} \geqslant m$ and $s = \max(p, 2)$.

COROLLARY 2.3. *For $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $f \in \mathbb{B}_p^r$, we have*

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs} E_{2^j}(f^{(r)})_{\varphi^r,p}^s \right)^{1/s} \leqslant C\omega_{m,r}^{\varphi}(f^{(r)}, 2^{-n})_p$$

$$, 2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs} \omega_{m+1,r}^{\varphi}(f^{(r)}, 2^{-j})_p^s \right)^{1/s} \leqslant C\omega_{m,r}^{\varphi}(f^{(r)}, 2^{-n})_p,$$

where $2^{j_0} \geqslant m$ and $s = \max(p, 2)$.

COROLLARY 2.4. *For $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $f \in \mathbb{B}_p^r$, we have*

$$t^m \left(\int_t^{1/m} \frac{\omega_{m+1,r}^{\varphi}(f^{(r)}, u)_p^s}{u^{ms+1}} \, du \right)^{1/s} \leqslant C\omega_{m,r}^{\varphi}(f^{(r)}, t)_p, \quad 0 < t \leqslant 1/m,$$

where $s = \max(p, 2)$.

3. Algebraic polynomial approximation in $\mathbb{L}_p$

In [**10**], we proved the following results analogous to Theorems $\widetilde{\mathrm{D}}_{\mathrm{r}}$, $\widetilde{\mathrm{I}}_{\mathrm{r}}$ and $\widetilde{\mathbb{C}}_{\mathrm{r}}$ (see also [**11**, Theorem 3.2] for the inverse result for $p = \infty$).

THEOREM $\mathbf{D_r}$. *If $f \in \mathbb{B}_p^r$, $1 \leqslant p \leqslant \infty$, then*

$$E_n(f)_p \leqslant c(k,r) n^{-r} \omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_p, \quad n \geqslant k + r. \tag{3.1}$$

Note that it follows from the DT estimates that if $f \in \mathbb{B}_p^r$, then

$$E_n(f)_p \leqslant c(r) n^{-r} \|f^{(r)} \varphi^r\|_p, \quad n \geqslant r,$$

which is asymptotically weaker than (3.1).

It is also known that if, for some $r \geqslant 1$, $f^{(r)} \in \mathbb{L}_p[-1,1]$, $1 \leqslant p \leqslant \infty$, then

$$E_n(f)_p \leqslant c(k,r) n^{-r} \omega_k^{\varphi}(f^{(r)}, n^{-1})_p, \quad n \geqslant k + r.$$

But we should emphasize that here we have to assume that $f^{(r)} \in \mathbb{L}_p[-1,1]$, as the DT-moduli are not well defined if the function is not in $\mathbb{L}_p[-1,1]$ and, clearly, $\omega_{k,r}^{\varphi}(f^{(r)}, n^{-1})_p$ is smaller than $\omega_k^{\varphi}(f^{(r)}, n^{-1})_p$.

THEOREM $\mathbf{I_r}$. *Let $r \in \mathbb{N}_0$, $k \geqslant 1$, and $N \in \mathbb{N}$, and let $\phi : [0,1] \to [0,\infty)$ be a nondecreasing function such that $\phi(0+) = 0$ and*

$$\int_0^1 r \frac{\phi(u)}{u^{r+1}} \, du < \infty.$$

If $f \in \mathbb{L}_p[-1,1]$, $1 \leqslant p \leqslant \infty$, and $E_n(f)_p \leqslant \phi(n^{-1})$, for all $n \geqslant N$, then f is a.e. identical with a function from $\mathbb{B}_p^r$, and

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant c(k,r) \int_0^t r \frac{\phi(u)}{u^{r+1}} \, du + c(k,r) t^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} \, du \\ + c(N,k,r) t^k E_{k+r}(f)_p, \quad t \in [0, 1/2].$$

If, in addition, $N \leqslant k + r$, then

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant c(k,r) \int_0^t r \frac{\phi(u)}{u^{r+1}} \, du + c(k,r) t^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} \, du, \quad t \in [0, 1/2].$$

Taking $N = 1$ and appropriately choosing the function ϕ, we get the following corollary of Theorem $\mathrm{I_r}$ in terms of the degrees of approximation.

COROLLARY 3.1. *Given $1 \leqslant p < \infty$, $k \in \mathbb{N}$, $r \in \mathbb{N}_0$. If $\sum_{n=1}^{\infty} r n^{r-1} E_n(f)_p < +\infty$, then f is a.e. identical with a function from $\mathbb{B}_p^r$, and*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant c \sum_{n > 1/t} r n^{r-1} E_n(f)_p + c t^k \sum_{1 \leqslant n \leqslant 1/t} n^{k+r-1} E_n(f)_p, \quad t \in [0, 1/2].$$

Theorem $\mathbf{C_r}$. *Let $r \in \mathbb{N}_0$, $\alpha > r$, $k \geqslant 1$ and $f \in \mathbb{B}_p^r$, $1 \leqslant p \leqslant \infty$. If $\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant t^{\alpha - r}$, then $E_n(f)_p \leqslant cn^{-\alpha}$, $n \geqslant k + r$.*

Conversely, if $r < \alpha < r + k$ and $f \in \mathbb{L}_p[-1, 1]$ and $E_n(f)_p \leqslant n^{-\alpha}$, $n \geqslant N$, then f is a.e. identical with a function from $\mathbb{B}_p^r$, and

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant c(\alpha, k, r)t^{\alpha - r} + c(N, k, r)t^k E_{k+r}(f)_p, \quad t \in [0, 1/2].$$

If, in addition, $N \leqslant k + r$, then $\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant c(\alpha, k, r)t^{\alpha - r}$.

4. Further characterizations

In addition to characterizations in the previous section, we can also characterize certain smoothness classes of functions via the growth of certain weighted norms of their polynomials of best approximation.

Theorem 4.1. *Let $f \in \mathbb{L}_p[-1, 1]$, $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r < \alpha < r + k$, and suppose that P_n denotes the $(n - 1)$st degree polynomial of best approximation of f in $\mathbb{L}_p[-1, 1]$. Then*

$$\|\varphi^{r+k} P_n^{(r+k)}\|_p \leqslant cn^{r+k-\alpha}, \quad n \geqslant r + k, \tag{4.1}$$

if and only if f is a.e. identical with a function from $\mathbb{B}_p^r$, and

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant ct^{\alpha - r}, \quad t > 0. \tag{4.2}$$

Proof. By virtue of [**6**, Theorem 7.3.1] we conclude that, for every $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$,

$$\|\varphi^{r+k} P_n^{(r+k)}\|_p \leqslant cn^{k+r} \omega_{k+r}^{\varphi}(f, n^{-1})_p.$$

Hence, if $f \in \mathbb{B}_p^r$ and (4.2) is valid, then (4.1) follows immediately from the inequality

$$\omega_{k+r}^{\varphi}(f, t)_p \leqslant ct^r \omega_{k,r}^{\varphi}(f^{(r)}, t)_p \tag{4.3}$$

which is an immediate consequence of Theorem 2.2.

Conversely, if (4.1) holds, then it follows by [**6**, Theorem 7.3.2] that $E_n(f)_p \leqslant cn^{-\alpha}$, $n \geqslant r + k$. Hence (4.2) follows from Theorem $\mathrm{C_r}$. □

We note that while inequality (4.3) cannot be reversed for a general function f, the following is an immediate consequence of Theorem $\mathrm{C_r}$.

Corollary 4.1. *Let $r \in \mathbb{N}_0$, $k \geqslant 1$, $f \in \mathbb{L}_p[-1, 1]$, $1 \leqslant p \leqslant \infty$, $r < \alpha < r + k$. If*

$$\omega_{r+k}^{\varphi}(f, t)_p \leqslant ct^{\alpha}, \quad t > 0,$$

then f is a.e. identical with a function from $\mathbb{B}_p^r$, and

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leqslant ct^{\alpha - r}, \quad t > 0.$$

5. Further results for Weighted DT moduli

The proofs (and therefore the results) of [**10**] may be extended to the weighted DT moduli with weight w which satisfies the conditions of [**6**, Section 6.1]. So, in particular, we have the hierarchy relations between the weighted moduli of the function (of course, provided its derivative exists), extending Theorem 2.2.

THEOREM 5.1. *Let $0 < r < k$, and assume that f is such that $f^{(r-1)}$ is locally absolutely continuous in $(-1, 1)$ and $w\varphi^r f^{(r)} \in \mathbb{L}_p[-1, 1]$, $1 \leqslant p \leqslant \infty$. Then*

$$\omega_k^\varphi(f, t)_{w,p} \leqslant ct^r \omega_{k-r}^\varphi(f^{(r)}, t)_{w\varphi^r,p}, \quad t > 0. \tag{5.1}$$

REMARK 5.1. The inequality (5.1) extends [**6**, Corollary 6.3.3(b)], as we do not require the condition of $\beta(c) \geqslant 1$, for $c = \pm 1$, that appears there.

PROOF. Recall that the main part modulus Ω_k^φ is defined in [**6**, (8.1.2)] by $\Omega_k^\varphi(f, t)_{w,p} := \sup_{0<h\leqslant t} \left\| w\Delta_{h\varphi}^k f \right\|_{\mathbb{L}_p[-1+2k^2h^2, -1+2k^2h^2]}$. Then, [**6**, (6.2.9)] implies that

$$\omega_k^\varphi(f, t)_{w,p} \leqslant c \int_0^t (\Omega_k^\varphi(f, \tau)_{w,p} / \tau) \, d\tau.$$

Also, by [**6**, (6.3.2)], we have $\Omega_k^\varphi(f, t)_{w,p} \leqslant ct\Omega_{k-1}^\varphi(f', t)_{w\varphi,p}$. Hence,

$$\begin{aligned}
\omega_k^\varphi(f, t)_{w,p} &\leqslant c \int_0^t \Omega_{k-1}^\varphi(f', \tau)_{w\varphi,p} \, d\tau \\
&\leqslant ct\Omega_{k-1}^\varphi(f', t)_{w\varphi,p} \leqslant ct\omega_{k-1}^\varphi(f', t)_{w\varphi,p},
\end{aligned}$$

where for the second inequality we used the monotonicity of $\Omega_{k-1}^\varphi(f', t)_{w\varphi,p}$, and for the third one we applied [**6**, (6.2.9)]. Applying this inequality r times we get the desired estimate. □

For the Jacobi weights $w = w_{\alpha,\beta}$ defined in (2.1), it was proved by Ky [**12**, Theorem 4] (see also Luther and Russo [**13**, Corollary 2.2]) that there is an $n_0 \in \mathbb{N}$ such that

$$E_n(f)_{w,p} \leqslant c\omega_k^\varphi(f, n^{-1})_{w,p}, \quad n \geqslant n_0. \tag{5.2}$$

Thus, by (5.1), we have the following Jackson-type result.

THEOREM 5.2. *Let $0 < r < k$ and assume that $f^{(r-1)}$ is locally absolutely continuous in $(-1, 1)$ and $w\varphi^r f^{(r)} \in \mathbb{L}_p[-1, 1]$, $1 \leqslant p \leqslant \infty$. Then*

$$E_n(f)_{w,p} \leqslant cn^{-r} \omega_{k-r}^\varphi(f^{(r)}, n^{-1})_{w\varphi^r,p}, \quad n \geqslant n_0.$$

It was proved in [**6**, Theorem 8.2.4] that

$$\omega_k^\varphi(f, t)_{w,p} \leqslant ct^k \sum_{0<n\leqslant 1/t} n^{k-1} E_n(f)_{w,p}, \quad t \leqslant t_0.$$

This readily implies that, if $0 < \alpha < k$ and $E_n(f)_{w,p} \leqslant n^{-\alpha}$, for $n \geqslant 1$, then

$$\omega_k^\varphi(f, t)_{w,p} \leqslant ct^\alpha, \quad t \leqslant t_0.$$

In fact, it is possible to prove the following more general result.

THEOREM 5.3. *Let $0 \leqslant r < \alpha < k$, and let f be such that $wf \in \mathbb{L}_p[-1,1]$, $1 \leqslant p \leqslant \infty$. If, for an $N \in \mathbb{N}$,*

$$E_n(f)_{w,p} \leqslant n^{-\alpha}, \quad n \geqslant N, \tag{5.3}$$

then f is a.e. identical with a function that has a locally absolutely continuous derivative $f^{(r-1)}$ in $(-1,1)$, and

$$\omega^{\varphi}_{k-r}(f^{(r)},t)_{w\varphi^r,p} \leqslant c(w,\alpha,k,r)t^{\alpha-r} + c(w,N,k,r)t^{k-r}E_k(f)_{w,p}, \quad t > 0.$$

In particular, if $N \leqslant k$, then $\omega^{\varphi}_{k-r}(f^{(r)},t)_{w\varphi^r,p} \leqslant c(w,\alpha,k,r)t^{\alpha-r}$, $t > 0$.

PROOF. Let $P_k \in \mathcal{P}_k$ be a polynomial of best approximation to f in the weighted norm $\|w\cdot\|_p$, and set $F := f - P_k$. Then $E_n(F)_{w,p} = \|wF\|_p = E_k(f)_{w,p}$, $n < k$, and $E_n(F)_{w,p} = E_n(f)_{w,p}$, $n \geqslant k$. Hence, in particular, $E_n(F)_{w,p} \leqslant E_k(f)_{w,p}$, for all $n \in \mathbb{N}$.

Combining [**6**, Theorem 8.2.1] and (5.3), we obtain

$$\begin{aligned}
\Omega^{\varphi}_k(F,t)_{w,p} &\leqslant ct^k \sum_{0<n\leqslant 1/t} n^{k-1}E_n(F)_{w,p} \\
&\leqslant ct^kN^kE_k(f)_{w,p} + ct^k \sum_{N\leqslant n\leqslant 1/t} n^{k-1}E_n(f)_{w,p} \\
&\leqslant c(N)t^kE_k(f)_{w,p} + ct^{\alpha}, \quad t > 0.
\end{aligned}$$

Hence,

$$\int_0^1 (\Omega^{\varphi}_k(F,\tau)_{w,p}/\tau^{r+1})\,d\tau \leqslant \int_0^1 \big(c\tau^{\alpha-r-1} + c(N)t^{k-r-1}E_k(f)_{w,p}\big)\,d\tau < \infty,$$

which, by [**6**, Theorem 6.3.1(a)], implies that $F^{(r-1)}$ is locally absolutely continuous in $(-1,1)$ and

$$\begin{aligned}
\Omega^{\varphi}_{k-r}(F^{(r)},t)_{w\varphi^r,p} &\leqslant c\int_0^t (\Omega^{\varphi}_k(F,\tau)_{w,p}/\tau^{r+1})\,d\tau \\
&\leqslant c\int_0^t \big(c\tau^{\alpha-r-1} + c(N)\tau^{k-r-1}E_k(f)_{w,p}\big)d\tau \\
&\leqslant ct^{\alpha-r} + c(N)t^{k-r}E_k(f)_{w,p}, \quad t > 0.
\end{aligned}$$

Finally, taking into account that

$$\omega^{\varphi}_{k-r}(F^{(r)},t)_{w\varphi^r,p} = \omega^{\varphi}_{k-r}(f^{(r)},t)_{w\varphi^r,p}, \quad t > 0,$$

we apply [**6**, (6.2.9)] to get

$$\begin{aligned}
\omega^{\varphi}_{k-r}(f^{(r)},t)_{w\varphi^r,p} &= \omega^{\varphi}_{k-r}(F^{(r)},t)_{w\varphi^r,p} \\
&\leqslant c\int_0^t (\Omega^{\varphi}_{k-r}(F^{(r)},\tau)_{w\varphi^r,p}/\tau)\,d\tau \\
&\leqslant ct^{\alpha-r} + c(N)t^{k-r}E_k(f)_{w,p}.
\end{aligned}$$

This completes the proof. □

Finally, we have the following result analogous to Corollary 4.1 which immediately follows from (5.2) and Theorem 5.3.

THEOREM 5.4. *Let* $wf \in \mathbb{L}_p[-1,1]$*,* $1 \leqslant p \leqslant \infty$ *and* $0 \leqslant r < \alpha < k$*. If*

$$\omega_k^{\varphi}(f,t)_{w,p} \leqslant ct^{\alpha}, \quad t > 0,$$

then f *is a.e. identical with a function that has a locally absolutely continuous derivative* $f^{(r-1)}$ *in* $(-1,1)$*, and* $\omega_{k-r}^{\varphi}(f^{(r)},t)_{w\varphi^r,p} \leqslant ct^{\alpha-r}$*,* $t > 0$*.*

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