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WEIGHTED MARKOV–BERNSTEIN INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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Abstract. We prove weighted Markov–Bernstein inequalities of the form

$$\int_{-\infty}^{\infty} |f'(x)|^p w(x)\, dx \leqslant C(\sigma+1)^p \int_{-\infty}^{\infty} |f(x)|^p w(x)\, dx$$

Here w satisfies certain doubling type properties, f is an entire function of exponential type $\leqslant \sigma$, $p > 0$, and C is independent of f and σ. For example, $w(x) = (1+x^2)^\alpha$ satisfies the conditions for any $\alpha \in \mathbb{R}$. Classical doubling inequalities of Mastroianni and Totik inspired this result.

1. Introduction

The classical Markov–Bernstein inequality for the unit circle asserts that for polynomials P of degree $\leqslant n$, and $0 < p \leqslant \infty$,

$$\|P'\|_{L_p(\Gamma)} \leqslant n\|P\|_{L_p(\Gamma)}. \tag{1.1}$$

Here Γ is the unit circle, and if $p < \infty$,

$$\|P\|_{L_p(\Gamma)} = \left(\int_{-\pi}^{\pi} |P(e^{i\theta})|^p d\theta \right)^{1/p}.$$

Of course, it was proved earlier for $1 \leqslant p \leqslant \infty$, and later for $0 < p < 1$ by Arestov [**1**]. There is a close cousin for entire functions f of exponential type $\leqslant \sigma$, and $0 < p \leqslant \infty$:

$$\|f'\|_{L_p(\mathbb{R})} \leqslant \sigma\|f\|_{L_p(\mathbb{R})}. \tag{1.2}$$

It too was earlier proved for $1 \leqslant p \leqslant \infty$, and later for $0 < p < 1$. See [**15**]. In fact, these inequalities are equivalent, and can be derived from each other–as follows, for example, from the methods of [**10**] where there is a similar equivalence between Marcinkiewicz–Zygmund and Plancherel–Polya inequalities. These are yet more

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illustrations of the classical link between approximation theory for polynomials and that for entire functions of exponential type, amply explored in the memoir of Ganzburg [**8**], and in the books of Timan [**17**], and Trigub and Belinsky [**18**], for example.

There is a vast literature on Markov–Bernstein inequalities, both for polynomials [**5**, **12**, **14**], and entire functions of exponential type. For the latter, there are Szegő type inequalities, and sharp inequalities for various subclasses of entire functions with special properties–see [**4**, **6**, **16**]. In another direction, weighted Bernstein inequalities involving inner functions, and model spaces have been investigated by Baranov [**2**, **3**].

For polynomials, one of the most beautiful results involves doubling weights, and is due to Mastroianni and Totik [**13**]. Recall the setting: let $W : [-\pi, \pi] \to [0, \infty)$ be measurable. Extend W as a 2π periodic function to the real line. We say that W is doubling if there is a constant L (called a doubling constant for W) such that for all intervals I, we have $\int_{2I} W \leqslant L \int_I W$. Here $2I$ is the interval with the same center as I, but with twice the length. A typical doubling weight is

$$W(t) = h(t) \prod_{j=1}^{k} |t - \beta_j|^{\gamma_j},$$

where h is bounded above and below by positive constants, and all $\{\beta_j\}$ are distinct and lie in $[-\pi, \pi]$, while all $\gamma_j > -1$. An immediate consequence of Theorem 4.1 in [**13**, p. 45] is that for $1 \leqslant p < \infty$,

$$\int_{-\pi}^{\pi} |P'(e^{i\theta})|^p \, W(\theta) \, d\theta \leqslant Cn^p \int_{-\pi}^{\pi} |P(e^{i\theta})|^p \, W(\theta) \, d\theta,$$

valid for $n \geqslant 1$ and all polynomials P of degree $\leqslant n$. This was extended to $0 < p < 1$ by Erdelyi [**7**]. The constant C depends only on p and the doubling constant L, not on the particular W.

In this paper, inspired by the results of Mastroanni, Totik, and Erdelyi, we prove weighted Markov–Bernstein inequalities. Our most general result follows.

THEOREM 1.1. *Let σ, $p > 0$, $r \in (0, 1]$, and let $w : \mathbb{R} \to [0, \infty)$ be a measurable function satisfying the following:*

(I) *The one-sided doubling condition about* 0*: there exists $L > 1$, such that for $|a| \geqslant r$,*

$$\left| \int_{a}^{2a} w \right| \leqslant L \left| \int_{a/2}^{a} w \right|. \tag{1.3}$$

(II) *The growth condition about integers: there exist B, $\beta \geqslant 1$ such that for $k \geqslant 0$ and $-1 \leqslant j \leqslant \max\left\{2k + 1, \frac{1}{r}\right\}$,*

$$\int_{jr}^{(j+1)r} w \leqslant B(1 + r|j - k|)^{\beta} \int_{kr}^{(k+1)r} w. \tag{1.4}$$

Assume also the analogous condition for $k < 0$. For $t \in \mathbb{R}$, let

$$w_r(t) = \frac{1}{2r} \int_{t-r}^{t+r} w(s)\, ds. \tag{1.5}$$

Then for entire functions f of exponential type $\leqslant \sigma$, we have

$$\int_{-\infty}^{\infty} |f'(t)|^p w_r(t)\, dt \leqslant C(\sigma + 1)^p \int_{-\infty}^{\infty} |f(t)|^p w_r(t)\, dt, \tag{1.6}$$

provided the right-hand side is finite. Here C depends on B, β, p and L, but is independent of σ, r, f, and the particular w.

COROLLARY 1.2. *Let $p > 0$. Assume that all the conditions of Theorem 1.1 hold for some $r_0 \in (0, 1)$, and all $r \in (0, r_0)$, with L, B and β independent of r. Then for $\sigma > 0$, and entire functions f of exponential type $\leqslant \sigma$, we have*

$$\int_{-\infty}^{\infty} |f'(t)|^p w(t)\, dt \leqslant C(\sigma + 1)^p \int_{-\infty}^{\infty} |f(t)|^p w(t)\, dt, \tag{1.7}$$

provided the right-hand side is finite. Here C depends on B, β, p and L, but is independent of σ, f, and the particular w.

COROLLARY 1.3. *Let σ, $p > 0$, and let $w : \mathbb{R} \to (0, \infty)$ be a measurable function satisfying the following: for some $M \geqslant 1$, we have for for $t \in \mathbb{R} \setminus \{0\}$ and both $\frac{1}{2} \leqslant \frac{s}{t} \leqslant 2$ and $|s - t| \leqslant 2$,*

$$\frac{1}{M} \leqslant \frac{w(s)}{w(t)} \leqslant M. \tag{1.8}$$

Then for entire functions f of exponential type $\leqslant \sigma$, we have

$$\int_{-\infty}^{\infty} |f'(t)|^p w(t)\, dt \leqslant C(\sigma + 1)^p \int_{-\infty}^{\infty} |f(t)|^p w(t)\, dt, \tag{1.9}$$

provided the right-hand side is finite. Here C depends on M, but is independent of σ, w and f.

COROLLARY 1.4. *Let σ, $p > 0$, and $\alpha \in \mathbb{R}$. Then for entire functions f of exponential type $\leqslant \sigma$, we have*

$$\int_{-\infty}^{\infty} |f'(t)|^p (1 + t^2)^\alpha dt \leqslant C(\sigma + 1)^p \int_{-\infty}^{\infty} |f(t)|^p (1 + t^2)^\alpha dt, \tag{1.10}$$

provided the right-hand side is finite. Here C is independent of σ and f.

To the best of our knowledge, even the inequalities in Corollary 1.4 are new. Almost all existing inequalities in the literature are unweighted, though they involve sharp constants as in (1.2). We note that if $1 = \lambda_1 < \lambda_2 < \cdots$, and $f(x) = \sum_{j=1}^{m} c_j \lambda_j^{-ix}$, we used orthogonal Dirichlet polynomials in [**11**] to prove

$$\left(\int_{-\infty}^{\infty} \frac{|f'(x)|^2}{1 + x^2} dx \right)^{1/2} \leqslant \left\{ \log \lambda_m + (\log \lambda_m)^{1/2} \right\} \left(\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1 + x^2} dx \right)^{1/2}.$$

Here one cannot replace $\log \lambda_m + (\log \lambda_m)^{1/2}$ by any factor smaller than $\log \lambda_m + C_1$ for some $C_1 > 0$. This inequality reflects the fact that f is entire of type $\leqslant \log \lambda_m$.

It is noteworthy that if one allows the weight to depend on the exponential type of the function, then it suffices to prove results for entire functions of exponential type 1. Indeed, suppose that for some weight w and all entire functions f of exponential type at most 1; we have

$$\int_{-\infty}^{\infty} |f'(t)|^p w(t)\, dt \leqslant C_1 \int_{-\infty}^{\infty} |f(t)|^p w(t)\, dt.$$

If now f is entire of exponential type $\leqslant \sigma$, and we apply this last inequality to $g(t) = f(t/\sigma)$, which does have type $\leqslant 1$, and then make a substitution $t = \sigma s$, we obtain for all entire functions f of exponential type $\leqslant \sigma$,

$$\int_{-\infty}^{\infty} |f'(s)|^p w(\sigma s)\, ds \leqslant C_1 \sigma^p \int_{-\infty}^{\infty} |f(s)|^p w(\sigma s)\, ds.$$

However, the goal of this paper is estimates in which the weight does not depend on σ.

We prove the results in Section 2. Throughout $C, C_1, C_2, \dots$ denote positive constants independent of f, σ, r. The same symbol does not necessarily denote the same constant in different occurrences.

2. Proofs of the results

Throughout, we let $S(t) = \sin \pi t/\pi t$ denote the sinc kernel. We will use the bounds $|S(t)| \leqslant \min\{1, 1/\pi|t|\}$. We begin by applying (1.2) to

$$g(t) = f(t)\Big[S\Big(\frac{t}{\ell}\Big) + iS\Big(\frac{t}{\ell} + \frac{1}{2}\Big)\Big]^{\ell}, \tag{2.1}$$

where ℓ is a fixed positive integer. This yields:

Lemma 2.1. *Let $\ell \geqslant 1$, $p > 0$, and f be entire of exponential type $\leqslant \sigma$. Then*

$$\int_{-\infty}^{\infty} |f'(t)|^p (1 + |t|)^{-\ell p} dt \leqslant C(\sigma + 1)^p \int_{-\infty}^{\infty} |f(t)|^p (1 + |t|)^{-\ell p} dt, \tag{2.2}$$

where C is independent of f and σ.

Proof. Let $h(t) = S\left(\frac{t}{\ell}\right) + iS\left(\frac{t}{\ell} + \frac{1}{2}\right)$, so that g of (2.1) satisfies $g = fh^{\ell}$. First note that for real t,

$$|h(t)| \leqslant \min\left\{2, \frac{\ell}{\pi|t|} + \frac{2\ell}{\pi(|t| + \ell/2)}\right\} \leqslant C(1 + |t|)^{-1}, \tag{2.3}$$

where C depends only on ℓ. By (1.2), and some simple calculations, also,

$$|h'(t)| \leqslant C(1 + |t|)^{-1}, \tag{2.4}$$

where again C depends only on ℓ. In the other direction, we see that

$$\begin{aligned} |h(t)|^2 &= \left(\frac{\sin \pi \frac{t}{\ell}}{\pi \frac{t}{\ell}}\right)^2 + \left(\frac{\cos \pi \frac{t}{\ell}}{\pi\left(\frac{t}{\ell} + \frac{1}{2}\right)}\right)^2 \\ &\geqslant \frac{\left(\sin \pi \frac{t}{\ell}\right)^2 + \left(\cos \pi \frac{t}{\ell}\right)^2}{\left(\pi\left(\left|\frac{t}{\ell}\right| + \frac{1}{2}\right)\right)^2} \geqslant C(1 + |t|)^{-2}. \end{aligned} \tag{2.5}$$

Then, recalling (2.1),

$$\begin{aligned} |f'(t)h(t)^{\ell}| &= |g'(t) - f(t)\ell h(t)^{\ell-1}h'(t)| \\ &\leqslant |g'(t)| + C|f(t)|(1+|t|)^{-\ell}, \end{aligned} \tag{2.6}$$

by (2.3) and (2.4). Now g is entire of exponential type $\leqslant \sigma + 1$, and (2.3) shows that

$$\int_{-\infty}^{\infty} |g(t)|^p dt \leqslant C \int_{-\infty}^{\infty} |f(t)|^p (1+|t|)^{-\ell p} dt < \infty,$$

so applying (1.2) to g gives

$$\int_{-\infty}^{\infty} |g'(t)|^p dt \leqslant (\sigma+1)^p \int_{-\infty}^{\infty} |g(t)|^p dt \leqslant C(\sigma+1)^p \int_{-\infty}^{\infty} |f(t)|^p (1+|t|)^{-\ell p} dt.$$

Together with (2.6), and (2.5), this yields

$$\begin{aligned} \int_{-\infty}^{\infty} |f'(t)|^p (1+|t|)^{-\ell p} dt &\leqslant C \int_{-\infty}^{\infty} \left|f'(t)h(t)^{\ell}\right|^p dt \\ &\leqslant C \int_{-\infty}^{\infty} \left(|g'(t)|^p + C|f(t)|^p (1+|t|)^{-\ell p}\right) dt \\ &\leqslant C\{(\sigma+1)^p + 1\} \int_{-\infty}^{\infty} |f(t)|^p (1+|t|)^{-\ell p} dt. \end{aligned}$$

So we have the result. □

From this we deduce:

LEMMA 2.2. *Let σ, $p > 0$, $\ell \geqslant 1$, and let $w : \mathbb{R} \to [0,\infty)$ be a measurable function. Let*

$$H(t) = \int_{-\infty}^{\infty} \frac{w(x)}{(1+|x-t|)^{\ell p}}\, dx, \quad t \in \mathbb{R}, \tag{2.7}$$

and assume that this is finite for $t \in \mathbb{R}$. Then for entire functions f of exponential type $\leqslant \sigma$ for which the right-hand side is finite,

$$\int_{-\infty}^{\infty} |f'(t)|^p H(t)\, dt \leqslant C(\sigma+1)^p \int_{-\infty}^{\infty} |f(t)|^p H(t)\, dt, \tag{2.8}$$

where C depends only on ℓ and p. In particular, it is independent of f, σ, w, H.

PROOF. For a given x, and f, apply Lemma 2.1 to the function $f(\cdot + x)$, so that we are translating the variable. Making a substitution $s = t + x$ yields

$$\int_{-\infty}^{\infty} |f'(s)|^p \frac{ds}{(1+|s-x|)^{\ell p}} \leqslant C(\sigma+1)^p \int_{-\infty}^{\infty} |f(s)|^p \frac{ds}{(1+|s-x|)^{\ell p}}.$$

Now multiply by $w(x)$ and integrate over all real x, and then interchange the integrals. The convergence of the right-hand side in (2.8), and the nonnegativity of the integrand justifies the interchange of integrals. □

Our final lemma before proving Theorem 1.1 involves upper and lower bounds on w_r.

LEMMA 2.3. *Assume the hypotheses of Theorem* 1.1. *Then for some* $C_1, C_2 > 0$ *that depend on* $L, B, \beta, \int_{-1}^{0} w, \int_{0}^{1} w$, *but not on* r, t, *nor on the particular* w,

$$C_2(1+|t|)^{-\beta} \leqslant w_r(t) \leqslant C_1(1+|t|)^{\log_2 L}. \tag{2.9}$$

PROOF. We first establish the lower bound. Let us assume first that $t \geqslant 0$ and choose $j_0 \geqslant 0$ such that $j_0 r \leqslant t < (j_0+1)r$. Note that then

$$\begin{aligned} j_0 r \geqslant t - r \ \text{ and } \ (j_0+1)r \leqslant t + r; \\ (j_0-1)r \leqslant t - r \ \text{ and } \ (j_0+2)r \geqslant t + r. \end{aligned} \tag{2.10}$$

Then, using (1.4),

$$\begin{aligned} \int_{t-r}^{t+r} w &\geqslant \int_{j_0 r}^{(j_0+1)r} w \geqslant B^{-1}(1+j_0 r)^{-\beta} \int_0^r w \\ &\geqslant B^{-1}(1+t)^{-\beta} \int_0^r w \geqslant B^{-1}(1+t)^{-\beta}(1+B(1+2^\beta))^{-1} \int_{-r}^{r} w, \end{aligned}$$

again by (1.4). Thus for $t \geqslant 0$, and some C depending only on B, β,

$$w_r(t) \geqslant C(1+t)^{-\beta} w_r(0). \tag{2.11}$$

Next, using (1.4),

$$\begin{aligned} \int_0^1 w &\leqslant \sum_{j=0}^{[1/r]} \int_{jr}^{(j+1)r} w \leqslant B\left(\int_0^r w\right) \sum_{j=0}^{[1/r]} (1+jr)^\beta \leqslant B\left(\int_0^r w\right) \int_0^{[1/r]+1} (1+sr)^\beta ds \\ &= B\left(\frac{1}{r}\int_0^r w\right) \int_0^{r[1/r]+r} (1+y)^\beta dy \leqslant B\left(\frac{1}{r}\int_0^r w\right) \int_0^2 (1+y)^\beta dy. \end{aligned}$$

A similar estimate holds for $\int_{-1}^{0} w$, so for some C depending only on B, β,

$$\int_{-1}^{1} w \leqslant C w_r(0). \tag{2.12}$$

Together with (2.11), this establishes the lower bound for $t \geqslant 0$, and of course $t < 0$ is similar. We turn to the upper bound. Again, we assume $t \geqslant 0$, and that j_0 is as above. We see using (2.10), and then (1.4), that

$$\int_{t-r}^{t+r} w \leqslant \int_{(j_0-1)r}^{(j_0+2)r} w \leqslant (1+2B2^\beta) \int_{j_0 r}^{(j_0+1)r} w.$$

We continue this using (1.4), as

$$\begin{aligned} &\leqslant (1+2^{\beta+1}B)\frac{1}{[1/r]+1} \sum_{k=j_0}^{j_0+[1/r]} B(1+|j_0-k|r)^\beta \int_{kr}^{(k+1)r} w \\ &\leqslant (1+2^{\beta+1}B)B2^\beta r \int_{j_0 r}^{(j_0+[1/r]+1)r} w. \end{aligned}$$

Thus we have shown that

$$w_r(t) \leqslant C \int_{j_0 r}^{j_0 r+2} w,$$

where C is independent of r, t, but depends on B and β. We continue this using (2.10), and then (1.3), as

$$\begin{aligned}
&\leqslant C\left[\int_0^1 w + \int_1^{t+2} w\right] \leqslant C\Bigg[\int_0^1 w + \sum_{0\leqslant k\leqslant \log_2(t+2)} \int_{2^k}^{2^{k+1}} w\Bigg] \\
&\leqslant C\left(\int_0^1 w\right)\Bigg[1 + \sum_{0\leqslant k\leqslant \log_2(t+2)} L^{k+1}\Bigg] \leqslant C\left(\int_0^1 w\right) L^{\log_2(t+2)} \\
&= C\left(\int_0^1 w\right)(t+2)^{\log_2 L}.
\end{aligned}$$

This gives the upper bound for $t \geqslant 0$, and the case $t < 0$ is similar. □

Proof of Theorem 1.1. Choose ℓ so large that

$$\log_2 L + \beta - \ell p \leqslant -2. \tag{2.13}$$

Note that this choice does not depend on w. Let H be as in Lemma 2.2. We estimate H above and below. Let us assume first that $t \geqslant 0$ and choose $j_0 \geqslant 0$ such that $j_0 r \leqslant t < (j_0+1)r$, so that (2.10) holds. Split

$$\begin{aligned}
H(t) &= \left(\int_{-\infty}^0 + \int_0^{\max\{(2j_0+1)r,1\}} + \int_{\max\{(2j_0+1)r,1\}}^{\infty}\right) \frac{w(s)}{(1+|s-t|)^{\ell p}} ds \\
&= I_1 + I_2 + I_3. \qquad (2.14)
\end{aligned}$$

We start with the central integral I_2 as it will contribute to both our upper and lower bounds. We use our growth condition (1.4) as well as that fact that for $s \in [jr,(j+1)r]$, we have $|s-t| \geqslant |j-j_0|r - r \geqslant \frac{1}{2}|j-j_0|r$ if $|j-j_0| \geqslant 2$. If $|j-j_0| \leqslant 2$, observe that $|j-j_0|r \leqslant 2$. Thus

$$\begin{aligned}
I_2 &\leqslant \sum_{j=0}^{\max\{2j_0+1,[1/r]\}} \int_{jr}^{(j+1)r} \frac{w(s)}{(1+|s-t|)^{\ell p}}\, ds \qquad (2.15)\\
&\leqslant \sum_{j=0}^{\max\{2j_0+1,[1/r]\}} \frac{1}{\left(\frac{1}{4}(1+|j-j_0|r)\right)^{\ell p}} \int_{jr}^{(j+1)r} w(s)\, ds \\
&\leqslant 4^{\ell p} B\Bigg(\int_{j_0 r}^{(j_0+1)r} w(s)\, ds\Bigg) \sum_{j=0}^{\max\{2j_0+1,[1/r]\}} \frac{1}{(1+|j-j_0|r)^{\ell p-\beta}} \\
&\leqslant 4^{\ell p} B\left(\int_{t-r}^{t+r} w(s)\, ds\right) \sum_{k=-\infty}^{\infty} \frac{1}{(1+|k|r)^{\ell p-\beta}} \\
&\leqslant 4^{\ell p+3} B r w_r(t) \int_{-\infty}^{\infty} \frac{1}{(1+|s|r)^{\ell p-\beta}}\, ds \leqslant C_1 w_r(t).
\end{aligned}$$

Here C_1 depends on B, β, ℓ, p but is independent of r and w. We have also used (2.13) and $L \geqslant 1$ to ensure the convergence of the integral in the second last line. Note that we could not simply use the upper bound in Lemma 2.3 for w_r, as we need the last right-hand side of (2.15) to involve $w_r(t)$. In the other direction, we see from (1.4) that

$$
\begin{aligned}
(2.16)\qquad I_2 &\geqslant \sum_{j=j_0}^{\max\{2j_0+1,[1/r]-1\}} \frac{1}{(1+|j-j_0|r+r)^{\ell p}} \int_{jr}^{(j+1)r} w(s)\,ds \\
&\geqslant B^{-1}\left(\int_{j_0 r}^{(j_0+1)r} w(s)\,ds\right) \sum_{j=j_0}^{\max\{2j_0+1,[1/r]-1\}} \frac{1}{(2+|j-j_0|r)^{\ell p+\beta}} \\
&\geqslant B^{-1}\left(\int_{j_0 r}^{(j_0+1)r} w(s)\,ds\right) \sum_{k=0}^{\max\{j_0+1,[1/r]-1-j_0\}} \frac{1}{(2+kr)^{\ell p+\beta}}.
\end{aligned}
$$

Here, using our growth condition (1.4), and then (2.10),

$$
(1+2B2^{\beta})\int_{j_0 r}^{(j_0+1)r} w(s)\,ds \geqslant \int_{(j_0-1)r}^{(j_0+2)r} w(s)ds \geqslant \int_{t-r}^{t+r} w(s)\,ds = 2rw_r(t),
$$

and

$$
\begin{aligned}
\sum_{k=0}^{\max\{j_0+1,[1/r]-1-j_0\}} \frac{1}{(2+kr)^{\ell p+\beta}} &\geqslant \int_0^{\max\{j_0+2,[1/r]-j_0\}} \frac{1}{(2+tr)^{\ell p+\beta}} dt \\
&= \frac{1}{r}\int_0^{\max\{(j_0+1)r, r[1/r]-j_0 r\}} \frac{1}{(2+s)^{\ell p+\beta}} ds \\
&\geqslant \frac{1}{r}\int_0^{1/2} \frac{1}{(2+s)^{\ell p+\beta}} ds,
\end{aligned}
$$

since if $(j_0+1)r \leqslant \frac{1}{2}$, then $r[1/r] - j_0 r \geqslant 1 - r - j_0 r \geqslant \frac{1}{2}$. Substituting the last two inequalities in (2.16), and using (2.15), we have shown that for $t \geqslant 0$,

$$
(2.17)\qquad C_1 w_r(t) \geqslant I_2 \geqslant C_2 w_r(t),
$$

where C_1 and C_2 depend on ℓ, p, β, B, but not on r or the particular w. Next, our doubling condition (1.3) gives

$$
\begin{aligned}
(2.18)\qquad I_1 &\leqslant \sum_{j=0}^{\infty} \int_{-2^{j+1}}^{-2^j} \frac{w(s)}{(1+|s|+t)^{\ell p}}\,ds + \frac{1}{(1+t)^{\ell p}}\int_{-1}^{0} w \\
&\leqslant \sum_{j=0}^{\infty} \frac{1}{(1+2^j+t)^{\ell p}} \int_{-2^{j+1}}^{-2^j} w(s)\,ds + \frac{1}{(1+t)^{\ell p}}\int_{-1}^{0} w \\
&\leqslant \sum_{j=0}^{\infty} \frac{L^{j+1}}{(1+2^j+t)^{\ell p}} \int_{-1}^{0} w + \frac{1}{(1+t)^{\ell p}}\int_{-1}^{0} w
\end{aligned}
$$

$$\leqslant \left(\frac{1}{(1+t)^{\ell p}} \sum_{0\leqslant j\leqslant \log_2(1+t)} L^{j+1} + L \sum_{j>\log_2(1+t)} \left(\frac{L}{2^{\ell p}}\right)^j + \frac{1}{(1+t)^{\ell p}}\right) \int_{-1}^0 w$$

$$\leqslant C\left(\frac{1}{(1+t)^{\ell p}} L^{\log_2(1+t)} + \left(\frac{L}{2^{\ell p}}\right)^{\log_2(1+t)}\right) \int_{-1}^0 w$$

$$\leqslant C \left(\int_{-1}^0 w\right) (1+t)^{\log_2 L - \ell p},$$

by (2.13). Here C depends only on p, ℓ, L. Next, let $N = \log_2 \max\{[(2j_0+1)r], 1\}$, and let $j \geqslant N$, and $s \in [2^j, 2^{j+1}]$. We claim that

$$1 + |s-t| \geqslant \tfrac{1}{3} 2^j. \tag{2.19}$$

If first $j_0 = 0$, then $N = 1$ and $t < r$, so $1 + |s-t| \geqslant 1 + 2^j - 1 = 2^j$. If $j_0 \geqslant 1$, then $(j_0+1)r \leqslant \frac{2}{3}(2j_0+1)r \leqslant \frac{2}{3}2^N$, so $|s-t| \geqslant 2^j - (j_0+1)r \geqslant 2^j - \frac{2}{3}2^N \geqslant \frac{1}{3}2^j$. Thus we have (2.19). Then our doubling hypothesis (1.3) gives

$$I_3 \leqslant \sum_{j=N}^{\infty} \int_{2^j}^{2^{j+1}} \frac{w(s)}{(1+|s-t|)^{\ell p}} ds \leqslant \sum_{j=N}^{\infty} \frac{1}{(3^{-1}2^j)^{\ell p}} \int_{2^j}^{2^{j+1}} w(s)\, ds$$

$$\leqslant \sum_{j=N}^{\infty} \frac{1}{(3^{-1}2^j)^{\ell p}} L^{j+1} \int_0^1 w \leqslant 3^{\ell p} L \left(\int_0^1 w\right) \sum_{j=N}^{\infty} \left(\frac{L}{2^{\ell p}}\right)^j$$

$$\leqslant (2) 3^{\ell p} L \left(\int_0^1 w\right) \left(\frac{L}{2^{\ell p}}\right)^N$$

$$\leqslant C\left(\int_0^1 w\right) \left(\max\left\{[(2j_0+1)r], 1\right\}\right)^{\log_2 L - \ell p}$$

$$\leqslant C\left(\int_0^1 w\right) (1+t)^{\log_2 L - \ell p}.$$

In the third last line, we used $L/2^{\ell p} \leqslant 1/4$, as follows from (2.13). In the last line, we used (2.10). Together with (2.14), (2.17), and (2.18), we have shown that for $t \geqslant 0$,

$$C_2 w_r(t) \leqslant H(t) \leqslant C_1 \left(w_r(t) + \left(\int_{-1}^1 w\right) (1+t)^{\log_2 L - \ell p}\right).$$

Next, from (2.11) and (2.12), we can continue this as

$$C_2 w_r(t) \leqslant H(t) \leqslant C_1 w_r(t) (1 + (1+t)^{\log_2 L - \ell p + \beta}) \leqslant C_3 w_r(t),$$

by (2.13). The case $t < 0$ is similar. Now the result follows from Lemma 2.2. □

We note that at least for $p \geqslant 1$, one can use the Markov–Bernstein inequalities in Theorem 1.1 to prove that there exists $\delta_0 \in (0,1)$ such that for $\sigma > 0$, and nonidentically vanishing entire functions f of exponential type $\leqslant \sigma$, we have

$$\frac{1}{2} \leqslant \int_{-\infty}^{\infty} |f(t)|^p w_{\delta_0/(\sigma+1)}(t) dt \Big/ \int_{-\infty}^{\infty} |f(t)|^p w(t)\, dt \leqslant \frac{3}{2}.$$

This gives one way to prove Corollary 1.2. However, we use a different method below.

Proof of Corollary 1.2. First note that Lemma 2.3 and our hypotheses imply that for some $C > 1$,

$$C^{-1}(1+|t|)^{-\beta} \leqslant w_r(t) \leqslant C(1+|t|)^{\log_2 L}, \quad r \in (0, r_0) \text{ and } t \in \mathbb{R}. \tag{2.20}$$

Here C is independent of r and t. Let $\sigma > 0$ and f be entire of type $\leqslant \sigma$, with the integral in the right-hand side of (1.7) finite. Let us choose k such that $kp \geqslant \beta + \log_2 L + 2$, and choose $\varepsilon > 0$, and set $g(t) = f(t)S(\varepsilon t)^k$. By Lebesgue's differentiation theorem, we have for a.e. $t \in \mathbb{R}$,

$$\lim_{r\to 0+} w_r(t)|g(t)|^p = w(t)|g(t)|^p.$$

Next, (2.20) shows that for $r \in (0, r_0)$ and all real t

$$\begin{aligned} w_r(t)|g(t)|^p &\leqslant C(1+|t|)^{\log_2 L}|f(t)|^p \min\{1, 1/\pi\varepsilon|t|\}^{kp} \\ &\leqslant C_1(1+|t|)^{\log_2 L - kp}|f(t)|^p \\ &\leqslant C_1(1+|t|)^{\log_2 L - kp+\beta} w(t)|f(t)|^p \leqslant C_1C_2 w(t)|f(t)|^p, \end{aligned}$$

by Lemma 2.3 and our choice of k. Here C_1 and C_2 are independent of r, f but depend on ε and w. Since $C_1C_2w(t)|f(t)|^p$ is independent of r and integrable by (1.7), Lebesgue's dominated convergence theorem gives

$$\lim_{r\to 0+} \int_{-\infty}^{\infty} w_r(t)|g(t)|^p dt = \int_{-\infty}^{\infty} w(t)|g(t)|^p dt.$$

Next, for each given $R > 0$, as g' is bounded in each finite interval, and w_r is bounded independently of r,

$$\lim_{r\to 0+} \int_{-R}^{R} w_r(t)|g'(t)|^p dt = \int_{-R}^{R} w(t)|g'(t)|^p dt.$$

Then as g has exponential type $\leqslant \sigma + k\varepsilon\pi$, Theorem 1.1 and the last two limits yield

$$\begin{aligned} &\int_{-R}^{R} w(t)\Big|\frac{d}{dt}(f(t)S(\varepsilon t))\Big|^p dt \\ &\qquad \leqslant C(\sigma + k\varepsilon\pi + 1)^p \int_{-\infty}^{\infty} w(t)|f(t)S(\varepsilon t)|^p dt \\ &\qquad \leqslant C(\sigma + k\varepsilon\pi + 1)^p \int_{-\infty}^{\infty} w(t)|f(t)|^p dt, \end{aligned}$$

recall that $|S| \leqslant 1$. Here C is independent of $\varepsilon, \sigma, f, R$. We can now let $\varepsilon \to 0+$, and use the fact that $S(\varepsilon t)$ converges uniformly for t in compact subsets of $\mathbb{C}$ to $S(0) = 1$. A similar statement then holds for the derivatives. We deduce that

$$\int_{-R}^{R} w(t)|f'(t)|^p dt \leqslant C(\sigma + 1)^p \int_{-\infty}^{\infty} w(t)|f(t)|^p dt.$$

Finally, let $R \to \infty$. □

PROOF. Proof of Corollary 1.3 We choose $r = 1$ in Theorem 1.1. Our condition (1.8) shows that for some $C > 1$ and all $t \in \mathbb{R}$,

$$M^{-1} \leqslant w_1(t)/w(t) \leqslant M. \tag{2.21}$$

That condition also gives for $a \geqslant 0$,

$$\int_a^{2a} w \leqslant M2aw(a) \leqslant 4M^2 \int_{a/2}^{a} w$$

and similarly,

$$\int_{-2a}^{-a} w \leqslant 4M^2 \int_{-a/2}^{0} w.$$

So we can choose $L = 4M^2$ in (1.3). Next, let $k \geqslant 0$ and $-1 \leqslant j \leqslant \max\left\{2k+1, \frac{1}{r}\right\} = 2k + 1$. We have to show that (1.4) holds for the given j, k and with $r = 1$. Firstly if $j = -1$ or 0, (1.8) gives

$$\int_j^{j+1} w \leqslant M^2 \int_1^2 w. \tag{2.22}$$

So now let us consider $1 \leqslant j \leqslant 2k + 1$. Let us first suppose that $j \leqslant k$, and choose $0 \leqslant n \leqslant \log_2 k$ such that

$$\frac{k}{2^{n+1}} \leqslant j \leqslant \frac{k}{2^n}.$$

Then by repeated use of (1.8),

$$\int_j^{j+1} w \leqslant Mw(j) \leqslant M^2 w\Big(\frac{k}{2^n}\Big) \leqslant M^{n+2} w(k) \leqslant M^{n+3} \int_k^{k+2} w. \tag{2.23}$$

Here

$$M^n = 2^{n \log_2 M} \leqslant (k/j)^{\log_2 M} = (1 + (k-j)/j)^{\log_2 M} \leqslant (1 + |k-j|)^{\log_2 M}.$$

Combined with (2.22) and (2.23), we have shown that for $-1 \leqslant j \leqslant k$,

$$\int_j^{j+1} w \leqslant M^5 (1 + |k-j|)^{\log_2 M} \int_k^{k+1} w.$$

Next, if $k < j \leqslant 2k + 1$,

$$\int_j^{j+1} w \leqslant M^2 w(k) \leqslant M^3 \int_k^{k+1} w \leqslant M^3 (1 + |k-j|)^{\log_2 M} \int_k^{k+1} w.$$

In summary, we have established (1.4) with $B = M^5$ and $\beta = \log_2 M$. Then, recalling(2.21), Theorem 1.1 gives the result. □

PROOF OF COROLLARY 1.4. It is easy to see that $w(x) = (1 + x^2)^\alpha$ satisfies (1.8), with, for example, $M = 17^{|\alpha|}$. □

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