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# ON A CONJECTURE OF NEVAI

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Abstract. It is shown that a conjecture concerning the derivatives of orthogonal polynomials, proved by Nevai in 1990 for generalized Jacobi weights, holds for doubling weights as well.

## 1. Introduction and main result

In 1990, an international conference on approximation theory was held at the University of South Florida, bringing together for the first time a large number of approximation theorists from the United States and Soviet countries. In the proceedings of this conference Nevai conjectured that the following theorem could be extended to the case in which the orthogonal polynomials $p_n(\alpha)$ are replaced by their derivatives.

Theorem 1.1 (Mátè, Nevai, Totik [**6**]). *Let $0 < p \leqslant \infty$. Then there is a constant $\mathcal{C}$ with the property that for every measure $\alpha$ supported in $[-1,1]$ such that $\alpha' > 0$ almost everywhere there, the inequality*

$$\left(\int_{-1}^{1}\left|\frac{f(t)}{\sqrt{\alpha'(t)}(1-t^2)^{1/4}}\right|^p dt\right)^{1/p} \leqslant \mathcal{C}\liminf_{n\to\infty}\left(\int_{-1}^{1}\bigl|f(t)p_n(\alpha,t)\bigr|^p dt\right)^{1/p}$$

*holds for every measurable function $f$ in $[-1,1]$.*

We emphasize that Theorem 1.1 is useful in several contexts (e.g., Fourier series in orthogonal systems, related interpolation and quadrature processes).

Conjecture (Nevai [**7**]). *Let $0 < p \leqslant \infty$. Then there is a constant $\mathcal{C}$ with the property that for every measure $\alpha$ supported in $[-1,1]$ such that $\alpha' > 0$ almost*

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Dedicated to Professor Giuseppe Mastroianni on the occasion of his retirement.

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$$\left(\int_{-1}^{1}\left|\frac{f(t)}{\sqrt{\alpha'(t)}(1-t^2)^{3/4}}\right|^p dt\right)^{1/p} \leqslant \mathcal{C}\liminf_{n\to\infty}\frac{1}{n}\left(\int_{-1}^{1}\left|f(t)p_n'(\alpha,t)\right|^p dt\right)^{1/p}$$

*holds for every measurable function $f$ in* $[-1,1]$.

In the same paper, Nevai proved that the conjecture is true in the case of sufficiently smooth generalized Jacobi weight functions, concluding that "Nevertheless, I do not know how to prove similar results for general weights or measures".

Theorem 1.2 (Nevai [**7**]). *Let*

$$U(x)=\prod_{k=1}^{m}|x-y_k|^{A_k} \quad \text{and} \quad W(x)=g(x)\prod_{k=1}^{m}|x-y_k|^{B_k}$$

*where $y_k\in[-1,1]$, $A_k>-1$, $B_k>-1$, and $g$ is a nonnegative function. If $r\geqslant 0$, $p\geqslant 1$ and $0<c_1\leqslant g(x)\leqslant c_2<\infty$ for $x\in[-1,1]$, then there is a positive constant $\mathcal{C}$ such that*

$$\int_{-1}^{1}\frac{U(x)}{W(x)^{\frac{r+p}{2}}(1-x^2)^{\frac{r+3p}{4}}}\,dx \leqslant \mathcal{C}\liminf_{n\to\infty}\frac{1}{n^p}\int_{-1}^{1}|p_n(W,x)|^r|p_n'(W,x)|^p U(x)\,dx.$$

Subsequently, Nevai and Xu proved Theorem 1.2 with the interval of integration $[-1,1]$ replaced by an arbitrary subinterval $\Delta\subseteq[-1,1]$, in order to show the convergence of the Hermite interpolating polynomial to a continuous and differentiable function in weighted $L^p$-norm (see [**8**]).

The aim of this paper is to prove that Theorem 1.2 holds with the generalized Jacobi weights replaced by doubling weights. To be more precise, we will prove the following equivalent statement.

Theorem 1.3. *Let $u,w$ be doubling weights, with $w>0$ a.e. in $[-1,1]$, and $\varphi(x)=\sqrt{1-x^2}$. If $r\geqslant 0$, $1\leqslant p<\infty$, then there exists a positive constant $\mathcal{C}$ such that*

$$\int_{-1}^{1}\frac{u(x)}{w(x)^{\frac{r+p}{2}}\varphi(x)^{\frac{r+p}{2}}}\,dx \leqslant \mathcal{C}\liminf_{m\to\infty}\frac{1}{m^p}\int_{-1}^{1}|p_m(w,x)|^r|p_m'(w,x)\,\varphi(x)|^p u(x)\,dx.$$

We note that, if $u\varphi^{-p}$ is a doubling weight, the statement can be rewritten as

$$\int_{-1}^{1}\frac{u(x)}{w(x)^{\frac{r+p}{2}}(1-x^2)^{\frac{r+3p}{4}}}\,dx \leqslant \mathcal{C}\liminf_{m\to\infty}\frac{1}{m^p}\int_{-1}^{1}|p_m(w,x)|^r|p_m'(w,x)|^p u(x)\,dx.$$

Before proving Theorem 1.3, in the next Section we recall some preliminary results.

## 2. Basic facts

In the sequel $\mathcal{C}$ will stand for a positive constant that can assume different values in each formula. Moreover, letting $A$ and $B$ be positive quantities depending on some parameters, we will write $A\sim B$ if there exists a positive constant $\mathcal{C}$

(independent of these parameters) such that $(A/B)^{\pm 1} \leqslant \mathcal{C}$. Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most $m$.

The definition and the main properties of the doubling weights can be found in Stein's book [**9**] (see also [**1, 2, 3, 4, 5**]). Here, we recall only the results used in the proof of Theorem 1.3. We say that a weight $u$ satisfies the doubling property if, for some constant $L$

$$\int_{2I} u(t)\,dt \leqslant L \int_I u(t)\,dt$$

for all intervals $I \subset [-1,1]$, where $2I$ denotes the twice enlarged $I$ (enlarged from its center). For instance, the above generalized Jacobi weights satisfy the doubling property. We emphasize that a doubling weight can vanish on a set of positive measure (see [**9**, Chap. I, Sect. 8.8]).

Let $u$ be a doubling weight. Setting

$$\Delta_m(x) = \frac{\varphi(x)}{m} + \frac{1}{m^2} = \frac{\sqrt{1-x^2}}{m} + \frac{1}{m^2},$$

we define

$$u_m(x) = \frac{1}{\Delta_m(x)} \int_{x-\Delta_m(x)}^{x+\Delta_m(x)} u(t)\,dt.$$

From the inequality

$$u_m(x) \leqslant \mathcal{C}\Big(1 + m|x-t| + m\big|\sqrt{1-x^2} - \sqrt{1-t^2}\,\big|\Big)^s u_m(t),$$

where $x, t \in [-1,1]$ and $s$ is a constant, it follows that (see [**4**])

$$u_m(x) \sim u_m(t), \qquad |x-t| \leqslant \mathcal{C}\Delta_m(x). \tag{1}$$

Moreover, in [**4**] Mastroianni and Totik proved that

$$u_{bm}(x) \sim u_m(x), \quad b \geqslant 1, \tag{2}$$

$$\int_{-1}^{1} |P_m(x)|^p u_m(x)\,dx \sim \int_{-1}^{1} |P_m(x)|^p u(x)\,dx \tag{3}$$

for any $P_m \in \mathbb{P}_m$ and for $p > 0$. Note that (3) holds also with $P_m$ replaced by a generalized polynomial of the form $\mathcal{T}_m(x) = \prod_i |x - x_i|^{A_i}$, $m = \sum_i A_i$, $A_i \geqslant 1$(see [**4**]).

In all the previous relations $\mathcal{C}$ and the constants in "$\sim$" are independent of $m$ and $P_m$.

Let $w$ be a doubling weight and $\{p_m(w)\}_{m\in\mathbb{N}}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. We denote by $x_k := x_{m,k}(w)$, $k = 1, \ldots, m$, the zeros of $p_m(w)$ located as $-1 < x_1 < x_2 < \cdots < x_m < 1$. Furthermore, we set $x_0 = -1$ and $x_{m+1} = 1$. Then the equivalence

$$\Delta x_k = x_{k+1} - x_k \sim \Delta_m(x_k) = \frac{\sqrt{1-x_k^2}}{m} + \frac{1}{m^2}, \quad k = 0, \ldots, m, \tag{4}$$

holds with the constants in "$\sim$" independent of $m$, as shown by Mastroianni and Totik in [**5**].

## 3. Proof of Theorem 1.3

We use arguments similar to those in [**7**, pp. 87–90].

Let $x \in [x_k, x_{k+1}]$, $k = 1, \dots, m$. Since $p_m(w, x_k) = 0$, we have

$$\begin{aligned} |p_m(w,x)|^{\frac{r}{p}+1} &\leqslant \left(\frac{r}{p}+1\right) \int_{x_k}^{x} |p_m(w,t)|^{\frac{r}{p}} |p_m'(w,t)|\, dt, \\ &\leqslant \left(\frac{r}{p}+1\right) \int_{x_k}^{x_{k+1}} |p_m(w,t)|^{\frac{r}{p}} |p_m'(w,t)|\, dt, \end{aligned}$$

and

$$|p_m(w,x)|^{r+p} \leqslant \left(\frac{r}{p}+1\right)^p (\Delta x_k)^{p-1} \int_{x_k}^{x_{k+1}} |p_m(w,t)|^r |p_m'(w,t)|^p dt,$$

having used the Hölder inequality in the case $p > 1$.

Then, multiplying both sides by $u_m(x)$ and using (1), we obtain

$$|p_m(w,x)|^{r+p} u_m(x) \leqslant \mathcal{C} (\Delta x_k)^{p-1} \int_{x_k}^{x_{k+1}} |p_m(w,t)|^r |p_m'(w,t)|^p u_m(t)\, dt,$$

and, integrating over $[x_k, x_{k+1}]$,

$$\int_{x_k}^{x_{k+1}} |p_m(w,x)|^{r+p} u_m(x)\, dx \leqslant \mathcal{C} (\Delta x_k)^p \int_{x_k}^{x_{k+1}} |p_m(w,t)|^r |p_m'(w,t)|^p u_m(t)\, dt.$$

By (4) and since $\Delta_m(x_k) \sim \Delta_m(t)$, it follows that

$$\int_{x_k}^{x_{k+1}} |p_m(w,x)|^{r+p} u_m(x)\, dx \leqslant \mathcal{C} \int_{x_k}^{x_{k+1}} |p_m(w,t)|^r |p_m'(w,t) \Delta_m(t)|^p u_m(t)\, dt$$

whence, summing on $k = 1, \dots, m$, we get

$$\int_{-1}^{1} |p_m(w,x)|^{r+p} u_m(x)\, dx \leqslant \mathcal{C} \int_{-1}^{1} |p_m(w,t)|^r |p_m'(w,t) \Delta_m(t)|^p u_m(t)\, dt.$$

Now, since $u$ is doubling, also the weight $\varphi^p u$ satisfies the doubling property and $(\varphi^p u)_m(x) \sim m^p \Delta_m(x)^p u_m(x)$. So, by (2) and (3), we get

$$\int_{-1}^{1} |p_m(w,x)|^{r+p} u(x)\, dx \leqslant \frac{\mathcal{C}}{m^p} \int_{-1}^{1} |p_m(w,t)|^r |p_m'(w,t) \varphi(t)|^p u(t)\, dt,$$

where $\mathcal{C}$ is independent of $m$.

Then, using Theorem 1.1, with $f$ and $p$ replaced by $u^{\frac{1}{r+p}}$ and $r+p$, respectively, we obtain

$$\int_{-1}^{1} \frac{u(x)}{w(x)^{\frac{r+p}{2}} \varphi(x)^{\frac{r+p}{2}}}\, dx \leqslant \mathcal{C} \liminf_{m\to\infty} \frac{1}{m^p} \int_{-1}^{1} |p_m(w,t)|^r |p_m'(w,t) \varphi(t)|^p u(t)\, dt,$$

which completes the proof.

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