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ENUMERATION OF CERTAIN CLASSES OF ANTICHAINS

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Abstract. An antichain is here regarded as a hypergraph that satisfies the following property: neither of every two different edges is a subset of the other one. The paper is devoted to the enumeration of antichains given on an n-set and having one or more of the following properties: being labeled or unlabeled; being ordered or unordered; being a cover (or a proper cover); and finally, being a T_0-, T_1- or T_2-hypergraph. The problem of enumeration of these classes comprises, in fact, different modifications of Dedekind's problem. Here a theorem is proved, with the help of which a greater part of these classes can be enumerated. The use of the formula from the theorem is illustrated by enumeration of labeled antichains, labeled T_0-antichains, ordered unlabeled antichains, and ordered unlabeled T_0-antichains. Also a list of classes that can be enumerated in a similar way is given. Finally, we perform some concrete counting, and give a table of digraphs that we used in the counting process.

1. Introduction

By a hypergraph we mean a finite nonempty set together with a finite multiset of its subsets. The elements of this set are called vertices and the subsets are called edges of the hypergraph. Quite naturally, we introduce the notion of the hypergraph with and without multiple edges. If a linear order is given on the multiset of edges of a hypergraph, we get an ordered hypergraph. An antichain is a hypergraph without multiple edges that satisfies the following property: neither of every two different edges is a subset of the other one. A hypergraph is a T_0-hypergraph if for every two different vertices there exists an edge that contains exactly one of them.

By a relative equivalence $\mathfrak{p}$ on a set X we mean a subset X' of the given set together with a relation of equivalence $\sim$ on this subset X'; so we have that $\mathfrak{p} = (X', \sim)$.

Let us fix an n-set V, and denote by $\mathfrak{H}$ the set of all hypergraphs on V, i.e., of all hypergraphs that have V as their set of vertices. Then on $\mathfrak{H}$ we take a relative equivalence $\mathfrak{p} = (\mathfrak{H}_\mathfrak{p}, \sim_\mathfrak{p})$. Antichains belonging to the set $\mathfrak{H}_\mathfrak{p}$ are called

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$\mathfrak{p}$-antichains. Supposing that $\mathfrak{p}$ satisfies some natural conditions we get that every class of the equivalence $\sim_{\mathfrak{p}}$ is either a subset of or disjoint from the set of all $\mathfrak{p}$-antichains. We do not differentiate $\mathfrak{p}$-antichains that belong to the same class of equivalence $\sim_{\mathfrak{p}}$, and our goal is to count the number of all classes of $\mathfrak{p}$-antichains.

We derive a formula (Theorem 3.3) for counting classes of $\mathfrak{p}$-antichains in the labeled case. A new class of digraphs, a class of all hedgehogs, is introduced, and the formula "goes" over the digraphs of this class. As an illustration of the use of the formula, examples of the enumeration of labeled antichains, labeled T_0-antichains, ordered unlabeled antichains, and ordered unlabeled T_0-antichains with a fixed number of edges are given.

The first of these examples, i.e., the case of labeled antichains, has connection with the well-known problem of Dedekind [**1**], a brief history of which can be found in [**2, 3**]. Though the problem has been considered in many papers, it remains open till now.

2. Basic notions and designations

Let X be a set. Denote by $|X|$ the cardinality of the set X, by $\mathfrak{B}(X)$ the power set of X. If $|X| = n$, then we say that X is an n-set.

For all integers $m_1, m_2 \in \mathbf{Z}$, $m_1 \leqslant m_2$, by $\overline{m_1, m_2}$ denote the integer interval $\{m_1, m_1+1, \dots, m_2\}$. Also, by $\overline{n}$ denote the set $\{1, \dots, n\}$ for every $n \in \mathbf{N}$, and let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

Following [**4**], by a *multiset* on a set S we mean an ordered pair consisting of S and a mapping $f\colon S \to \mathbf{N}_0$. Let $\boldsymbol{a} = (S, f)$ be a multiset; the value $f(s)$ is called *multiplicity* of $s \in S$ in $\boldsymbol{a}$. If it is clear from the context which function f is meant, we use the notation $\|s\|$ instead of $f(s)$. For some $s \in S$ we write $s \in \boldsymbol{a}$ if $\|s\| > 0$. If $\|s\| = 0$ for every $s \in S$, then the multiset $\boldsymbol{a}$ is called the *empty multiset*. By the *cardinality* of the multiset $\boldsymbol{a}$ we mean the number $|\boldsymbol{a}| = \sum_{s \in S} \|s\|$, and it is called an *$m$-multiset* if $|\boldsymbol{a}| = m$. Let $\boldsymbol{b} = (S, g)$ be another multiset. We write $\boldsymbol{a} \subseteq \boldsymbol{b}$ if $f(s) \leqslant g(s)$ for every $s \in S$.

Let us introduce some notions from the graph theory which we are going to use in the paper. By *unordered hypergraph* or simply *hypergraph* we mean an ordered pair $H = (V, \mathcal{E})$, where V is a finite nonempty set and $\mathcal{E}$ a finite multiset on $\mathfrak{B}(V)$. Let us call elements of the set V *vertices*, and members of the multiset $\mathcal{E}$ *edges* of the given hypergraph H. In what follows the set of vertices of a hypergraph H will be also denoted by VH, and the multiset of its edges by $\mathcal{E}H$. If $|\mathcal{E}| = m$ and $|V| = n$, then we call hypergraph H an *(m, n)-hypergraph*. We say that a hypergraph H is a *hypergraph without multiple edges* if $\|e\| = 1$ for every $e \in \mathcal{E}H$.

If in the above given definition instead of a multiset $\mathcal{E}$ we take an m-tuple $(e_1, \dots, e_m)$, where $e_i \subseteq V$ $(i \in \overline{m})$, then we have an *ordered hypergraph*, that is, if $|V| = n$, we have an *ordered (m, n)-hypergraph.* The sets e_i, $i \in \overline{m}$, are its edges.

Let e be an edge of a hypergraph H (ordered or unordered). We say that e is an *empty edge* if $e = \emptyset$, and e is a *unit edge* if $e = VH$.

Note that a graph may be regarded as a special case of a hypergraph. In what follows, we shall often denote by VG the set of vertices V, and by EG the set of edges E of the graph $G = (V, E)$. The same notation is used for a digraph $D = (V, E)$: by VD we mean set of vertices V, and by ED the set of edges E.

Let $H = (V, \mathcal{E})$ be a hypergraph or an ordered hypergraph. We say that a vertex $v \in V$ of H is *incident to* an edge $e \in \mathcal{E}$ of H (or e is *incident to* v) if $v \in e$. A vertex v is called an *isolated vertex* in H if there is no edge in H which is incident to v. A set $V' \subseteq V$ is a set of *adjacent* vertices in H if there exists an edge $e \in \mathcal{E}$ such that $V' \subseteq e$.

Denote by $\mathcal{H}(V)$ [$\vec{\mathcal{H}}(V)$] the set of all hypergraphs [ordered hypergraphs] that have a set V as their set of vertices. Let H_1 and H_2 be two hypergraphs from $\mathcal{H}(V)$. These hypergraphs are *equal*, $H_1 \equiv H_2$, if $\mathcal{E}H_1 = \mathcal{E}H_2$. They are *isomorphic*, $H_1 \simeq H_2$, if there is a bijection $\iota : V \to V$ such that $e \in \mathcal{E}H_1$ iff $\iota(e) \in \mathcal{E}H_2$ for every $e \in \mathfrak{B}(V)$, and $\|e\| = \|\iota(e)\|$ for every $e \in \mathcal{E}H_1$. These two relations are equivalence relations on $\mathcal{H}(V)$. By *labeled* [*unlabeled*] hypergraph (on V) we mean a class of equivalence $\equiv$ [$\simeq$]. Isomorphic [equal] hypergraphs have the same number of edges, and because of that we can speak about an *unlabeled* [*labeled*] (m, n)-hypergraph. Analogously, we can introduce the notion of an *unlabeled* [*labeled*] *ordered* (m, n)-hypergraph.

Let H be a hypergraph [an ordered hypergraph]. It is called an *antichain* if it is a hypergraph [an ordered hypergraph] without multiple edges and if $e_1 \not\subseteq e_2$ for every $e_1, e_2 \in \mathcal{E}H$, $e_1 \neq e_2$. It is called a *cover* if there is no isolated vertex in H. If a cover does not contain the unit edge, we say that it is a *proper cover*.

By analogy with the notions of T_0-, T_1- and T_2-spaces from general topology let us introduce similar notions for hypergraphs. A hypergraph (an ordered hypergraph) H is:

a) a T_0-*hypergraph* iff for every two different vertices $u, v \in V$ there exists an edge e from H such that $(u \in e \wedge v \notin e) \vee (u \notin e \wedge v \in e)$,
b) a T_1-*hypergraph* iff for every pair $(u, v) \in V^2$ of different vertices there is an edge e from H, such that $(u \in e \wedge v \notin e)$,
c) a T_2-*hypergraph* iff for every pair $(u, v) \in V^2$ of different vertices there exist edges e_1, e_2 from H, such that $(u \in e_1 \wedge v \in e_2 \wedge e_1 \cap e_2 = \emptyset)$.

It is clear that if a hypergraph belongs to one of the above classes, then an isomorphic hypergraph is also from the same class. Thus we may say that an unlabeled hypergraph is an antichain, cover and so on.

Let us fix an infinite set $V_\infty = \{v_1, v_2, \ldots\}$. Put $V_i = \{v_1, \ldots, v_i\}$ for every $i \in \mathbf{N}$. Introduce classes of labeled [unlabeled] (m, n)-hypergraphs $T_{a_1 a_2 a_3 a_4 a_5}(m, n)$ $(0 \leqslant a_1, a_2, a_3 \leqslant 1;\ 0 \leqslant a_4 \leqslant 2;\ 0 \leqslant a_5 \leqslant 3)$ on V_n in the following way. The parameters a_i, $1 \leqslant i \leqslant 5$, have the following meaning:

	a_1	a_2	a_3	a_4	a_5
0	labeled	ordered	antichain	proper-cover	T_2
1	unlabeled	unordered	$\emptyset$	cover	T_1
2	—	—	—	$\emptyset$	T_0
3	—	—	—	—	$\emptyset$

In the table the symbol $\emptyset$ means that the corresponding property is not taken into account. The introduced classes are in fact the constituents of the previously defined classes of hypergraphs. For example, the class $T_{00022}(m,n)$ consists of all labeled ordered T_0-hypergraphs on V_n with m edges which are antichains. Class $T_{10213}(m,n)$ consists of all unlabeled ordered (m,n)-hypergraphs on V_n which are covers.

Some of introduced parameters are not completely independent. For example, if a hypergraph is a T_1- or T_2-hypergraph, and $n>1$, then it is also a cover; if it is an antichain, and $m>1$, then it is also a proper cover.

We consider, basically, classes $\mathcal{A}_{i_1i_2i_3}(m,n) = T_{0i_10i_2i_3}(m,n)$ $(0 \leqslant i_1 \leqslant 1$, $0 \leqslant i_2 \leqslant 2$, $1 \leqslant i_3 \leqslant 3)$, and our aim is to find their cardinality. Let us put $t_{a_1a_2a_3a_4a_5}(m,n) = |T_{a_1a_2a_3a_4a_5}(m,n)|$ and $\alpha_{i_1i_2i_3}(m,n) = |\mathcal{A}_{i_1i_2i_3}(m,n)|$.

If it is clear from the context which set is taken as a set of vertices for a hypergraph $H=(V,\mathcal{E})$, we use the notation $\mathcal{E}$ instead of $(V,\mathcal{E})$. Denote by $\mathcal{H}(m,n)$ the set of all ordered hypergraphs with m edges and with the set of vertices V_n. Let us put $\mathfrak{H}(n) = \cup_{m=1}^{\infty}\mathcal{H}(m,n)$.

Let $H=(e_1,\dots,e_m)$ be a labeled ordered hypergraph from the set $\mathcal{H}(m,n)$. Let us define the *incidence matrix* $M_H = [a_{ij}]_{m\times n}$ of H as a matrix for which $a_{ij}=1$ if $v_j \in e_i$, and $a_{ij}=0$, otherwise. Let c_j $(1 \leqslant j \leqslant n)$ be the j-th column of M_H. Then we sometimes represent matrix M_H by n-tuple $(c_1,c_2,\dots,c_n)$.

For every labeled ordered hypergraph $H \in \mathcal{H}(m,n)$ with incidence matrix M_H a *dual* labeled ordered hypergraph H^T is defined as the hypergraph from the set $\mathcal{H}(n,m)$ whose incidence matrix is M_H^T, where M_H^T is transpose of M_H.

Denote by $\mathcal{D}_m$ the class of all labeled digraphs with the set V_m as their set of vertices. Every digraph with m vertices that we are going to consider further is an element of the set $\mathcal{D}_m$. The unique digraph without edges which belongs to $\mathcal{D}_m$ is denoted by $\emptyset_m$.

Let $D \in \mathcal{D}_m$, and let $H=(e_1,\dots,e_m) \in \mathcal{H}(m,n)$. We say that D *correlates with* H if for every $i,j \in \overline{m}$, $i \neq j$, from $(v_i,v_j) \in ED$ follows that $e_i \subseteq e_j$. By f_H denote the function $f_H : V_m \to \mathfrak{B}(V_n)$ such that $f_H(v_i)=e_i$ for every $i \in \overline{m}$. Denote by $\mathcal{H}(D,n)$ the set of all hypergraphs H' from the set $\mathcal{H}(m,n)$ such that D correlates with H'. It is clear that $\mathcal{H}(m,n)=\mathcal{H}(\emptyset_m,n)$. Denote by $\hat{\mathcal{H}}(D,n)$ the set of all hypergraphs without multiple edges from $\mathcal{H}(D,n)$.

Let $H=(e_1,\dots,e_m) \in \mathcal{H}(m,n)$. Denote by $\overline{H}$ a hypergraph $(e_{k_1},\dots,e_{k_{m'}})$, $1=k_1<k_2<\cdots<k_{m'}$, such that $e_{k_i} \neq e_{k_j}$ for every $i,j \in \overline{m'}$, $i \neq j$, and for every $i \in \overline{m}$ there exists $j \in \overline{m'}$ such that $k_j \leqslant i$ and $e_i = e_{k_j}$. It is obvious that for every H the hypergraph $\overline{H}$ is unique.

By *a relative equivalence on* a set A we mean a pair $\mathfrak{p}=(A_\mathfrak{p},\sim_\mathfrak{p})$, where $A_\mathfrak{p} \subseteq A$ and $\sim_\mathfrak{p}$ is equivalence on $A_\mathfrak{p}$. Let B be a subset of A. We say that B is $\mathfrak{p}$-*subset* of A if the set $A_\mathfrak{p} \cap B \cap a$ is equal to a or $\emptyset$ for every class $a \subseteq A_\mathfrak{p}$ of equivalence $\sim_\mathfrak{p}$. Let us put $B_\mathfrak{p} = B \cap A_\mathfrak{p}$, $B/\mathfrak{p} = \{a \in A_\mathfrak{p}/\sim_\mathfrak{p} \mid B \cap a \neq \emptyset\}$, and $\alpha_\mathfrak{p}(B) = |B/\mathfrak{p}|$. If we write $x \sim_\mathfrak{p} y$ for some $x,y \in A$, then it means that there exists an $a \in A_\mathfrak{p}/\sim_\mathfrak{p}$ such that $x,y \in a$.

Let $\mathfrak{p}$ be a relative equivalence on $\mathfrak{H}(n)$. We say that $\mathfrak{p}$ is *regular* on $\mathfrak{H}(n)$, if

1) the sets $\mathcal{H}(D,n)$ and $\hat{\mathcal{H}}(D,n)$ are $\mathfrak{p}$-subsets of $\mathfrak{H}(n)$ for every $m \in \mathbf{N}$ and $D \in \mathcal{D}_m$;
2) $H \sim_{\mathfrak{p}} H'$ then $\overline{H} \sim_{\mathfrak{p}} \overline{H'}$ for every $H, H' \in \mathcal{H}(m,n)$.

Let $\mathfrak{p}$ be regular relative equivalence on $\mathfrak{H}(n)$. Then we take that

$$\mathcal{H}_{\mathfrak{p}}(D,n) = (\mathcal{H}(D,n))_{\mathfrak{p}}, \quad \hat{\mathcal{H}}_{\mathfrak{p}}(D,n) = (\hat{\mathcal{H}}(D,n))_{\mathfrak{p}}, \quad \mathfrak{H}_{\mathfrak{p}}(D,n) = \mathcal{H}(D,n)/\mathfrak{p}$$

and $\hat{\mathfrak{H}}_{\mathfrak{p}}(D,n) = \hat{\mathcal{H}}(D,n)/\mathfrak{p}$. Also denote $\lambda_{\mathfrak{p}}(D,n) = \alpha_{\mathfrak{p}}[\mathcal{H}(D,n)]$ and $\hat{\lambda}_{\mathfrak{p}}(D,n) = \alpha_{\mathfrak{p}}[\hat{\mathcal{H}}(D,n)]$.

Denote by $\mathcal{H}^{(i)}(D,n)$, $i \in \overline{|VD|}$, the set of all hypergraphs from $\mathcal{H}(D,n)$ that have exactly i different edges. We say that a regular relative equivalence $\mathfrak{p} = (\mathfrak{H}_{\mathfrak{p}}(n), \sim_{\mathfrak{p}})$ on $\mathfrak{H}(n)$ is *strong*, if

1) $\mathcal{H}^{(i)}(D,n)$ is $\mathfrak{p}$-subset of $\mathfrak{H}(n)$ for every $D \in \mathcal{D}_m$ and every $i \in \overline{m}$,
2) for every $H', H'' \in \mathcal{H}^{(i)}_{\mathfrak{p}}(D,n) = (\mathcal{H}^{(i)}(D,n))_{\mathfrak{p}}$, $H' \sim_{\mathfrak{p}} H''$, and every $v', v'' \in V_m$, $f_{H'}(v') \subseteq f_{H'}(v'')$ iff $f_{H''}(v') \subseteq f_{H''}(v'')$.

3. On the number of $\mathfrak{p}$-antichains

Let $\mathfrak{p} = (\mathfrak{H}_{\mathfrak{p}}(n), \sim_{\mathfrak{p}})$ be a relative equivalence on $\mathfrak{H}(n)$. By $\mathfrak{p}$-*antichain* we mean every antichain belonging to $\mathfrak{H}_{\mathfrak{p}}(n)$. Denote by $\mathcal{A}_{\mathfrak{p}}(m,n)$ the set of all $\mathfrak{p}$-antichains that belong to the class $\mathcal{H}_{\mathfrak{p}}(m,n) = \mathcal{H}_{\mathfrak{p}}(\emptyset_m, n)$. Let $\alpha_{\mathfrak{p}}(m,n) = \alpha_{\mathfrak{p}}(\mathcal{A}_{\mathfrak{p}}(m,n))$.

THEOREM 3.1. *Let $\mathfrak{p}$ be a regular relative equivalence on $\mathfrak{H}(n)$. Then the set $\mathcal{A}_{\mathfrak{p}}(m,n)$ is $\mathfrak{p}$-subset of $\mathfrak{H}(n)$ and it holds that*

$$\alpha_{\mathfrak{p}}(m,n) = \sum_{D \in \mathcal{D}_m} (-1)^{|ED|} \lambda_{\mathfrak{p}}(D,n) = \sum_{D \in \mathcal{D}_m} (-1)^{|ED|} \hat{\lambda}_{\mathfrak{p}}(D,n).$$

PROOF. We say that a $H = (e_1, \ldots, e_m) \in \mathcal{H}(m,n)$ possesses the property p_{ij} ($i, j \in \overline{m}$, $i \neq j$) if $e_i \subseteq e_j$. Let $H \in \mathcal{H}_{\mathfrak{p}}(m,n)$. Then $H \in \mathcal{A}_{\mathfrak{p}}(m,n)$ iff H does not possess any of the properties p_{ij}, $i, j \in \overline{m}$, $i \neq j$. Note that $\mathcal{A}_{\mathfrak{p}}(m,n) \subseteq \hat{\mathcal{H}}_{\mathfrak{p}}(m,n)$.

Now observe arbitrary r of such properties $p_{i_1 j_1}, \ldots, p_{i_r j_r}$. Let D be a digraph from $\mathcal{D}_m$ such that $ED = \{(v_{i_1}, v_{j_1}), \ldots, (v_{i_r}, v_{j_r})\}$. It is clear that the set of all hypergraphs $H \in \mathcal{H}(m,n)$ which possess the properties $p_{i_1 j_1}, \ldots, p_{i_r j_r}$ is actually the set $\mathcal{H}(D,n)$. Now, it is clear that

$$\mathcal{A}_{\mathfrak{p}}(m,n) = \mathcal{H}_{\mathfrak{p}}(m,n) \smallsetminus \bigcup_{D \in \mathcal{D}_m \smallsetminus \emptyset_m} \mathcal{H}(D,n) = \mathcal{H}_{\mathfrak{p}}(m,n) \smallsetminus \bigcup_{D \in \mathcal{D}_m \smallsetminus \emptyset_m} \mathcal{H}_{\mathfrak{p}}(D,n)$$
$$= \hat{\mathcal{H}}_{\mathfrak{p}}(m,n) \smallsetminus \bigcup_{D \in \mathcal{D}_m \smallsetminus \emptyset_m} (\mathcal{H}_{\mathfrak{p}}(D,n) \cap \hat{\mathcal{H}}_{\mathfrak{p}}(m,n)) = \hat{\mathcal{H}}_{\mathfrak{p}}(m,n) \smallsetminus \bigcup_{D \in \mathcal{D}_m \smallsetminus \emptyset_m} \hat{\mathcal{H}}_{\mathfrak{p}}(D,n).$$

The first part of the theorem follows from this relation, and, also, from the relation, using the inclusion-exclusion principle, we can get now the second part of the theorem. □

Denote by $\mathcal{D}'_m$ the set of all acyclic digraphs from $\mathcal{D}_m$, and by $\mathcal{D}''_m$ the set of all digraphs from $\mathcal{D}'_m$ in which there is a path of the length $\geqslant 2$. Let us call the elements of the set $\mathcal{J}_m = \mathcal{D}'_m \smallsetminus \mathcal{D}''_m$ *hedgehogs*. Note that a connected hedgehog J is an oriented bipartite graph with distinguished blocks $V_1(J)$ and $V_2(J)$ such that for every edge $(u, v) \in E(J)$ it holds that $u \in V_1(J)$ and $v \in V_2(J)$. Note also that the notion of the hedgehog is close to the notion of 2-graduate posets [**5**]. Now let us give an improvement of the formula from Theorem 3.2.

THEOREM 3.2. $\alpha_{\mathfrak{p}}(m,n) = \sum_{D \in \mathcal{J}_m} (-1)^{|ED|} \hat{\lambda}_{\mathfrak{p}}(D,n)$.

PROOF. Let D be a digraph from $\mathcal{D}_m \setminus \mathcal{D}'_m$. As D is from $\mathcal{D}_m \setminus \mathcal{D}'_m$, then D has at least one simple cycle. Let $v_{i_1} v_{i_2} \dots v_{i_k}$, where $v_{i_1} = v_{i_k}$ and $(v_{i_j}, v_{i_{j+1}}) \in ED$ for every $j \in \overline{k-1}$, be one of these cycles. Now if $H = (e_1, \dots, e_m) \in \mathcal{H}_{\mathfrak{p}}(D,n)$, then $e_{i_1} \subseteq e_{i_2} \subseteq \cdots \subseteq e_{i_k} \subseteq e_{i_1}$, so it follows that $e_{i_1} = e_{i_2} = \cdots = e_{i_k}$. Thus $H \notin \hat{\mathcal{H}}_{\mathfrak{p}}(D,n)$, and we have that $\hat{\mathcal{H}}_{\mathfrak{p}}(D,n) = \emptyset$. Thus we have equation

$$\sum_{D \in \mathcal{D}_m \setminus \mathcal{D}'_m} (-1)^{|ED|} \hat{\lambda}_{\mathfrak{p}}(D,n) = 0. \tag{3.1}$$

Let us consider the class of digraphs $\mathcal{D}''_m$. Order all the ordered pairs of different vertices from V_m^2 in a sequence: $(u'_1, u''_1), (u'_2, u''_2), \dots, (u'_{a_0}, u''_{a_0})$, $a_0 = m(m-1)$. Break the set $\mathcal{D}''_m$ into disjoint classes $(\mathcal{D}''_m)_i$, $i \in \overline{a_0}$, so that the digraph $D \in \mathcal{D}''_m$ will belong to class $(\mathcal{D}''_m)_i$ iff the following condition is satisfied: D does not belong to the set $\cup_{l=1}^{i-1} (\mathcal{D}''_m)_l$ and in D there is a directed path of the length $\geqslant 2$ with the beginning in u'_i and with the end in the vertex u''_i.

It is clear that for every $i \in \overline{a_0}$ we can break the class $(\mathcal{D}''_m)_i$ into 2-sets, such that the digraphs from such a 2-set differ only in the fact that one contains the edge (u'_i, u''_i) and the other does not. Also, it is clear that for every such 2-set $\{D', D''\}$ the equation $\hat{\mathcal{H}}_{\mathfrak{p}}(D',n) = \hat{\mathcal{H}}_{\mathfrak{p}}(D'',n)$ is fulfilled, and consequently we have that $\hat{\lambda}_{\mathfrak{p}}(D',n) = \hat{\lambda}_{\mathfrak{p}}(D'',n)$. Then it is clear that the respective summands in the sum from the statement differ only in the sign, so they are annuled in the sum, i.e., holds the equation

$$\sum_{D \in \mathcal{D}''_m} (-1)^{|ED|} \hat{\lambda}_{\mathfrak{p}}(D,n) = \sum_{i=1}^{a_0} \sum_{D \in (\mathcal{D}''_m)_i} (-1)^{|ED|} \hat{\lambda}_{\mathfrak{p}}(D,n) = 0. \tag{3.2}$$

The statement of the theorem now follows from (3.1), (3.2) and Theorem 3.1, and the fact that the sets $\mathcal{J}_m$, $\mathcal{D}''_m$ and $\mathcal{D}_m \setminus \mathcal{D}'_m$ form a partition of the set $\mathcal{D}_m$. $\square$

Note that every subgraph of a hedgehog is a hedgehog. Let D be a hedgehog. Then denote by $\mathrm{Ex}(D)$ the set of all vertices from which at least one edge goes out, by $\mathrm{En}(D)$ the set of all vertices in which at least one edge goes in, and by $\mathrm{Is}(D)$ the set of all isolated vertices of D. It is clear that $\mathrm{Out}(D) = \mathrm{Ex}(D) \cup \mathrm{Is}(D) \neq \emptyset$ and $\mathrm{In}(D) = \mathrm{En}(D) \cup \mathrm{Is}(D) \neq \emptyset$.

LEMMA 3.1. $\sum_{D \in \mathcal{J}_m} (-1)^{|ED|} = (-1)^{m-1}$.

PROOF. The statement will be proved by mathematical induction on number m. It is easy to verify that the formula holds for $m = 1$ and for $m = 2$. Let us assume that it holds for every natural number $\leqslant m$ and let us show that then it holds for $m + 1$. Break $\mathcal{J}_{m+1}$ into two parts $\mathcal{J}'_{m+1} = \{D \in \mathcal{J}_{m+1} \mid v_{m+1} \in \mathrm{In}\,(D)\}$ and $\mathcal{J}''_{m+1} = \{D \in \mathcal{J}_{m+1} \mid v_{m+1} \in \mathrm{Ex}\,(D)\}$.

Let us consider the class $\mathcal{J}'_{m+1}$ and break it into subclasses $[\mathcal{J}'_{m+1}]_i$, $i \in \overline{a_1}$, so that any two digraphs $D, D' \in \mathcal{J}'_{m+1}$ belong to the same class $[\mathcal{J}'_{m+1}]_{i_0}$ for some $i_0 \in \overline{a_1}$ iff $(V_m, ED \setminus (V_m \times \{v_{m+1}\})) = (V_m, ED' \setminus (V_m \times \{v_{m+1}\})) = D'_{i_0}$. Then

$$\sigma_1 = \sum_{D \in \mathcal{J}'_{m+1}} (-1)^{|ED|} = \sum_{i=1}^{a_1} \sum_{D \in [\mathcal{J}'_{m+1}]_i} (-1)^{|ED|}$$
$$= \sum_{i=1}^{a_1} (-1)^{|ED'_i|} \sum_{E' \subseteq \mathrm{Out}\,(D'_i) \times \{v_{m+1}\}} (-1)^{|E'|}.$$

As $\mathrm{Out}\,(D'_i) \neq \emptyset$ for every $i \in \overline{a_1}$, then

$$\sum_{E' \subseteq \mathrm{Out}\,(D'_i) \times \{v_1\}} (-1)^{|E'|} = 0, \qquad \text{for every } i \in \overline{a_1},$$

i.e., $\sigma_1 = 0$.

Now consider the class $\mathcal{J}''_{m+1}$ and break it into subclasses $[\mathcal{J}''_{m+1}]_i$, $i \in \overline{a_2}$, so that any two digraphs $D, D' \in \mathcal{J}''_{m+1}$ belong to the same class $[\mathcal{J}''_{m+1}]_{i_0}$ for some $i_0 \in \overline{a_2}$ iff $(V_m, ED \setminus (\{v_{m+1}\} \times V_m)) = (V_m, ED' \setminus (\{v_{m+1}\} \times V_m)) = D''_{i_0}$. Note that the set of all D''_i, $i \in \overline{a_2}$, is the set $\mathcal{J}_m$. Let $E''_i = \{v_{m+1}\} \times \mathrm{In}\,(D''_i)$ for every $i \in \overline{a_2}$. By the induction hypothesis we have

$$\sigma_2 = \sum_{D \in \mathcal{J}''_{m+1}} (-1)^{|ED|} = \sum_{i=1}^{a_2} \sum_{D \in [\mathcal{J}''_{m+1}]_i} (-1)^{|ED|} = \sum_{i=1}^{a_2} (-1)^{|ED''_i|}$$
$$\times \sum_{E' \subseteq E''_i,\, E' \neq \emptyset} (-1)^{|E'|} = \sum_{i=1}^{a_2} (-1)^{|ED''_i|} \Big[-1 + \sum_{E' \subseteq E''_i} (-1)^{|E'|} \Big]$$
$$= (-1) \sum_{i=1}^{a_2} (-1)^{|ED''_i|} = (-1) \sum_{D \in \mathcal{J}_m} (-1)^{|ED|} = (-1)(-1)^{m-1} = (-1)^m.$$

Finally we can write $\sum_{D \in \mathcal{J}_{m+1}} (-1)^{|ED|} = \sigma_1 + \sigma_2 = 0 + (-1)^m = (-1)^m$. □

Let X be a set. An ordered m-tuple $(Y_1, \ldots, Y_m)$, where Y_i, $i \in \overline{m}$, are all the blocks of a partition of the set X, is called an *ordered partition* of the set X.

Let $\leqslant$ be a linear ordering of the set X, and let π be a partition of the set X into m blocks. Denote by Y_1 the partition block that contains the minimal element of the set X. Let us assume that blocks $Y_1, \ldots, Y_k$, $1 \leqslant k < m$, of the partition π are taken. Then by Y_{k+1} denote the partition block that contains the minimal element of the set $X \setminus \bigcup_{j=1}^{k} Y_j$. The ordered partition $\vec{\pi} = (Y_1, \ldots, Y_m)$ is called *the ordering of the partition π in respect to the linear ordering $\leqslant$*.

Take a hypergraph $H \in \mathcal{H}^{(i)}(D, n)$. It defines a partition of the set V_m into i blocks in such a way that two elements $v', v'' \in V_m$ belong to the same block iff $f_H(v') = f_H(v'')$; denote this partition by $\pi(H, D)$. Denote by $\vec{\pi}(H, D) = (V_m^{(1)}(H), \ldots, V_m^{(i)}(H))$ the corresponding ordering of the partition $\pi(H, D)$ in respect to the linear ordering $\leqslant_m$ ($v_i \leqslant_m v_j$ iff $i \leqslant j$).

Let $H = (e_1, \ldots, e_m)$ be an ordered hypergraph. Denote by $\subseteq_H$ the partial ordering on the set $\langle EH \rangle = \{e_j \mid 1 \leqslant j \leqslant m\}$ defined by the relation $\subseteq$. Let $\mathfrak{p} = (\mathfrak{H}_\mathfrak{p}(n), \sim_\mathfrak{p})$ be a strong relative equivalence on $\mathfrak{H}(n)$. Also, let H' and H'' be two hypergraphs from $\mathcal{H}^{(i)}_\mathfrak{p}(D, n)$ for some $D \in \mathcal{D}_m$ such that $H' \sim_\mathfrak{p} H''$. Then, it is easy to show that $\vec{\pi}(H', D) = \vec{\pi}(H'', D)$, and that $\subseteq_{H'}$ and $\subseteq_{H''}$ are isomorphic. The latter fact implies correctness of the following notions. Let H be a hypergraph from the set $\mathcal{H}^{(i)}_\mathfrak{p}(D, n)$ for some $D \in \mathcal{D}_m$. Consider the set $[H] \in \mathfrak{H}_\mathfrak{p}(n)/{\sim_\mathfrak{p}}$ where by $[H]$ we mean the class of the equivalence $\sim_\mathfrak{p}$ on $\mathfrak{H}_\mathfrak{p}(n)$ containing H. If $\subseteq_H$ is nonempty, we call the class $[H]$ a *complex class*, in the opposite case we call it a *simple class*. The set of all complex classes is denoted by $\mathfrak{H}^{(i)}_1(\mathfrak{p}, n)$ and the set of all simple classes by $\mathfrak{H}^{(i)}_0(\mathfrak{p}, n)$.

Let $\mathfrak{h}$ be a complex class, and take some $H \in \mathfrak{h}$. Let c be a chain of the length $\geqslant 1$ in $(\langle EH \rangle, \subseteq_H)$. The minimal and maximal element of the chain c are denoted, respectively, by $e_{\min}(c)$ and $e_{\max}(c)$. It is clear that for each $D \in \mathcal{J}_m$ such that $H \in \mathcal{H}^{(i)}(D, n)$, it holds that $f_H^{-1}[e_{\min}(c)] \smallsetminus \mathrm{En}(D) \neq \emptyset$ and $f_H^{-1}[e_{\max}(c)] \smallsetminus \mathrm{Ex}(D) \neq \emptyset$; otherwise $e_{\min}(c)$ and $e_{\max}(c)$ could not be, respectively, minimal and maximal element of the chain c. Let $f_H^{-1}[e_{\min}(c)]$ and $f_H^{-1}[e_{\max}(c)]$ be, respectively, the i_c-th and the j_c-th block of the ordered partition $\vec{\pi}(H, D)$, and take that

$$C(H) = \{\, (i_c, j_c) \mid c \text{ is a chain in } (\langle EH \rangle, \subseteq_H) \text{ of the length } \geqslant 1\}.$$

Let (i_H, j_H) be the minimal element of the set $C(H)$ in respect to the lexicographic ordering of this set. It is easy to see that for every $H' \in \mathfrak{h}$ holds that $(i_{H'}, j_{H'}) = (i_H, j_H)$. Because of that by $(i_\mathfrak{h}, j_\mathfrak{h})$ we mean (i_H, j_H) for an arbitrary $H \in \mathfrak{h}$. Now it is easy to prove the following assertion.

Lemma 3.2. *Let $\mathfrak{h} \in \mathfrak{H}^{(i)}_1(\mathfrak{p})$. Then for every $H_1, H_2 \in \mathfrak{h}$ we have*

$$V_m^{(i_{H_1})}(H_1) = V_m^{(i_{H_2})}(H_2) = \hat{V}_1(\mathfrak{h}) \quad \text{and} \quad V_m^{(j_{H_1})}(H_1) = V_m^{(j_{H_2})}(H_2) = \hat{V}_2(\mathfrak{h}).$$

Lemma 3.3. *If $\mathfrak{p}$ is a strong relative equivalence on $\mathfrak{H}(n)$, then*

$$\sum_{D \in \mathcal{J}_m} (-1)^{|ED|} \lambda^{(i)}_\mathfrak{p}(D, n) = (-1)^{m-i} \cdot S(m, i) \cdot \alpha_\mathfrak{p}(i, n),$$

where $\lambda^{(i)}_\mathfrak{p}(D, n) = \alpha_\mathfrak{p}(\mathcal{H}^{(i)}(D, n))$, and $S(n, k)$ are Stirling numbers of the second kind.

Proof. For a given ordered hypergraph H, $\mathcal{E}H = i$, let us denote by $\mathcal{J}_m(H)$ the set of all $D \in \mathcal{J}_m$ satisfying the condition $H \in \mathcal{H}^{(i)}(D, n)$. It is obvious that from $H' \sim_\mathfrak{p} H''$ $(H', H'' \in \mathcal{H}^{(i)}(m, n) = \mathcal{H}^{(i)}(\emptyset_m, n))$ it follows that $\mathcal{J}_m(H') = \mathcal{J}_m(H'')$. By $\mathcal{J}_m(\mathfrak{h})$ $(\mathfrak{h} \in \mathcal{H}^{(i)}(m, n)/\mathfrak{p})$ we mean the set $\mathcal{J}_m(H)$ where H is a hypergraph from $\mathfrak{h}$. Now we have

$$\sum_{D \in \mathcal{J}_m} (-1)^{|ED|} \lambda^{(i)}_\mathfrak{p}(D, n) = \sum_{\mathfrak{h} \in \mathcal{H}^{(i)}(m,n)/\mathfrak{p}} \; \sum_{\mathfrak{h} \in \mathcal{H}^{(i)}_m/\mathfrak{p}\, D \in \mathcal{J}_m(\mathfrak{h})} (-1)^{|ED|}$$

$$= \sum_{\mathfrak{h} \in \mathfrak{H}^{(i)}_1(\mathfrak{p})} \; \sum_{\mathfrak{h} \in \mathfrak{H}^{(i)}_1(\mathfrak{p}) D \in \mathcal{J}_m(\mathfrak{h})} (-1)^{|ED|} + \sum_{\mathfrak{h} \in \mathfrak{H}^{(i)}_0(\mathfrak{p})} \; \sum_{\mathfrak{h} \in \mathfrak{H}^{(i)}_0(\mathfrak{p}) D \in \mathcal{J}_m(\mathfrak{h})} (-1)^{|ED|}.$$

In the above expression denote the first sum after the last sign = by ω_1, and the second one by ω_2.

Let $\mathfrak{h} \in \mathfrak{H}_1^{(i)}(\mathfrak{p})$. Let us break the set $\mathcal{J}_m(\mathfrak{h})$ into disjoint classes in the following way: D' and D'' belong to the same class $\mathfrak{d}$ iff

$$\hat{V}_1(\mathfrak{h}) \setminus \mathrm{En}(D') = \hat{V}_1(\mathfrak{h}) \setminus \mathrm{En}(D'') = U_1(\mathfrak{d}),$$
$$\hat{V}_2(\mathfrak{h}) \setminus \mathrm{Ex}(D') = \hat{V}_2(\mathfrak{h}) \setminus \mathrm{Ex}(D'') = U_2(\mathfrak{d})$$

and $ED' \setminus E_{\mathfrak{d}} = ED'' \setminus E_{\mathfrak{d}}$, where $E_{\mathfrak{d}} = U_1(\mathfrak{d}) \times U_2(\mathfrak{d})$; denote the corresponding relation of equivalence on $\mathcal{J}_m(\mathfrak{h})$ by ρ. Then

$$\begin{aligned}\omega_1 &= \sum_{\mathfrak{h}\in\mathfrak{H}_1^{(i)}(\mathfrak{p})} \sum_{\mathfrak{h}\in\mathfrak{H}_1^{(i)}(\mathfrak{p})\mathfrak{d}\in\mathcal{J}_m(\mathfrak{h})/\rho} \sum_{\mathfrak{h}\in\mathfrak{H}_1^{(i)}(\mathfrak{p})D\in\mathfrak{d}} (-1)^{|ED|} \\ &= \sum_{\mathfrak{h}\in\mathfrak{H}_1^{(i)}(\mathfrak{p})} \sum_{\mathfrak{h}\in\mathfrak{H}_1^{(i)}(\mathfrak{p})\mathfrak{d}\in\mathcal{J}_m(\mathfrak{h})/\rho} (-1)^{|E_{\mathfrak{d}}|} \sum_{\mathfrak{h}\in\mathfrak{H}_1^{(i)}(\mathfrak{p})E'\subseteq E_{\mathfrak{d}}} (-1)^{|E'|} = 0.\end{aligned}$$

Since $\mathfrak{p}$ is a regular relative equivalence, then exactly one class $\mathfrak{a}_{\mathfrak{h}}$ from the set $\mathcal{A}_{\mathfrak{p}}(i,n)/\mathfrak{p}$ corresponds to every simple class $\mathfrak{h}$. Now it is clear that the simple class $\mathfrak{h}$ is completely determined by the class $\mathfrak{a}_{\mathfrak{h}}$ and by the partition $\pi(\mathfrak{h})$ of the set V_m into i blocks. There are $\alpha_{\mathfrak{p}}(i,n)$ such classes and there are $S(m,i)$ such partitions. Denote blocks of the partition $\pi(\mathfrak{h})$ by $V_j(\mathfrak{h})$, $j \in \bar{i}$. Then it is clear that in every digraph $D \in \mathcal{J}_m(\mathfrak{h})$ there does not exist an edge that connects a vertex from the set $V_k(\mathfrak{h})$ with a vertex from the set $V_l(\mathfrak{h})$, $k,l \in \bar{i}$, $k \neq l$, that is to say, $ED \cap (V_k(\mathfrak{h}) \times V_l(\mathfrak{h})) = \emptyset$ for every $k,l \in \bar{i}$, $k \neq l$. Put $D_k' = (V_k(\mathfrak{h}), ED \cap V_k^2(\mathfrak{h}))$ and $m_k = |V_k(\mathfrak{h})|$; $k \in \bar{i}$. By D_k, $k \in \bar{i}$, denote the digraph from $\mathcal{J}_{m_k}$ that is isomorphic to D_k'. It is clear that when D passes the set $\mathcal{J}_m$, then the digraph D_k passes the whole set $\mathcal{J}_{m_k}$ for every $k \in \bar{i}$. Using Lemma 3.1 we have

$$\begin{aligned}\sum_{D\in\mathcal{J}_m(\mathfrak{h})} (-1)^{|ED|} &= \sum_{D_1\in\mathcal{J}_{m_1}} \cdots \sum_{D_i\in\mathcal{J}_{m_i}} (-1)^{|ED_1|} \cdots (-1)^{|ED_i|} \\ &= \sum_{D_1\in\mathcal{J}_{m_1}} \cdots \sum_{D_{i-1}\in\mathcal{J}_{m_{i-1}}} (-1)^{|ED_1|} \cdots (-1)^{|ED_{i-1}|} \sum_{D_i\in\mathcal{J}_{m_i}} (-1)^{|ED_i|} \\ &= (-1)^{m_i-1} \sum_{D_1\in\mathcal{J}_{m_1}} \cdots \sum_{D_{i-1}\in\mathcal{J}_{m_{i-1}}} (-1)^{|ED_1|} \cdots (-1)^{|ED_{i-1}|} = \cdots \\ &= (-1)^{m_1-1} \cdots (-1)^{m_i-1} = (-1)^{m-i}.\end{aligned}$$

Now it is clear that

$$\omega = \omega_2 = \sum_{\mathfrak{a}_{\mathfrak{h}}\in\mathcal{A}_{\mathfrak{p}}(i,n)/\mathfrak{p}} \sum_{\pi(\mathfrak{h})} \sum_{D\in\mathcal{J}_m(\mathfrak{h})} (-1)^{|ED|} = (-1)^{m-i} S(m,i)\alpha_{\mathfrak{p}}(i,n). \qquad \square$$

Theorem 3.3. *If $\mathfrak{p}$ is a strong relative equivalence on $\mathfrak{H}(n)$, then*

$$\alpha_{\mathfrak{p}}(m,n) = \sum_{i=1}^{m} |s(m,i)| \sum_{D\in\mathcal{J}_i} (-1)^{|ED|} \lambda_{\mathfrak{p}}(D,n),$$

where $s(n,k)$ are Stirling numbers of the first kind.

PROOF. Let $\beta_{\mathfrak{p}}(m,n)=\sum_{D\in\mathcal{J}_m}(-1)^{|ED|}\lambda_{\mathfrak{p}}(D,n)$. Then from Lemma 3.3 and the equality $\lambda_{\mathfrak{p}}(D,n)=\sum_{i=1}^{m}\lambda_{\mathfrak{p}}^{(i)}(D,n)$ it follows that

$$\beta_{\mathfrak{p}}(m,n)=\sum_{i=1}^{m}\sum_{D\in\mathcal{J}_m}(-1)^{|ED|}\lambda_{\mathfrak{p}}^{(i)}(D,n)=\sum_{i=1}^{m}(-1)^{m-i}\cdot S(m,i)\cdot\alpha_{\mathfrak{p}}(i,n).$$

Applying the Stirling inversion [**4**] to the previous formula we get the required equation

$$\alpha_{\mathfrak{p}}(m,n)=\sum_{i=1}^{m}|s(m,i)|\cdot\beta_{\mathfrak{p}}(i,n)=\sum_{i=1}^{m}|s(m,i)|\sum_{D\in\mathcal{J}_i}(-1)^{|ED|}\lambda_{\mathfrak{p}}(D,n).\qquad\square$$

Denote by $\tilde{\mathcal{A}}_{\mathfrak{p}}(m,n)$ the set of all unordered antichains that correspond to ordered antichains from the set $\mathcal{A}_{\mathfrak{p}}(m,n)$. As in an antichain there are no multiple edges, then it is obvious that

$$\tilde{\alpha}_{\mathfrak{p}}(m,n)=(1/m!)\,\alpha_{\mathfrak{p}}(m,n),\tag{3.3}$$

where $\tilde{\alpha}_{\mathfrak{p}}(m,n)=|\tilde{\mathcal{A}}_{\mathfrak{p}}(m,n)|$, and from Theorem 3.3 we have the following statement.

THEOREM 3.4. *If $\mathfrak{p}$ is a strong relative equivalence on $\mathfrak{H}(n)$, then*

$$\tilde{\alpha}_{\mathfrak{p}}(m,n)=\frac{1}{m!}\sum_{i=1}^{m}|s(m,i)|\sum_{D\in\mathcal{J}_i}(-1)^{|ED|}\lambda_{\mathfrak{p}}(D,n).$$

For every $k_1,\dots,k_m\in\mathbf{N}_0$, let $(jk_j)_{\overline{m}}=1k_1+2k_2+\cdots+mk_m$. Denote by

$$Y_n(x_1,x_2,\dots,x_n)=\sum_{(jk_j)_{\overline{n}}=n}B(k_1,\dots,k_n)\,x_1^{k_1}x_2^{k_2}\dots x_n^{k_n}$$

Bell polynomial [**6**]. Let $\mathfrak{p}$ be a strong relative equivalence on $\mathfrak{H}(n)$. We say that $\mathfrak{p}$ *allows partitioning* if $\lambda_{\mathfrak{p}}(D,n)=\lambda_{\mathfrak{p}}(D_1,n)\,\lambda_{\mathfrak{p}}(D_2,n)\dots\lambda_{\mathfrak{p}}(D_k,n)$, whenever $D=D_1\cup D_2\cup\cdots\cup D_k$. Denote by $\mathcal{J}_i^c$ the set of all connected hedgehogs with i vertices. Then, in the case of a strong relative equivalence which allows partitioning, Theorem 3.3 can be reformulated in the following statement:

THEOREM 3.5. *If $\mathfrak{p}$ is a strong relative equivalence on $\mathfrak{H}(n)$ which allows partitioning, then*

$$\tilde{\alpha}_{\mathfrak{p}}(m,n)=\frac{1}{m!}\sum_{i=1}^{m}|s(m,i)|\,Z(i,n)=\frac{1}{m!}\sum_{i=1}^{m}|s(m,i)|\,Y_i(\beta(1,n),\beta(2,n),\dots,\beta(i,n))$$

where $\beta(j,n)=\sum_{D\in\mathcal{J}_j^c}(-1)^{|ED|}\lambda_{\mathfrak{p}}(D,n)$ for every $j\in\overline{i}$.

PROOF. Let π be a partition of the set V_i of the type $1^{k_1}\dots i^{k_i}$; by $\mathcal{P}_i(k_1,\dots,k_i)$ denote the set of all such partitions, and let $|\pi|=k_1+\cdots+k_i$. Define a function $f_\pi:\overline{|\pi|}\to\overline{i}$ in the following way: the value $f_\pi(r)$ $(r\in\overline{|\pi|})$ is equal to the ordinal number of the first nonpositive number in the sequence $r-k_1, r-k_1-k_2, \dots, r-k_1-\dots-k_i$. Then we get

$$\sum_{D\in\mathcal{J}_i}(-1)^{|ED|}\lambda_{\mathfrak{p}}(D,n) = \sum_{(jk_j)_{\overline{i}}=i}\ \sum_{\pi\in\mathcal{P}_i(k_1,\dots,k_i)}\ \sum_{D_i\in\mathcal{J}^c_{f_\pi(i)},\, i\in\overline{|\pi|}}(-1)^{|E(D_1\cup\dots\cup D_{|\pi|})|}\lambda_{\mathfrak{p}}(D_1\cup\dots\cup D_{|\pi|},n).$$

But as

$$\begin{aligned}\sum_{D_i\in\mathcal{J}^c_{f_\pi(i)},\, i\in\overline{|\pi|}}&(-1)^{|E(D_1\cup\dots\cup D_{|\pi|})|}\lambda_{\mathfrak{p}}(D_1\cup\dots\cup D_{|\pi|},n)\\ &= \sum_{D_i\in\mathcal{J}^c_{f_\pi(i)},\, i\in\overline{|\pi|}}(-1)^{|ED_1|}\lambda_{\mathfrak{p}}(D_1,n)\cdots(-1)^{|ED_{|\pi|}|}\lambda_{\mathfrak{p}}(D_{|\pi|},n)\\ &= \sum_{D_1\in\mathcal{J}^c_{f_\pi(1)}}(-1)^{|ED_1|}\lambda_{\mathfrak{p}}(D_1,n)\cdots\sum_{D_{|\pi|}\in\mathcal{J}^c_{f_\pi(|\pi|)}}(-1)^{|ED_{|\pi|}|}\lambda_{\mathfrak{p}}(D_{|\pi|},n)\\ &= \Big[\sum_{D\in\mathcal{J}^c_1}(-1)^{|ED|}\lambda_{\mathfrak{p}}(D,n)\Big]^{k_1}\cdots\Big[\sum_{D\in\mathcal{J}^c_i}(-1)^{|ED|}\lambda_{\mathfrak{p}}(D,n)\Big]^{k_i}\\ &= [\beta(1,n)]^{k_1}\cdots[\beta(i,n)]^{k_i},\end{aligned}$$

we have, finally, that

$$\begin{aligned}\sum_{D\in\mathcal{J}_i}(-1)^{|ED|}\lambda_{\mathfrak{p}}(D,n) &= \sum_{(jk_j)_{\overline{i}}=i}\ \sum_{\pi\in\mathcal{P}_i(k_1,\dots,k_i)}[\beta(1,n)]^{k_1}\cdots[\beta(i,n)]^{k_i}\\ &= \sum_{(jk_j)_{\overline{i}}=i}B(k_1,k_2,\dots,k_i)\,[\beta(1,n)]^{k_1}\cdots[\beta(i,n)]^{k_i} = Y(\beta(1,n),\dots,\beta(i,n)). \quad\square\end{aligned}$$

For application of the formula from Theorem 3.3 it would be very useful to have some simple necessary conditions for a relative equivalence to be a strong relative equivalence. Let us give one of such conditions. For a given hypergraph H by $\mathfrak{M}_H$ denote the set of columns of the matrix M_H.

THEOREM 3.6. *Let* $\mathfrak{p}$ *be a relative equivalence. If* $\mathfrak{M}_{H'} = \mathfrak{M}_{H''}$ *for every* $H', H'' \in \mathfrak{H}_{\mathfrak{p}}(n)$, $H' \sim_{\mathfrak{p}} H''$, *then* $\mathfrak{p}$ *is a strong relative equivalence.*

4. Enumeration of some classes of antichains

In this section we are going to show how to calculate some of the numbers $\alpha_{i_1i_2i_3}(m,n)$ using the above obtained formulas. First of all, let us introduce some notions and give some results that we use in what follows.

Let us introduce two vertex colorings of graphs or digraphs with two colors, red and green. By $\uparrow$-coloring of a graph or a digraph we mean a coloring of its vertices with red and green such that adjacent vertices cannot be colored red (here $\uparrow$ denotes Sheffer's stroke or alternative denial), and by $\Rightarrow$-coloring of a digraph we mean a coloring of this digraph with red and green color such that there is not a vertex colored red from which an edge goes to a vertex colored green. The reason for such naming of these colorings can be easily seen if the red color is replaced by 1 (logically true) and the green by 0 (logically false). For the same reason, a "regular" coloring of a graph with two colors, such that two adjacent vertices are not colored with the same color, could be called a $\underline{\vee}$-coloring, where $\underline{\vee}$ is the operation of exclusive disjunction.

Let D be a digraph. Denote by $\eta_{\Rightarrow}(D)$ $[\eta_{\uparrow}(D)]$ the number of all $\Rightarrow$-colorings [$\uparrow$-colorings] of the digraph. Let G be a graph. Denote by $\eta_{\uparrow}(G)$ the number of all $\uparrow$-colorings of G. For given digraph D by $\overline{D}$ denote the graph obtained from D by canceling orientation of its edges. It is easy to prove the following statement.

PROPOSITION 4.1. *For an arbitrary hedgehog $D \in \mathcal{J}_m$ the equation $\eta_{\Rightarrow}(D) = \eta_{\uparrow}(D) = \eta_{\uparrow}(\overline{D})$ holds.*

It is easy to see that the set of red colored vertices in an $\eta_{\uparrow}$-coloring of a digraph (graph) is an independent set of its vertices. Let us also note that the notion of hedgehog is very close to the notion of bipartite graph. Namely, graph $\overline{D}$ is a bipartite graph for every connected hedgehog D. Thus Proposition 4.1 allows to cancel orientations of edges in hedgehogs, and practically to pass from hedgehogs to bipartite graphs, and to consider $\uparrow$-colorings instead of $\Rightarrow$-colorings. Turning to a new type of coloring is not purely formal, but it also seems convenient for the following reasons. The number of all $\uparrow$-colorings of a graph G can be calculated in the following way. Denote by G_v^1 the graph that is obtained from G when the point v and all its incident edges are rejected, and by G_v^2 the graph that is obtained from G when v and all its adjacent vertices are rejected, and all the edges incident to some of the rejected vertices are discarded, too. Introduce, formally, a graph without edges and vertices, denote it by $\emptyset_0$, and take that $\eta_{\uparrow}(\emptyset_0) = 1$. It is easy to prove the following statements [**2**].

PROPOSITION 4.2 (Decomposition lemma). $\eta_{\uparrow}(G) = \eta_{\uparrow}(G_v^1) + \eta_{\uparrow}(G_v^2)$.

Using the above statement it is easy to calculate numbers $\eta_{\uparrow}(G)$ for special cases of graph G.

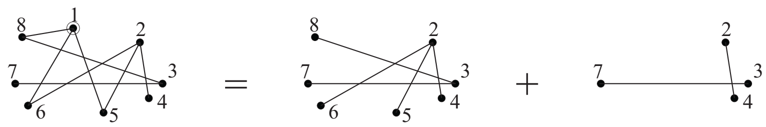

FIGURE 1. An illustration of Decomposition lemma

EXAMPLE 4.1. Let K_n and $K_{m,n}$ be respectively a complete graph with n vertices and a complete bipartite graph with partition classes consisting of m and n elements. Then $\eta_{\uparrow}(K_n) = n+1$ and $\eta_{\uparrow}(K_{m,n}) = 2^m + 2^n - 1$. If P_n is a path with n vertices, then we get Fibonacci sequence $\eta_{\uparrow}(P_n) = \eta_{\uparrow}(P_{n-1}) + \eta_{\uparrow}(P_{n-2})$, $n = 2, 3, \ldots$, $\eta_{\uparrow}(P_0) = \eta_{\uparrow}(\emptyset_0) = 1$, $\eta_{\uparrow}(P_1) = 2$. If Z_n is an n-cycle, then we get that $\eta_{\uparrow}(Z_n) = \eta_{\uparrow}(P_{n-1}) + \eta_{\uparrow}(P_{n-3})$, $n \geqslant 3$, and, consequently, that $\eta_{\uparrow}(Z_{n+2}) = \eta_{\uparrow}(Z_{n+1}) + \eta_{\uparrow}(Z_n)$, where $\eta_{\uparrow}(Z_3) = 4$ i $\eta_{\uparrow}(Z_4) = 7$.

The proof of the following statement is trivial.

PROPOSITION 4.3. *If $C_1, C_2, \dots, C_s$ are all components (maximal connected subgraphs) of the graph G, then $\eta_{\uparrow}(G) = \eta_{\uparrow}(C_1)\eta_{\uparrow}(C_2)\dots\eta_{\uparrow}(C_s)$.*

Now, let us return to our main problem of finding numbers $\alpha_{i_1 i_2 i_3}(m,n)$. As (3.3) implies that $\alpha_{1 i_2 i_3}(m,n) = (1/m!)\alpha_{0 i_2 i_3}(m,n)$, the finding of the numbers $\alpha_{1 i_2 i_3}(m,n)$ ($i_2 \in \overline{0,2}$, $i_3 \in \overline{0,3}$) can be reduced to the calculation of the numbers $\alpha_{0 i_2 i_3}(m,n)$. Let us show how to calculate some of the numbers $\alpha_{0 i_2 i_3}(m,n)$ using appropriate strong relative equivalence $\mathfrak{p}_{i_2 i_3}$. Denote by $\lambda_{i_2 i_3}(D,n)$ the number $\lambda_{\mathfrak{p}_{i_2 i_3}}(D,n)$. So as not to overload the text with unnecessary details, we will adduce formulas only for the numbers $\lambda_{i_2 i_3}(D,n)$, as the numbers $\alpha_{0 i_2 i_3}(m,n)$ can be calculated by the use of the main formula from Theorem 3.4.

It is easy to note that if $H \in \mathcal{H}(D,n)$, then every column of the matrix M_H defines a $\Rightarrow$-coloring of the digraph D. Also, it is clear that every n-tuple of $\Rightarrow$-colorings defines a hypergraph from $\mathcal{H}(D,n)$. So, $|\mathcal{H}(D,n)| = \eta^n_{\Rightarrow}(D)$. Now from Proposition 4.1 it follows that if D is a hedgehog, then every column of the matrix M_H defines a $\uparrow$-coloring for every $H \in \mathcal{H}(D,n)$. Corollaries 4.1–4.4 follow easily from this observation.

COROLLARY 4.1. *If $\mathfrak{p}_{23} = (\mathfrak{H}(n), =)$, then $\lambda_{23}(D,n) = \eta^n_{\Rightarrow}(D) = \eta^n_{\uparrow}(D)$ for every $D \in \mathcal{J}_m$.*

Class $\mathcal{A}_{023}$ is closely connected with the Post class $\mathcal{M}$ of all monotone Boolean functions. Let us explain this connection.

For every two binary n-tuples $(a_1, \dots, a_n)$, and $(b_1, \dots, b_n)$, we write

$$(a_1, \dots, a_n) \leqslant (b_1, \dots, b_n) \quad \text{if} \quad a_i \leqslant b_i \quad \text{for every} \quad i = 1, \dots, n.$$

If $(a_1, \dots, a_n) \leqslant (b_1, \dots, b_n)$ and there is some i_0, $0 \leqslant i_0 \leqslant n$, such that $a_{i_0} < b_{i_0}$, then we write that $(a_1, \dots, a_n) < (b_1, \dots, b_n)$. A Boolean function f of n variables is monotone if for every two binary n-tuples, $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$, from $(a_1, \dots, a_n) \leqslant (b_1, \dots, b_n)$ it follows that $f(a_1, \dots, a_n) \leqslant f(b_1, \dots, b_n)$; denote by $\mathcal{M}(n)$ the class of all such functions. Then $\mathcal{M} = \cup_{n=1}^{\infty} \mathcal{M}(n)$.

The problem of counting the number of elements of the classes $\mathcal{M}(n)$ has a long history and is known as the Dedekind's problem. It was formulated by Dedekind in [**1**] as far back as in 1897 as the problem of determining the number of elements in a free distributive lattice $FD(n)$ on n generators (see also, [**5**]). In terms of the set theory the problem is equivalent to the problem of determining the number of all antichains on an n-set. The solution of the problem can be attempted in the following way.

Let f be an arbitrary monotone Boolean function of n variables. A binary n-tuple $(a_1, a_2, \dots, a_n)$ is a *lower unit* for f if $f(a_1, a_2, \dots, a_n) = 1$ and for every $(b_1, b_2, \dots, b_n) < (a_1, a_2, \dots, a_n)$ it holds that $f(b_1, b_2, \dots, b_n) = 0$. It is easy to note that the function f is uniquely defined by the set $\mathbf{U}(f)$ of its lower units.

Denote by $\mathcal{M}(m,n)$ the set of all monotone Boolean functions of n variables with exactly m lower units ("mincuts"). Let $\alpha(m,n) = |\mathcal{M}(m,n)|$. As it is known from famous Sperner's lemma [4], m takes values from 0 to $\binom{n}{\lfloor n/2\rfloor}$, so

$$|\mathcal{M}(n)| = \sum_{m=0}^{\binom{n}{\lfloor n/2\rfloor}} |\mathcal{M}(m,n)|.$$

Therefore, we can enumerate the class $\mathcal{M}(n)$ in such a way as to enumerate classes $\mathcal{M}(m,n)$. A brief history of the problem of enumeration of the classes $\mathcal{M}(m,n)$ can be found in [**2, 3**]. Let us deduce the main result from [**3**]. Let $\boldsymbol{a} = (a_1,\dots,a_n)$ be a binary n-tuple. By $V(\boldsymbol{a})$ denote the subset of the set V_n such that $v_i \in V(\boldsymbol{a})$ iff $a_i = 1$.

Let f be a monotone Boolean function from the set $\mathcal{M}(m,n)$. Denote by H_f hypergraf $(V_n, \mathcal{E}(f))$, where $\mathcal{E}(f) = \{V(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathbf{U}(f)\}$. Note that H_f is an unordered labeled (m,n)-antichain (an (m,n)-hypergraph that is an antichain). Now, it is easy to see that the correspondence $f_H \to H$ defines a bijection between $\mathcal{M}(m,n)$ and the set $\mathcal{A}_{123}$. Using Corollary 4.1 and Theorem 3.4 we have the following statement.

Theorem 4.1. $\alpha_{123}(m,n) = \dfrac{1}{m!}\displaystyle\sum_{i=1}^{m} |s(m,i)| \sum_{D\in\mathcal{J}_m} (-1)^{|ED|}\eta_{\uparrow}^{n}(D).$

If for a labeled ordered (m,n)-antichain H we observe a corresponding dual labeled ordered hypergraph H^T, then it is easy to see that it is a labeled ordered T_1-hypergraph with m vertices and n hyperedges, and, consequently, it is also a cover. Thus, $t_{00111}(m,n) = \alpha_{023}(n,m)$.

Let us consider the class $\mathcal{A}_{022}(m,n)$ of all ordered (m,n)-T_0-antichains.

Corollary 4.2. *If $\mathfrak{p}_{22} = (\mathfrak{H}'(n)), =)$, where $\mathfrak{H}'(n)$ is the set of all hypergraphs $H \in \mathfrak{H}(n)$ such that in M_H all columns are different, then*

$$\lambda_{22}(D,n) = [\eta_{\uparrow}(D)]_n = \eta_{\uparrow}(D)(\eta_{\uparrow}(D)-1)\dots(\eta_{\uparrow}(D)-n+1)$$

for every $D \in \mathcal{J}_m$.

Now consider the class $T_{10023}(m,n)$ of all ordered unlabeled (m,n)-antichains. Then we have the following statement.

Corollary 4.3. *If $\mathfrak{p} = (\mathfrak{H}(n), \sim)$, where $H_1 \sim H_2$ iff the multisets defined by the columns of the matrices of the hypergraphs H_1 and H_2 are equal, then*

$$\lambda_{\mathfrak{p}}(D,n) = \binom{\eta_{\uparrow}(D)+n-1}{n} \quad \textit{for every} \quad D \in \mathcal{J}_m.$$

For the class $T_{10022}(m,n)$ of all ordered unlabeled (m,n)-T_0-antichains we have

Corollary 4.4. *If $\mathfrak{p} = (\mathfrak{H}'(n), \sim)$, where $H_1 \sim H_2$ iff $\mathfrak{M}_{H_1} = \mathfrak{M}_{H_2}$, then*

$$\lambda_{\mathfrak{p}}(D,n) = \binom{\eta_{\uparrow}(D)}{n} \quad \textit{for every} \quad D \in \mathcal{J}_m.$$

Let us note that we get the formulas for the corresponding "cover" cases if in the above formulas we replace $\eta_\uparrow(D)$ by $\eta_\uparrow(D)-1$ because the incidence matrix of a cover does not contain zero column. Thus we solve the cases of the classes $\mathcal{A}_{023}(m,n)$, $\mathcal{A}_{022}(m,n)$, $T_{10023}(m,n)$, $T_{10022}(m,n)$, $\mathcal{A}_{013}(m,n)$, $\mathcal{A}_{012}(m,n)$, $T_{10013}(m,n)$ and $T_{10012}(m,n)$. Using (3.3) we also get the corresponding formulas for the classes $\mathcal{A}_{123}(m,n)$, $\mathcal{A}_{122}(m,n)$, $\mathcal{A}_{113}(m,n)$, and $\mathcal{A}_{112}(m,n)$.

5. Calculations, examples and data

It is easy to see that the following proposition holds.

Proposition 5.1. *For every hedgehog D, $\eta_\uparrow(D)=\eta_\uparrow(D^{-1})$.*

Let $\mathcal{J}^c_{s,t}$ be a set of all connected hedgehogs D such that $|\mathrm{Ex}(D)|=s$ and $|\mathrm{En}(D)|=t$. If we consider the case of strong relative equivalence on $\mathfrak{H}(n)$ from Corollary 4.1 by using Theorem 3.3 and Theorem 3.5 we get the following proposition:

Proposition 5.2. *It holds that*

$$\alpha_{123}(m,n)=\frac{1}{m!}\sum_{i=1}^{m}|s(m,i)|\,Z(i,n), \tag{5.1}$$

where

$$Z(i,n)=Y_i(\beta(1,n),\beta(2,n),\dots,\beta(i,n)),$$

$$\beta(1,n)=2^n,\quad \beta(j,n)=\sum_{k=1}^{j-1}\binom{j}{k}b(k,j-k),\ 2\leqslant j\leqslant i,$$

$$b(s,t)=\sum_{D\in\mathcal{J}^c_{s,t}}(-1)^{|ED|}\eta^n_\uparrow(D),\ s,t\geqslant 1.$$

Let $k_1,\dots,k_i\in\mathbf{N}_0$ be nonnegative numbers such that $(jk_j)_{\overline{i}}=i$. Denote by $\mathcal{J}(k_1,\dots,k_i)$ the set of all hedgehogs $D_1\cup\dots\cup D_{k_1+\dots+k_i}$ such that $D_i\in\mathcal{J}_{f_\pi(i)}$ for every $i\in\overline{k_1+\dots+k_i}$, where f_π is the function defined in the proof of Theorem 3.5 for the partition type $1^{k_1}2^{k_2}\dots i^{k_i}$. By $\mathcal{J}(k_1,\dots,k_i;l)$ denote the set of all hedgehogs D from $\mathcal{J}(k_1,\dots,k_i)$ such that $\eta_\uparrow(D)=l$. Let $j(k_1,\dots,k_i;l)=|\mathcal{J}(k_1,\dots,k_i;l)|$. It is clear that the following proposition holds.

Proposition 5.3. *If $\mathfrak{p}$ is a strong relative equivalence on $\mathfrak{H}(n)$, satisfying $\lambda_\mathfrak{p}(D,n)=g(\eta_\uparrow(D))$ for every $D\in\mathcal{J}$, then the number $\alpha_{123}(m,n)$ is equal to*

$$\frac{1}{m!}\sum_{i=1}^{m}\sum_{(jk_j)_{\overline{i}}=i}\sum_{l=1}^{2^i}|s(m,i)|B(k_1,\dots,k_i)\,j(k_1,\dots,k_i;l)g(l).$$

Example 5.1. Using Example 4.1 we get

$$b(1,t)=\sum_{J\in\mathcal{J}^c_{1,t}}(-1)^{|EJ|}\eta^n_\uparrow(J)=(-1)^{|EK_{1,t}|}\eta^n_\uparrow(K_{1,t})=(-1)^t(2^t+1)^n,$$

and consequently we have that $b(1,1) = -3^n$, $b(1,2) = 5^n$, $b(1,3) = -9^n$, $b(1,4) = 17^n$, $b(1,5) = -33^n$, and $b(1,6) = 65^n$.

It is clear that $b(s,t) = b(t,s)$. So, in order to calculate $\alpha_{123}(m,n)$, $1 \leqslant m \leqslant 7$, it is sufficient to consider classes $\mathcal{J}^c_{2,2}$, $\mathcal{J}^c_{2,3}$, $\mathcal{J}^c_{2,4}$, $\mathcal{J}^c_{3,3}$, $\mathcal{J}^c_{2,5}$ and $\mathcal{J}^c_{3,4}$. In the given table beside each graph there are two numbers, the upper one gives the number of its isomorphic graphs, and the lower one gives the corresponding number $\eta_\uparrow$, which is easily calculated with the help of Decomposition Lemma; the graphs are classified by the degree of the vertices of their lower parts (they are given below the graphs in the form of the corresponding tuples). Define the operation | in the following way: for every $a, b \in \mathbf{N}$, let $a|b = a \cdot b^n$ (here n is fixed). Now using formula (5.1) and the table (Fig. 2), we get

$$\begin{aligned}
&b(1,1) = -1|3; \quad b(1,2) = 1|5; \quad b(1,3) = -1|9, \quad b(2,2) = -4|8 + 1|7;\\
&b(1,4) = 1|17, \quad b(2,3) = 6|14 + 6|13 - 6|12 + 1|11;\\
&b(1,5) = -1|33, \quad b(2,4) = -24|23 - 8|26 + 12|21 + 12|22 - 8|20 + 1|19,\\
&b(3,3) = -18|22 - 36|21 + 18|19 + 6|18 - 18|22 - 9|24 + 18|19 + 36|20-\\
&\qquad 18|17 - 9|18 - 9|18 + 9|16 - 1|15;\\
&b(1,6) = 1|65,\\
&b(2,5) = 10|50 + 40|43 + 30|41 - 20|42 - 60|39 + 20|38 + 20|37 - 10|36 + 1|35,\\
&b(3,4) = 144|36 + 36|40 + 72|34 + 72|37 - 72|31 - 72|33 - 72|34 - 72|31 + 24|31+\\
&\qquad 18|29 + 36|28 + 24|35 + 72|38 + 12|44 - 144|32 - 72|34 - 72|36 - 36|33+\\
&\qquad 72|30 + 72|32 + 144|29 + 36|29 - 72|27 - 12|30 - 24|26 + 36|30 + 18|32-\\
&\qquad 36|27 - 72|28 + 36|25 + 18|26 + 12|26 - 12|24 + 1|23
\end{aligned}$$

and

$$\begin{aligned}
&\beta(1,n) = 1|2, \quad \beta(2,n) = -2|3, \quad \beta(3,n) = 6|5, \quad \beta(4,n) = -8|9 - 24|8 + 6|7,\\
&\beta(5,n) = 10|17 + 120|14 + 120|13 - 120|12 + 20|11,\\
&\beta(6,n) = -12|33 - 240|26 - 180|24 - 720|23 - 360|22 - 360|21 + 480|20+\\
&\qquad 750|19 - 240|18 - 360|17 + 180|16 - 20|15,\\
&\beta(7,n) = 14|65 + 420|50 + 840|44 + 1680|43 - 840|42 + 1260|41 + 2520|40-\\
&\qquad 2520|39 + 5880|38 + 5880|37 + 4620|36 + 1722|35 - 5040|34-\\
&\qquad 7560|33 - 3780|32 - 8400|31 + 6720|30 + 13860|29 - 2520|28-\\
&\qquad 7560|27 + 420|26 + 2520|25 - 840|24 + 70|23.
\end{aligned}$$

Replacing x_i with $\beta(i,n)$ in Bell polynomials $Y_i(x_1, \dots, x_i)$, $i = 1, 2, \dots, 7$,

$$\begin{aligned}
&Y_1 = x_1, \quad Y_2 = x_1^2 + x_2, \quad Y_3 = x_1^3 + 3x_1x_2 + x_3,\\
&Y_4 = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4,\\
&Y_5 = x_1^5 + 10x_1^3x_2 + 10x_1^2x_3 + 15x_1x_2^2 + 5x_1x_4 + 10x_2x_3 + x_5,\\
&Y_6 = x_1^6 + 15x_1^4x_2 + 45x_1^2x_2^2 + 15x_2^3 + 20x_1^3x_3 + 60x_1x_2x_3 + 10x_3^2 + 15x_1^2x_4+\\
&\qquad 15x_2x_4 + 6x_1x_5 + x_6,\\
&Y_7 = x_1^7 + 21x_1^5x_2 + 105x_1^3x_2^2 + 105x_1x_2^3 + 35x_1^4x_3 + 210x_1^2x_2x_3 + 105x_2^2x_3+\\
&\qquad 70x_1x_3^2 + 35x_1^3x_4 + 105x_1x_2x_4 + 35x_3x_4 + 21x_1^2x_5 + 21x_2x_5 + 7x_1x_6 + x_7,
\end{aligned}$$

we get respectively the values

$$\begin{aligned}
&Z(1,n) = 1|2, \quad Z(2,n) = 1|4 - 2|3, \quad Z(3,n) = 1|8 - 6|6 + 6|5,\\
&Z(4,n) = 1|16 - 12|12 + 24|10 + 4|9 - 24|8 + 6|7,\\
&Z(5,n) = 1|32 - 20|24 + 60|20 + 20|18 + 10|17 - 120|16 - 120|15+\\
&\qquad 150|14 + 120|13 - 120|12 + 20|11,\\
&Z(6,n) = 1|64 - 30|48 + 120|40 + 60|36 + 60|34 - 12|33 - 360|32 - 720|30+\\
&\qquad 810|28 + 120|27 + 480|26 + 360|25 - 180|24 - 720|23 - 240|22-\\
&\qquad 540|21 + 480|20 + 750|19 - 240|18 - 360|17 + 180|16 - 20|15,\\
&Z(7,n) = 1|128 - 42|96 + 210|80 + 140|72 + 210|68 - 84|66 + 14|65 - 840|64-\\
&\qquad 2520|60 + 2730|56 + 840|54 + 840|52 - 420|51 + 2940|50 + 1260|48-\\
&\qquad 5040|46 + 840|45 - 1260|44 + 1680|43 - 9660|42 + 1260|41 + 840|40-\\
&\qquad 7560|39 + 11130|38 + 5880|37 + 7980|36 + 2982|35 - 7560|34-\\
&\qquad 8400|33 - 2520|32 - 8400|31 + 6580|30 + 13860|29 - 2520|28-\\
&\qquad 7560|27 + 420|26 + 2520|25 - 840|24 + 70|23,
\end{aligned}$$

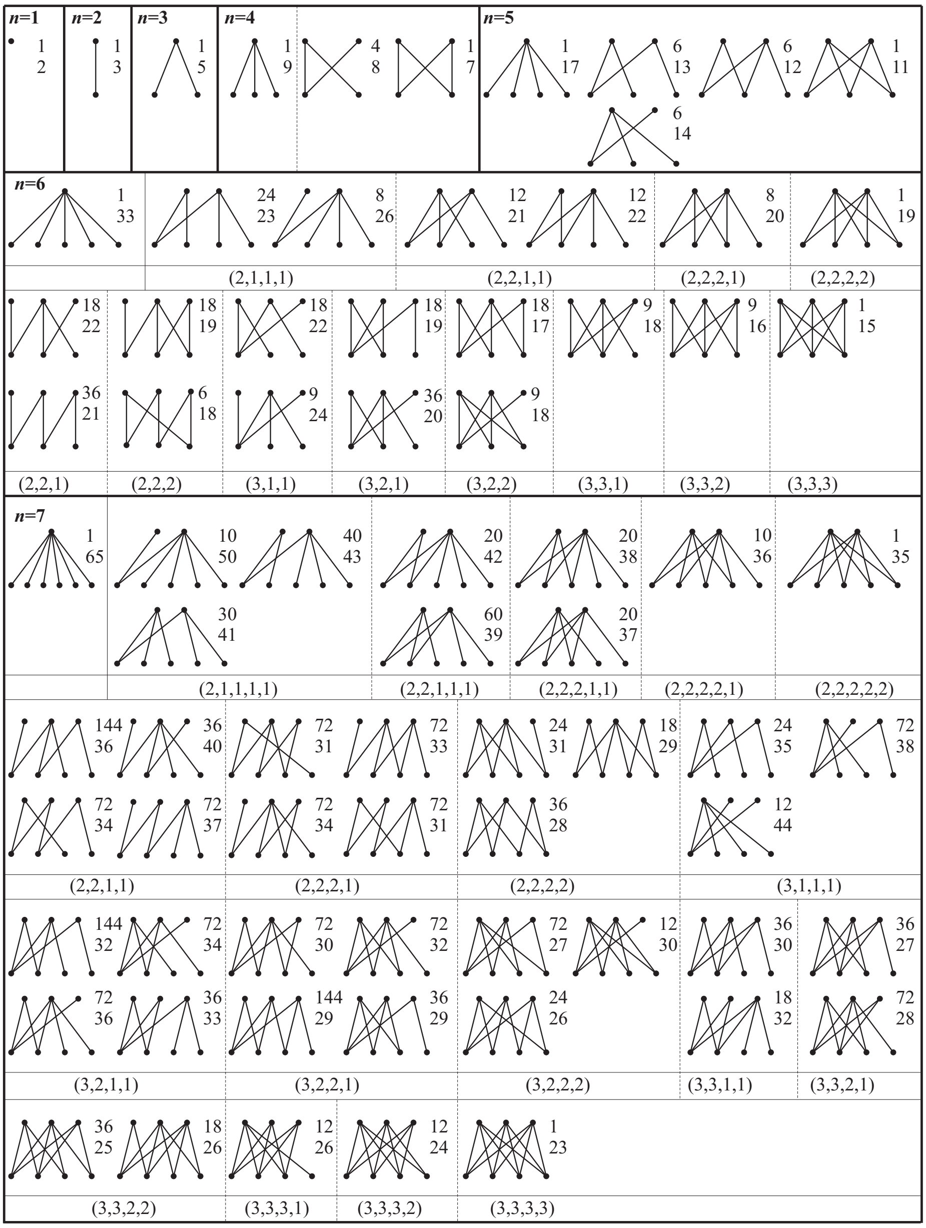

Figure 2. Nonisomorphic hedgehogs (having up to 7 vertices) with the corresponding number of isomorphic copies, and the number of their ↑-colorings

and from this we finally calculate formulas $\hat{\alpha}(m,n) = \alpha_{123}(m,n)$ $(1 \leqslant m \leqslant 7)$:

$$
\begin{aligned}
\hat{\alpha}(1,n) = \quad & 1|2,\\
\hat{\alpha}(2,n) = \quad & (1|4 - 2|3 + 2)/2!,\\
\hat{\alpha}(3,n) = \quad & (1|8 - 6|6 + 6|5 + 3|4 - 6|3 + 2|2)/3!,\\
\hat{\alpha}(4,n) = \quad & (1|16 - 12|12 + 24|10 + 4|9 - 18|8 + 6|7 - 36|6+\\
& 36|5 + 11|4 - 22|3 + 6|2)/4!,\\
\hat{\alpha}(5,n) = \quad & (1|32 - 20|24 + 60|20 + 20|18 + 10|17 - 110|16 - 120|15 + 150|14+\\
& 120|13 - 240|12 + 20|11 + 240|10 + 40|9 - 205|8 + 60|7 - 210|6+\\
& 210|5 + 50|4 - 100|3 + 24|2)/5!,\\
\hat{\alpha}(6,n) = \quad & (1|64 - 30|48 + 120|40 + 60|36 + 60|34 - 12|33 - 345|32 - 720|30+\\
& 810|28 + 120|27 + 480|26 + 360|25 - 480|24 - 720|23 - 240|22-\\
& 540|21 + 1380|20 + 750|19 + 60|18 - 210|17 - 1535|16 - 1820|15+\\
& 2250|14 + 1800|13 - 2820|12 + 300|11 + 2040|10 + 340|9 - 1815|8+\\
& 510|7 - 1350|6 + 1350|5 + 274|4 - 548|3 + 120|2)/6!,\\
\hat{\alpha}(7,n) = \quad & (1|128 - 42|96 + 210|80 + 140|72 + 210|68 - 84|66 + 14|65 - 819|64-\\
& 2520|60 + 2730|56 + 840|54 + 840|52 - 420|51 + 2940|50 + 630|48-\\
& 5040|46 + 840|45 - 1260|44 + 1680|43 - 9660|42 + 1260|41+\\
& 3360|40 - 7560|39 + 11130|38 + 5880|37 + 9240|36 + 2982|35-\\
& 6300|34 - 8652|33 - 9905|32 - 8400|31 - 8540|30 + 13860|29+\\
& 14490|28 - 5040|27 + 10500|26 + 10080|25 - 8120|24 - 15050|23-\\
& 5040|22 - 11340|21 + 20580|20 + 15750|19 - 1540|18 - 5810|17-\\
& 16485|16 - 21420|15 + 26250|14 + 21000|13 - 29820|12 + 3500|11+\\
& 17640|10 + 2940|9 - 16016|8 + 4410|7 - 9744|6 + 9744|5 + 1764|4-\\
& 3528|3 + 720|2)/7!.
\end{aligned}
$$

Riviere [**7**] found the formulas for $\hat{\alpha}(m,n)$, $1 \leqslant m \leqslant 3$. Cvetković [**8**] calculated the number $\hat{\alpha}(4,n)$ by using computer, in fact by using the method of exhaustive search. Arocha [**9**] gave explicit formulas for the numbers $\hat{\alpha}(5,n)$ and $\hat{\alpha}(6,n)$. The above formulas, as well as the corresponding formulas for $\hat{\alpha}(m,n)$, $8 \leqslant m \leqslant 10$, together with their values for small n are presented in [**10**]. Using formula (5.1) and data from [**11**] for bipartite graphs the formulas for $\hat{\alpha}(m,n)$ could be generated by computer up to $m = 15$.

We obtained the above formulas using Proposition 5.2, but it is easy to see that the formulas have the form of the formula given in Theorem 3.5, so by changing the meaning of the operation | we can get the corresponding formulas for all other classes given in Section 4. For example, we can get the number of all labeled ordered T_0-$(3,n)$-antichains from $\hat{\alpha}(3,n)$, if we put that $a|b = a \cdot [b]_n$, and we obtain

$$\alpha_{022}(3,n) = ([8]_n - 6[6]_n + 6[5]_n + 3[4]_n - 6[3]_n + 2[2]_n)/3!.$$

We get the number of all unlabeled ordered $(3,n)$-antichains from $\hat{\alpha}(3,n)$, if we put that $a|b = a \cdot C^n_{b+n-1}$, and delete 3! in the denominator; so we obtain that

$$t_{10023}(3,n) = C^n_{n+7} - 6C^n_{n+5} + 6C^n_{n+4} + 3C^n_{n+3} - 6C^n_{n+2} + 2C^n_{n+1}.$$

We get the number of all unlabeled ordered $(3,n)$-T_0-antichains from $\hat{\alpha}(3,n)$, if we put that $a|b = a \cdot C^n_b$, and delete 3! in denominator. So we have that

$$t_{10022}(3,n) = C^n_8 - 6C^n_6 + 6C^n_5 + 3C^n_4 - 6C^n_3 + 2C^n_2.$$

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