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COMMUTATORS ON L^2-SPACES

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Abstract. Given a normal operator N on a Hilbert space and an operator X for which the commutator $K = XN - NX$ belongs to the Hilbert–Schmidt class, we discuss the possibility to represent X as a sum of a Cauchy transform corresponding to K in the spectral representation of N and an operator commuting with N.

This article is devoted to recovering an operator from its commutator with a normal operator on a Hilbert space. By the spectral theorem, every normal operator can be realized as an operator M_z of multiplication by the independent variable z in a certain L^2-space on the complex plane. Given an integral operator K with kernel $k(\xi, z)$, it is easily seen that the integral operator X with kernel $\frac{k(\xi,z)}{\xi-z}$, if it is well defined, satisfies $XM_z - M_zX = K$. However, one should clarify the action of X and find an appropriate formal definition of it. In general, if $K = XM_z - M_zX$ for some bounded operator X, there may exist a family of operators which can be viewed as integral operators with kernel $\frac{k(\xi,z)}{\xi-z}$, while the question about the natural choice of a single operator remains obscure.

An operator commutes with a normal operator if and only if it acts as multiplication by a function in the spectral representation of the normal operator. Hence if the difference of two operators is a multiplication operator, then their commutators with the normal operator coincide. Our general question can be formulated as follows: *Given an operator X such that $XM_z - M_zX = K$ is an integral operator, is it possible to write X as a sum of a Cauchy transform constructed by K and an operator of multiplication*? It is also natural to ask, *what are conditions on an operator K, under which K can be represented as a commutator*?

The same idea can be applied to a pair of normal operators N_1, N_2 acting on Hilbert spaces H_1, H_2, respectively. Suppose that K is a rank-one operator: $K = (\cdot, h_1)h_2$ with $h_1 \in H_1$, $h_2 \in H_2$. Then by the spectral theorem, for $i = 1, 2$, there exist Borel measures μ_i on the complex plane such that the restrictions of the operators N_i to the reducing subspaces generated by h_i in H_i are unitarily

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equivalent to the operators of multiplication by the independent variable in $L^2(\mu_i)$, in which the vectors h_i correspond to the functions identically equal to 1. One can thus think that $H_1 = L^2(\mu_1)$, $H_2 = L^2(\mu_2)$, $K = (\cdot, 1)1$. If the Cauchy transform

$$h \mapsto \int \frac{h(\xi)\, d\mu(\xi)}{\xi - z}, \qquad h \in L^2(\mu_1), \tag{0.1}$$

is a well-defined operator from $L^2(\mu_1)$ to $L^2(\mu_2)$ and we denote it by X, then it obviously satisfies $XM_z - M_zX = K$.

Our work can be naturally divided into three steps (the order of which sometimes does not correspond to the structure of the article). The first step consists of an interpretation of the problem about commutators in Hilbert spaces in terms of integral (in general, singular) operators on L^2-spaces.

The second step is a regularization of singular integral operators and the question about the uniform boundedness of the norms for the family of regularized operators. Since the regularized operators may fail to converge, it is important to know that the limit set is not empty, which is conveniently guaranteed by the uniform boundedness of their norms.

After the talk given by the author at St. Petersburg seminar, in which the results of this article were presented, R. Bessonov drew our attention to paper [**7**], whose results cover a part of our work on this step. In [**4**] we mentioned an important special case of this result without a proof, which remained unpublished during almost ten years. It is contained in our Theorem 1.1.

The third step is the convergence problem for the regularized integral operators. In some cases the convergence may follow directly from the uniform boundedness of norms or the regularized operators. We formulate open questions about conditions on μ, under which, like in the case of measures on the circle or on the line, we always have the convergence in $L^2(\mu)$. Also we discuss various radial functions, which can be used for the regularization, and, by using Wiener's tauberian theorem, we find a condition guaranteeing that the convergence for one of them yields that for each of them.

1. Normal operators

An operator N is said to be normal if it commutes with its adjoint: $N^*N = NN^*$. The generic normal operator of multiplicity 1 is unitarily equivalent to the operator M_z of multiplication by the independent variable on $L^2(\mu)$, where μ is a Borel measure on the complex plane. We consider only bounded operators, which means that μ is compactly supported. Every normal operator is an orthogonal direct sum of normal operators of multiplicity 1.

Let N_1, N_2 be (bounded) normal operators on Hilbert spaces H_1, H_2, respectively. Suppose that an operator $K : H_1 \to H_2$ can be written as a commutator

$$K = XN_1 - N_2X. \tag{1.1}$$

Our goal is to recover the operator X from K. In general, the operator X, if it exists, is not unique. Namely, any bounded operator Y intertwining N_1 and N_2, which means that $YN_1 = N_2Y$, can be added to X. Such intertwining operators can be realized as operators of multiplication by bounded functions with respect to spectral decompositions of the normal operators.

The case of two different normal operators N_1, N_2 can be reduced to the case of a single operator N. Namely, one can set $N = N_1 \oplus N_2$ and take $\left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ K & 0 \end{smallmatrix}\right)$ in place of X and K, respectively.

For a wide class of measures the Cauchy transform is a regular integral operator, and this property can be used for constructing X satisfying (1.1). Assume that for the measure μ_1, the masses of disks of radius r centered at μ_2-almost all points z, do not exceed $p(r)$. If $\int_0^\infty \frac{p(r)}{r^2}\,dr < \infty$, then (0.1) is a regular integral for bounded functions h. Indeed,

$$\int \frac{|h(\xi)|\,d\mu_1(\xi)}{|\xi - z|} \leqslant \|h\|_\infty \cdot \int_0^\infty \frac{p(r)}{r^2}\,dr < \infty.$$

Since the support of μ_1 is compact, one can think that the function p is constant for large argument. Thus, only the behavior of p near zero is of importance. For the planar Lebesgue measure, $p(r)$ behaves as r^2; regular integrals appear also for $p(r) = r^{1+\delta}$ with $\delta > 0$.

Deeper results on the convergence for measures on the unit circle and on the real line, are given by scattering theory for unitary and selfadjoint operators. Namely, if the spectral measure of a unitary operator U_1 is absolutely continuous relative to the Lebesgue measure on the unit circle and $K = U_2X - XU_1$ is of trace class, then the strong limits Y_+, Y_- of the sequence $U_2^n X U_1^{-n}$ exist as $n \to \pm\infty$, and then $Y_\pm U_1 = U_2 Y_\pm$. At the same time,

$$X - U_2^n X U_1^{-n} = \begin{cases} -\sum_{k=1}^n U_2^{k-1} K U_1^{-k}, & n > 0, \\ \sum_{k=1}^{-n} U_2^{-k} K U_1^{k-1}, & n < 0. \end{cases}$$

Given an operator K for which the limit operators exist, each of them can be taken as X.

Below in Theorem 4.1 we adapt the above construction of wave operators to the multidimensional settings. This gives us a sufficient condition on the measure for the convergence under natural restrictions on the commutator. It would be interesting to find a characterization of such measures, see Problem 2.1.

Suppose that the commutator $K = XM_z - M_zX$ is an integral operator on $L^2(\mu)$,

$$(1.2) \qquad (Kh)(z) = \int k(\xi, z)\, h(\xi)\, d\mu(\xi), \quad h \in L^2(\mu).$$

An operator of the form $\sum(\cdot, u_k)v_k$ can be rewritten as an integral operator with kernel $\sum \overline{u_k(\xi)}\, v_k(z)$. Any operator K on $L^2(\mu)$ from the Hilbert–Schmidt class $\mathfrak{S}_2$ can be represented as an integral operator with kernel $k \in L^2(\mu \times \mu)$, that is,

$$\int |k(\xi, z)|^2\, d\mu(\xi)\, d\mu(z) < \infty.$$

This makes natural to work with commutators that belong to the Hilbert–Schmidt class. Actually, we often consider even more narrow classes.

Let K be an integral operator on $L^2(\mu)$ defined by (1.2). The formula for the corresponding operator X may have the form

$$h \mapsto \int \frac{k(\xi, z)}{\xi - z}\, h(\xi)\, d\mu(\xi),$$

if we find a way to define the integrals on the right-hand side. Consider the family of approximating regular integral operators B_ϵ with parameter $\epsilon > 0$, $\epsilon \searrow 0$:
(1.3)

$$(B_\epsilon h)(z) = \int \frac{|\xi - z|^2}{|\xi - z|^2 + \epsilon^2}\, \frac{k(\xi, z)}{\xi - z}\, h(\xi)\, d\mu(\xi) = \int \frac{\bar\xi - \bar z}{|\xi - z|^2 + \epsilon^2}\, k(\xi, z)\, h(\xi)\, d\mu(\xi).$$

If $k \in L^2(\mu\times\mu)$, then the kernels of the integral operators B_ϵ belong to $L^2(\mu\times\mu)$ as well. Hence all operators B_ϵ belong to the Hilbert–Schmidt class; for us it is important that they are well defined.

THEOREM 1.1. *Assume that $k \in L^2(\mu \times \mu)$ and the operators B_ϵ on $L^2(\mu)$ are defined by* (1.3). *Then the norms of the operators B_ϵ are uniformly bounded if and only if K can be written as a commutator $K = XM_z - M_zX$ for some bounded operator X on $L^2(\mu)$.*

Here this result will be obtained as a special case of Theorem 4.2 below.

PROOF OF THE '*only if*' PART. Apply the operators B_ϵ to the function $h \equiv 1$; by the assumption, we obtain a bounded family of vectors. Find a sequence (ϵ_n) such that the vectors $B_{\epsilon_n}1$ have a weak limit. We will show that then the weak limit of $B_{\epsilon_n}h$ exists for every h; the limit operator will be taken as X.

The set of such vectors h is invariant under M_z; this follows from the fact that the commutators of the operators B_ϵ with M_z have the form

$$((B_\epsilon M_z - M_z B_\epsilon)h)(z) = \int \frac{|\xi - z|^2}{|\xi - z|^2 + \epsilon^2}\, k(\xi, z)\, h(\xi)\, d\mu(\xi),$$

which tends to $(Kh)(z)$ as $\epsilon \searrow 0$. This also implies that $XM_z - M_zX = K$.

Similarly, the commutators $B_\epsilon M_{\bar z} - M_{\bar z} B_\epsilon$ with the operator $M_{\bar z}$ of multiplication by $\bar z$ tend to the integral operator with kernel $\frac{\bar\xi - \bar z}{\xi - z} k(\xi, z) \in L^2(\mu \times \mu)$, whence the set where the limit exists is also invariant under $M_{\bar z}$. By the assumption, the norms of the operators B_ϵ are uniformly bounded, hence the limit exists for vectors h from a closed subspace. The minimal closed subspace of $L^2(\mu)$ that contains the constant function and is invariant under M_z and $M_{\bar z}$ is the whole space $L^2(\mu)$. □

The '*if*' part will be established in the proof of Theorem 4.2.

From the proof one can see that the resulting operator X is determined by the limit of a convergent sequence $B_{\epsilon_n}h$ with $h \equiv 1$ and satisfies $XM_z - M_zX = K$. The uniform boundedness of the norms of B_ϵ guarantees that the limit set is not empty. If it consists of a single operator, then the whole family of operators B_ϵ weakly tends to a limit as $\epsilon \searrow 0$, which can naturally be viewed as the Cauchy transform corresponding to the commutator. In this case any operator with the given commutator can be represented as a sum of the Cauchy transform and an operator of multiplication by a bounded function. In the case where the limit set contains more than one element, the differences between any two limit operators commute with M_z, and thus they are operators of multiplication.

From our previous results [**6**], it follows that in general the operators B_ϵ may fail to have a weak limit. We discuss this below in Section 2. However, in some special cases the limit does exist.

COROLLARY 1.1. *Let μ_1, μ_2 be mutually singular Borel measures with compact supports on the complex plane. Take a bounded operator $X : L^2(\mu_1) \to L^2(\mu_2)$. Assume that the commutator $XM_z - M_zX$ is a Hilbert–Schmidt intergal operator with kernel k. Then the operators*

$$h \mapsto \int \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2 + \epsilon^2} k(\xi, z)\, h(\xi)\, d\mu_1(\xi) = \int \frac{|\xi - z|^2}{|\xi - z|^2 + \epsilon^2} \frac{k(\xi, z)}{\xi - z} h(\xi)\, d\mu_1(\xi)$$

viewed as operators from $L^2(\mu_1)$ to $L^2(\mu_2)$, tend to X in the weak operator topology as $\epsilon \searrow 0$.

PROOF. To fulfil the assumptions of the Theorem, set $\mu = \mu_1 + \mu_2$ and adapt the case of two mutually singular measures to the case of a single measure as was described above. If the operators B_ϵ do not tend to X, a sequence of ϵ_n in the proof of the theorem can be chosen so that the limit operator will not coincide with X. The difference between it and X is a nonzero operator commuting with M_z; this contradicts the assumption that μ_1 and μ_2 are mutually singular. □

2. The planar analogs of results for the line

Consider special cases of approximation based on formula (1.3). If μ lies on the real line, the numbers ξ and z are real, and the operator M_z is selfadjoint. We have

$$\begin{aligned}\frac{\bar{\xi} - \bar{z}}{|\xi - z|^2 + \epsilon^2} &= \frac{1}{2}\left(\frac{1}{\xi - z + i\epsilon} + \frac{1}{\xi - z - i\epsilon}\right) \\ &= -\frac{1}{2}\int_0^\infty \left(\frac{e^{-it(\xi - z)} - 1}{\xi - z} + \frac{e^{it(\xi - z)} - 1}{\xi - z}\right) \epsilon e^{-\epsilon t}\, dt.\end{aligned}$$

If $XM_z - M_zX$ is the integral operator with kernel $k(\xi, z)$, then

$$\begin{aligned}\exp(itM_z)X\exp(-itM_z) - X &= \int_0^t \frac{d\left(\exp(isM_z)X\exp(-isM_z)\right)}{ds}\, ds \\ &= -i\int_0^t \exp(isM_z)(XM_z - M_zX)\exp(-isM_z)\, ds\end{aligned} \tag{2.1}$$

is the integral operator with kernel

$$-\,i\int_0^t \exp(is(z - \xi))\, ds \cdot k(\xi, z) = \frac{\exp(-it(\xi - z)) - 1}{\xi - z} k(\xi, z). \tag{2.2}$$

Similarly, the operator $\exp(itM_z)X\exp(-itM_z) - X$ is the integral operator with kernel

$$\frac{\exp(it(\xi - z)) - 1}{\xi - z} k(\xi, z).$$

Therefore,

$$B_\epsilon = X - \frac{1}{2}\int_0^\infty \big(\exp(itM_z)X\exp(-itM_z) + \exp(-itM_z)X\exp(itM_z)\big)\,\epsilon e^{-\epsilon t}\,dt.$$

If μ is absolutely continuous relative to the Lebesgue measure, then from scattering theory we know that the family of operators $\exp(itM_z)X\exp(-itM_z)$ has strong limits as $t \to \pm\infty$. Then the limits of the summands in the brackets are the past and future wave operators; the averaged limits exist as well, and the integral on the right-hand side tends to the half-sum of the past and future wave operators.

If μ is a singular continuous measure, the averaged limit may fail to exist if the commutator has rank two [**6**]; in the case of rank-one commutators, the operators B_ϵ do have a weak limit, cf. [**4**].

For the case where μ is a measure on the unit circle, we have $|\xi| = |z| = 1$ and the operator M_z is unitary. We have

$$\frac{\bar{\xi} - \bar{z}}{|\xi - z|^2 + \epsilon^2} = \frac{r}{1+r}\left(\frac{1}{r\xi - z} + \frac{1}{\xi - rz}\right),$$

where r is a solution to the equation $r^2 - (2+\epsilon^2)r + 1 = 0$; r tends to 1 as $\epsilon \searrow 0$. The construction, similarly to that for the real line, admits an analogous interpretation via the half-sum of past and future wave operators, which are the limits of the sequence $M_z^n X M_z^{-n}$ as $n \to \pm\infty$.

Now we return to the general case of compactly supported measures on the complex plane. Recall that a rank-one operator $K = (\cdot, u)v$ on $L^2(\mu)$ is the integral operator with kernel $k(\xi, z) = \overline{u(\xi)}\,v(z)$.

Conjecture 2.1 (cf. the Conjecture in [**4**]). *Let X be an operator on $L^2(\mu)$ whose commutator with M_z is a rank-one integral operator with kernel $k(\xi, z) = \overline{u(\xi)}\,v(z)$. Then the operators B_ϵ have a limit in the weak operator topology as $\epsilon \searrow 0$.*

This conjecture is true if μ is a Borel measure on the unit circle; this follows from the existence of the limits almost everywhere established in [**4**], or this can be proved by a reduction to the case of mutually singular measures as in Corollary 1.1. As we said above, in general the result does not hold for commutators of higher rank; however, like in the cases of measures on a line or on a circle, this essentially depends on μ.

Problem 2.1. *Characterize the class of compactly supported Borel measures μ on the complex plane, for which the uniform boundedness of the operators B_ϵ on $L^2(\mu)$ implies the weak convergence of them whenever the commutator K belongs to the trace class.*

The following sufficient condition will be established in Theorem 4.1: for almost all unimodular complex numbers α with respect to the Lebesgue measure on the unit circle, the projections of μ to the lines $\{t\alpha : t \in \mathbb{R}\}$ are absolutely continuous with respect to the Lebesgue measure on the lines.

Now consider another property connected with commutators on L^2-spaces. For a Borel measure μ, the mapping from the trace class of operators on $L^2(\mu)$ to the space $L^1(\mu)$ defined by

$$\sum(\cdot, u_k)v_k \mapsto \sum \bar{u}_k v_k \quad \Big(u_k, v_k \in L^2(\mu),\ \sum_k \|u_k\|\,\|v_k\| < \infty\Big) \tag{2.3}$$

is well defined. This is a consequence of the fact that for a trace class operator $K = \sum(\cdot, u_k)v_k$, the traces of the operators $M_f K = \sum(\cdot, u_k) f v_k$, $f \in L^\infty(\mu)$, where M_f is the operator of multiplication by f, are equal to the integrals

$$\sum \int f v_k\, \bar{u}_k\, d\mu = \int f \cdot \Big(\sum \bar{u}_k v_k\Big)\, d\mu$$

and thus uniquely determine the function $\sum \bar{u}_k v_k$. Obviously, this mapping is continuous and its norm does not exceed 1.

If μ is a Borel measure on the unit circle which is singular relative to the Lebesgue measure, then the fact that a trace class operator $K = \sum(\cdot, u_k)v_k$ can be represented as a commutator $K = XM_z - M_z X$ for some bounded operator X on $L^2(\mu)$ yields [**5**]

$$\sum_k \bar{u}_k v_k = 0 \qquad \mu - \text{a.e.};$$

that is, K belongs to the kernel of the mapping (2.3).

Problem 2.2. *Find a description of Borel measures μ with compact support on the complex plane, for which all commutators with the operator M_z belong to the kernel of the mapping* (2.3).

For discrete measures Problems 2.1 and 2.2 are trivial, because the property from Problem 2.2 is obviously fulfilled, and this easily implies the convergence as in Problem 2.1. For measures on the real line and on the unit circle, the classes of measures from Problems 2.1 and 2.2 consist of all measures that have zero singular continuous part and of all singular measures, respectively. Thus, for measures that have no discrete part, these classes are complementary: every measure having no point masses can be uniquely represented as a sum of two measures that possess these two properties.

It is natural to ask if the analogy with measures on the line or on the circle holds in the following questions about planar measures.

– *Can every Borel measure with compact support on the complex plane be represented as a sum of two measures satisfying the properties described in Problems* 2.1 *and* 2.2, *respectively? For measures that have no point masses, is this representation unique?*

– *If there exists a trace class operator $K = XM_z - M_z X = \sum(\cdot, u_k)v_k$ on $L^2(\mu)$, for which $\sum \bar{u}_k v_k \neq 0$ μ-almost everywhere, is it possible to find a rank-one operator $K = \tilde{X}M_z - M_z\tilde{X} = (\cdot, u)v$ on $L^2(\mu)$ with $\bar{u}v \neq 0$ μ-almost everywhere?*

– *Given a measure μ concentrated on a set of zero linear Hausdorff measure $\mathcal{H}_1$, does there exist an operator X on $L^2(\mu)$ whose commutator K with M_z* (*written as* *an integral operator with kernel* $k(\xi, z)$) *is small* (*of rank two, of trace class*), *and for which the operators* B_ϵ *fail to weakly converge?*

3. The mutidimensional case

A normal operator can be viewed as a pair of commuting selfadjoint operators: $N = A_1 + iA_2$; this corresponds to the natural identification of $\mathbb{C}$ with $\mathbb{R}^2$. Similarly, one can consider n-tuples $(A_1, \dots, A_n)$, and then our model will be determined by a Borel measure on $\mathbb{R}^n$ in place of $\mathbb{C}$, cf. [**2**]. In the scalar case A_j is the operator of multiplication by the j-th coordinate function on $L^2(\mu)$.

In the general case of a family of an arbitrary n-tuple of commuting selfadjoint operators one should consider the space of vector-valued functions

$$H = \int \oplus H(s)\, d\mu(s),$$

where μ is a Borel measure on $\mathbb{R}^n$, and $H(s)$ are auxiliary Hilbert spaces defined for μ-almost all s. The spectral measure of the family $A_1, \dots, A_n$ is the projection-valued function defined on measurable subsets of $\mathbb{R}^n$ taking a subset of $\mathbb{R}^n$ to the operator of multiplication by its indicator. The fact that an integral operator on H belongs to the Hilbert–Schmidt class means that its kernel is an operator-valued function, whose values are Hilbert–Schmidt operators, the Hilbert-Schmidt norms of which are square-summable with respect to $\mu \times \mu$. Analogs of our results hold in the general case, the generalizations are straightforward, but the formulas become much more complicated; that is why we consider only the scalar case where $H = L^2(\mu)$.

Lemma 3.1. *Let* μ *be a Borel measure in* $\mathbb{R}^n$ *with compact support. Assume that* X *is an operator on* $L^2(\mu)$, *whose commutators* $XM_{x_j} - M_{x_j}X$ *with the operators* M_{x_j} *of multiplication by the coordinate functions* x_j ($j = 1, \dots, n$) *belong to the Hilbert–Schmidt class. Then there exists a function* l *of two variables* $x, y \in \mathbb{R}^n$ *such that*

$$\iint |l(x,y)|^2 \cdot |x-y|^2 \, d\mu(x)\, d\mu(y) < \infty \tag{3.1}$$

and the commutators $XM_{x_j} - M_{x_j}X$ *are the integral operators with kernels of the form* $(x_j - y_j)l(x, y)$.

Thus, l is an analog of the kernel of X as if it were an integral operator, that is, the analog of the function $\frac{k(\xi,z)}{\xi - z}$ for normal operators. If X is a regular integral operator with kernel l, the formulas for commutators become trivial. However, they hold if this is not the case.

The Lemma can be compared with the fact for normal operators that if $k(\xi, z)$ is the kernel of $XM_z - M_zX$ written as the integral operator, then $XM_{\bar z} - M_{\bar z}X$ is the integral operator with kernel $\frac{\bar\xi - \bar z}{\xi - z} k(\xi, z)$:

$$\big((XM_{\bar z} - M_{\bar z}X)h\big)(z) = \int \frac{\bar\xi - \bar z}{\xi - z}\, k(\xi, z)\, h(\xi)\, d\mu(\xi).$$

According to the theory of double operator integrals, to get the operator $XM_f - M_f X$ from $XM_z - M_z X$, the kernel of the integral operator should be multiplied by $\frac{f(\xi)-f(z)}{\xi - z}$, see [**3**]. We mentioned the special case of this general fact for $f(z) = \bar{z}$.

Proof. The operations of taking the commutators with M_{x_j} for all possible j form a commutative family:

$$(XM_{x_j} - M_{x_j}X)M_{x_{j'}} - M_{x_{j'}}(XM_{x_j} - M_{x_j}X) \\ = (XM_{x_{j'}} - M_{x_{j'}}X)M_{x_j} - M_{x_j}(XM_{x_{j'}} - M_{x_{j'}}X).$$

Therefore, if $XM_{x_j} - M_{x_j}X$, $XM_{x_{j'}} - M_{x_{j'}}X$ are integral operators with kernels $k_j, k_{j'}$, then $k_j(x,y)\,(x_{j'} - y_{j'}) = k_{j'}(x,y)\,(x_j - y_j)$ for $\mu\times\mu$-almost all pairs $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then l can be defined by

$$l(x,y) = \frac{k_j(x,y)}{x_j - y_j},$$

which does not depend on the choice of j whenever $x_j \neq y_j$. If $x = y$, then, obviously, the kernels of all commutators vanish at (x,x) whenever μ has a point mass at x, and then one can set $l(x,x) = 0$.

We have

$$\iint |l(x,y)|^2 \cdot |x-y|^2\, d\mu(x)\, d\mu(y) = \iint |l(x,y)|^2 \cdot \sum_j |x_j - y_j|^2\, d\mu(x)\, d\mu(y) \\ = \sum_j \iint |k_j(x,y)|^2\, d\mu(x)\, d\mu(y) < \infty,$$

and (3.1) follows. □

4. The approximation and the uniform boundedness

Take a family $A_1, \dots, A_n$ of pairwise commuting bounded selfadjoint operators on a Hilbert space, and let X be a bounded operator on the same space. For $s = (s_j) \in \mathbb{R}^n$ define

$$A_{[s]} = \sum_j s_j A_j \quad \text{and} \quad X_s = \exp(i\, A_{[s]})\, X\, \exp(-i\, A_{[s]}).$$

Define a mapping Ξ from $L^1(\mathbb{R}^n)$ to the space of all bounded linear operators on the Hilbert space under consideration; namely, for $\gamma \in L^1(\mathbb{R}^n)$ set

$$\Xi(\gamma) = \int X_s\, \gamma(s)\, d\lambda_n(s),$$

where λ_n is the Lebesgue measure on $\mathbb{R}^n$. The norm of the mapping Ξ does not exceed 1, because the operators $\exp(i\, A_{[s]})$ are unitary, whence $\|X_s\| = \|X\|$. Fix a function $\gamma \in L^1(\mathbb{R}^n)$; often we assume that $\int_{\mathbb{R}^n} \gamma\, d\lambda_n = 1$. For $\epsilon > 0$ define the functions γ_ϵ by

$$\gamma_\epsilon(s) = \frac{1}{\epsilon^n}\, \gamma\left(\frac{s}{\epsilon}\right);$$

obviously, $\|\gamma_\epsilon\|_{L^1} = \|\gamma\|_{L^1}$. We are interested in the behavior of the operators $\Xi(\gamma_\epsilon)$ as $\epsilon \searrow 0$. Their norms are uniformly bounded by $\|\gamma\|_{L^1}$.

A sufficient condition for the convergence can be obtained as a consequence of the classical result of scattering theory. For a measure μ on $\mathbb{R}^n$ and for a point ω on the unit sphere $\mathbb{S}_{n-1}$ of $\mathbb{R}^n$, consider the projection of μ to the line $\{t\omega : t \in \mathbb{R}\}$. Equivalently, one can consider the measure on $\mathbb{R}$ whose mass on a measurable subset $e \subset \mathbb{R}$ is the μ-measure of the set $\{x \in \mathbb{R}^n : \langle x, \omega\rangle \in e\}$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$. If for $s \in \mathbb{R}^n$ the commutator $XA_{[s]} - A_{[s]}X$ belongs to the trace class and the projection of μ to the line $\{t\omega : t \in \mathbb{R}\}$ is absolutely continuous with respect to the Lebesgue measure on the line, the strong limit of the operators $X_{s/\epsilon}$ exists as $\epsilon \searrow 0$. By integrating over $\mathbb{S}_{n-1}$ with respect to the $(n-1)$-dimensional Lebesgue measure σ on it, we obtain the following result.

THEOREM 4.1. *Assume that for the spectral measure of a family $A_1, \ldots, A_n$ of commuting selfadjoint operators, the projections to σ-almost all lines $\{t\omega : t \in \mathbb{R}\}$ are absolutely continuous with respect to the Lebesgue measure on the line, and X is an operator on $L^2(\mu)$ whose commutators with the operators M_{x_j}, $j = 1, \ldots, n$, belong to the trace class. Then the operators $\Xi(\gamma_\epsilon)$ have a strong limit as $\epsilon \searrow 0$.*

Notice that if μ has a nonzero mass on a set of zero linear Hausdorff measure $\mathcal{H}_1$, then all projections of μ to lines have masses on sets of zero Lebesgue measure.

Now we show that the operators $X - \Xi(\gamma_\epsilon)$ generalize the operators B_ϵ defined above. We will work on the space $L^2(\mu)$, for which we have

$$X_s = \exp(i\, M_{\langle x,s\rangle})X \,\exp(-i\, M_{\langle x,s\rangle}),$$

where $M_{\langle x,s\rangle}$ is the operator of multiplication by $\langle x, s\rangle$. The general case can be obtained by considering spaces of vector-valued functions in place of the scalar ones.

Take a function $\gamma \in L^1(\mathbb{R}^n)$, for which $\int \gamma \, d\lambda_n = 1$ and

$$\int |x|\, |\gamma(x)| \, d\lambda_n(x) < \infty. \tag{4.1}$$

Define $\Omega = 1 - \hat{\gamma}$, where $\hat{\gamma}$ is the Fourier transform of γ:

$$\hat{\gamma}(x) = \int \exp(-i\,\langle x, y\rangle)\, \gamma(y)\, d\lambda_n(y).$$

Take a function l of two variables $x, y \in \mathbb{R}^n$ satisfying (3.1), and consider the family of regular integral operators B_ϵ with parameter $\epsilon > 0$, $\epsilon \searrow 0$:

$$(B_\epsilon h)(z) = \int \Omega\left(\frac{x-y}{\epsilon}\right) l(x,y)\, h(\xi)\, d\mu(\xi). \tag{4.2}$$

Since $\gamma \in L^1(\mathbb{R}^n)$, we have $\Omega \to 1$ at infinity, and condition (4.1) implies that $\Omega(s) = O(s)$ as $s \to 0$. Therefore, the operators B_ϵ belong to the Hilbert–Schmidt class for every $\epsilon > 0$.

If l is as in Lemma 3.1 and γ satisfies (4.1), then $\Xi\gamma$ is the operator

$$h \mapsto \hat{\gamma}(0) \cdot (Xh)(x) + \int (\hat{\gamma}(x-y) - \hat{\gamma}(0))\, l(x,y)\, h(y)\, d\mu(y). \tag{4.3}$$

To verify this formula, it suffices to prove it for the Dirac measure at an arbitrary point $s \in \mathbb{R}^n$. Similarly to (2.1), (2.2), $X_s - X$ is the integral operator with kernel $(\exp(i\langle x-y, s\rangle) - 1) \cdot l(x,y)$.

Suppose that $\hat{\gamma}(0) = \int \gamma \, d\lambda_n = 1$ and $\Omega = 1 - \hat{\gamma}$. Then operator (4.3) can be rewritten as

$$h \mapsto (Xh)(x) - \int \Omega(x-y)\, l(x,y)\, h(y)\, d\mu(y),$$

which is the operator $X - B_\epsilon$ with $\epsilon = 1$. For an arbitrary ϵ, it is not difficult to check the relation $B_\epsilon = X - \int X_s\, \gamma(\epsilon s)\, d\lambda_n(s)$, whence

$$\|B_\epsilon\| \leqslant (1 + \|\gamma\|_1) \cdot \|X\|. \tag{4.4}$$

Now consider the modified Bessel function K_0 defined by the formula

$$K_0(r) = \int_1^\infty \frac{e^{-rt}}{\sqrt{t^2-1}}\, dt$$

(see [**1**], section 7.3.4, formula (15) with $\nu = 0$); clearly, $K_0(r) > 0$ for any $r \geqslant 0$. For $\gamma(\omega) = K_0(|\omega|)$, $\omega \in \mathbb{C}$, we have

$$\hat{\gamma}(\zeta) = \int \exp(-i\,\langle \zeta, \omega\rangle)\, K_0(|\omega|)\, d\lambda(\omega) = \frac{1}{|\zeta|^2 + 1}, \tag{4.5}$$

where λ is the planar Lebesgue measure on $\mathbb{C}$ and $\langle\cdot,\cdot\rangle$ is the real scalar product of two complex numbers: $\langle \zeta, \omega\rangle = \operatorname{Re}(\zeta\bar{\omega})$. We omit the proof of relation (4.5). Therefore,

$$\Omega(\zeta) = 1 - \hat{\gamma}(\zeta) = \frac{|\zeta|^2}{1 + |\zeta|^2}, \tag{4.6}$$

$$\Omega\Big(\frac{\xi - z}{\epsilon}\Big) = \frac{|\xi - z|^2}{|\xi - z|^2 + \epsilon^2}, \tag{4.7}$$

which gives us the special case of the operators B_ϵ considered in Theorem 1.1.

Theorem 4.2. *Assume that l is a function of $x, y \in \mathbb{R}^n$ satisfying* (3.1), *and the operators B_ϵ are defined by* (4.2), *where $\Omega = 1 - \hat{\gamma}$, $\gamma \in L^1(\mathbb{R}^n)$, $\int \gamma\, d\lambda_n = 1$. Then the following are equivalent:*

1) *the norms of the operators B_ϵ are bounded uniformly in ϵ;*
2) *the norms of the operators B_ϵ fail to tend to infinity as $\epsilon \searrow 0$;*
3) *there exists a bounded operator X on $L^2(\mu)$ as in Lemma* 3.1.

As we said above, Theorem 4.2 is essentially contained in [**7**]. Since the operator B_ϵ defined by (1.3) is a special case of (4.2), Theorem 1.1 is a special case of Theorem 4.2.

Proof. The implication 1) $\implies$ 2) is trivial. The proof of the implication 2) $\implies$ 3) is the same as the proof of the '*only if*' part of Theorem 1.1. The implication 3) $\implies$ 1) follows directly from estimate (4.4). ☐

5. Radial regularizations

An important class of regularizations is that based on radial functions γ. A function is called radial if it is constant on every sphere centered at the origin. Suppose that $\gamma \in L^1(\mathbb{R}^n)$ is a radial function. This is equivalent to the fact that γ can be written in the form

$$\gamma(x) = \frac{f(\log|x|)}{|x|^n}, \qquad x \in \mathbb{R}^n,$$

for some $f \in L^1(\mathbb{R})$. Indeed,

$$\begin{aligned}\|\gamma\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \frac{|f(\log|x|)|}{|x|^n}\, d\lambda_n(x) \\ &= A \cdot \int_0^\infty |f(\log r)|\, \frac{dr}{r} = A \cdot \int_{-\infty}^\infty |f(t)|\, dt = A \cdot \|f\|_{L^1(\mathbb{R})},\end{aligned}$$

where A is the full mass of the $(n-1)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^n$.

For the function γ_ϵ, $\gamma_\epsilon(x) = \epsilon^{-n}\,\gamma(x/\epsilon)$, we have

$$\gamma_\epsilon(x) = \epsilon^{-n} \cdot \frac{f(\log|x|/\epsilon)}{|x/\epsilon|^n} = \frac{f(\log|x| + \log 1/\epsilon)}{|x|^n},$$

which corresponds to the shifted function f, namely, to $f(t+\log\frac{1}{\epsilon})$ in place of $f(t)$.

We study the convergence of the operators $\Xi(\gamma_\epsilon)$ as $\epsilon \searrow 0$, which for radial functions γ can be rewritten in terms of the corresponding functions $f \in L^1(\mathbb{R})$. Suppose that we have the convergence for some function f. Then the convergence holds for all shifts of f, and hence also for all functions from the closed subspace of $L^1(\mathbb{R})$ generated by the shifts of f. The condition on f to generate the whole space $L^1(\mathbb{R})$ is given by Wiener's tauberian theorem [**8**], namely, this is equivalent to the fact that the Fourier transform $\mathcal{F}f$ of f,

$$(\mathcal{F}f)(a) = \int_{\mathbb{R}} e^{iat} f(t)\, dt,$$

does not vanish on the real line. In this case the convergence for the function γ corresponding to f yields that for any radial function $\gamma \in L^1(\mathbb{R}^n)$.

If condition (4.1) is fulfilled, we obtain the construction of regularized integral operators. Given a radial function Ω on $\mathbb{R}^n$ that tends to 1 at infinity, we should first find the radial function γ whose Fourier transform is $1-\Omega$, then we construct the function f, and, finally, we find its Fourier transform on the real line. It would be useful to find a direct formula for $\mathcal{F}f$ and to rewrite the property $(\mathcal{F}f)(a) \neq 0$ in terms of Ω.

Now we return to the case of normal operators. The approximating operators B_ϵ are the integral operators on $L^2(\mu)$ of the form

$$(B_\epsilon h)(z) = \int \Omega\left(\frac{\xi - z}{\epsilon}\right) \frac{k(\xi, z)}{\xi - z}\, h(\xi)\, d\mu(\xi),$$

where μ is a Borel measure on the complex plane, $\Omega = 1 - \hat{\gamma}$ for some radial function $\gamma \in L^1(\mathbb{C})$ with $\int \gamma \, d\lambda = 1$. To get rid of problems with the definition of the operators B_ϵ, we also require the property

$$\int |\omega| \, |\gamma(\omega)| \, d\lambda(x) < \infty, \tag{5.1}$$

which is the analog of (4.1). For the special case $\gamma(\omega) = K_0(|\omega|)$ we obtain the settings of Theorem 1.1, see formulas (4.5)–(4.7). It turns out that the convergence for this special case yields the convergence for any Ω with the described properties.

PROPOSITION 5.1. *If the operators B_ϵ with Ω defined by* (4.6) *converge, then the convergence holds for the operators B_ϵ with an arbitrary function $\Omega = 1-\hat{\gamma}$, where γ is a radial function on the complex plane with $\int \gamma \, d\lambda = 1$, for which condition* (5.1) *is fulfilled.*

SKETCH OF THE PROOF. For Ω defined by (4.6) we have

$$f(\log r) = r^2 \cdot K_0(r), \qquad K_0(r) = \int_1^\infty \frac{e^{-rt}}{\sqrt{t^2-1}} \, dt.$$

One can prove the formula for the Fourier transform of f:

$$(\mathcal{F}f)(a) = 2^{ia} \cdot \Gamma(1 + ia/2)^2.$$

It shows that $\mathcal{F}f$ does not vanish on the real line, as required. □

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