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HÖLDER'S REVERSE INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. We establish a new reverse Hölder integral inequality and its discrete version. As applications, we prove Radon's, Jensen's reverse and weighted power mean inequalities and their discrete versions.

1. Introduction

The well-known classical Hölder inequality can be stated as follows.

THEOREM 1.1. *Let a_i and b_i $(i = 1, \dots, n)$ be positive real sequences. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\Big(\sum a_i^p\Big)^{1/p}\Big(\sum b_i^q\Big)^{1/q} \geqslant \sum a_i b_i. \tag{1.1}$$

Here and in what follows $\sum$ means $\sum_{i=1}^n$.

The integral version of (1.1) is the following.

THEOREM 1.2. *Let $f(x)$ and $g(x)$ be positive continuous functions on $[a, b]$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\Big(\int_a^b f^p(x)\,dx\Big)^{1/p}\Big(\int_a^b g^q(x)\,dx\Big)^{1/q} \geqslant \int_a^b f(x)\,g(x)\,dx. \tag{1.2}$$

Hölder's inequality plays an important role in different branches of modern mathematics such as classical real and complex analysis, numerical analysis, probability and statistics, differential equations and et al. In recent years some authors [**1**,**2**,**6**,**7**,**9**,**10**,**16**–**19**] have given considerable attention to Hölder's inequality

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together by its integral version and various generalizations. Some reverse versions were recent established [**4**, **13**, **15**, **18**].

We establish a new reverse Hölder integral inequality and with its discrete form. As applications, we prove Radon's, Jensen's reverse and weighted power mean inequalities.

2. Hölder's reverse inequalities

LEMMA 2.1. *If a, b are positive real numbers and $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$, then* (*see* [**14**])

$$S\Big(\frac{a}{b}\Big)a^{1/p}b^{1/q} \geqslant \frac{a}{p}+\frac{b}{q}, \tag{2.1}$$

where

$$S(h)=\frac{h^{1/(h-1)}}{e\log h^{1/(h-1)}}, \quad h\neq 1.$$

THEOREM 2.1 (Hölder's reverse inequality). *Let $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. If $f(x)$ and $g(x)$ are non-negative continuous functions and $f^{1/p}(x)\,g^{1/q}(x)$ is integrable on $[a,b]$, then*

$$\Big(\int_a^b f(x)^p dx\Big)^{1/p}\Big(\int_a^b g(x)^q dx\Big)^{1/q} \leqslant \int_a^b S\Big(\frac{Yf^p(x)}{Xg^q(x)}\Big)\cdot f(x)g(x)\,dx, \tag{2.2}$$

where

$$X=\int_a^b f^p(x)\,dx, \quad Y=\int_a^b q^q(x)\,dx,$$

and $S(h)$ is as in Lemma 2.1.

PROOF. Let

$$a=\frac{f^p(x)}{X}, \quad b=\frac{g^q(x)}{Y}.$$

By using Lemma 2.1, we have

$$S\Big(\frac{Yf^p(x)}{Xg^q(x)}\Big)\cdot\frac{f(x)g(x)}{X^{1/p}Y^{1/q}} \geqslant \frac{1}{p}\frac{f^p(x)}{X}+\frac{1}{q}\frac{g^p(x)}{Y}.$$

Therefore

$$\frac{\int_a^b S\big(\frac{Yf^p(x)}{Xg^q(x)}\big)f(x)g(x)dx}{X^{1/p}Y^{1/q}} \geqslant \frac{1}{p}\frac{\|f(x)\|_p^p}{X}+\frac{1}{q}\frac{\|g(x)\|_p^p}{Y}=1.$$

This proof is completed. □

REMARK 2.1. Obviously, inequality (2.2) is just an inverse of inequality (1.2). Moreover, let $f(x)$ and $g(x)$ reduce to positive real sequences a_i and b_i ($i=1,\dots,n$), respectively and with appropriate changes in the proof of (2.1), we have

$$\Big(\sum a_i^p\Big)^{1/p}\Big(\sum b_i^q\Big)^{1/q} \leqslant \sum S\Big(\frac{Y'a_i^p}{X'b_i^q}\Big)a_ib_i,$$

where $X'=\sum a_i^p$, $Y'=\sum b_i^q$.

This is just an inverse of the well-known Hölder's inequality (1.1).

3. Applications of Hölder's reverse inequality

THEOREM 3.1 (Radon's reverse integral inequality). *Let $f(x)$ and $g(x)$ be positive and continuous functions. If $m > 0$, then*

$$\int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx \leqslant \frac{\left(\int_a^b S\left(\frac{\tilde{Y} f^{m+1}(x)}{\tilde{X} g^{m+1}(x)}\right) f(x)\,dx\right)^{m+1}}{\left(\int_a^b g(x)\,dx\right)^m}, \tag{3.1}$$

where

$$\tilde{X} = \int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx, \quad \tilde{Y} = \int_a^b g(x)\,dx.$$

PROOF. Let $p = m+1, q = (m+1)/m$ and replacing $f(x)$ and $g(x)$ by $u(x)$ and $v(x)$ in (2.2), respectively, we have

$$\left(\int_a^b u(x)^{m+1}\,dx\right)^{1/(m+1)} \left(\int_a^b v(x)^{(m+1)/m} dx\right)^{m/(m+1)} \leqslant \int_a^b S\left(\frac{\hat{Y} u^{m+1}(x)}{\hat{X} v^{(m+1)/m}(x)}\right) u(x) v(x)\,dx, \tag{3.2}$$

where $\hat{X} = \int_a^b u^{m+1}(x)\,dx$, $\hat{Y} = \int_a^b v^{(m+1)/m}(x)\,dx$. Taking $u(x) = \left(\frac{f(x)}{g(x)}\right)^{1/(m+1)}$, $v(x) = f^{m/(m+1)}(x)\, g^{1/(m+1)}(x)$ in (3.2), we obtain

$$\int_a^b S\left(\frac{\bar{Y}}{\bar{X} g^{(m+1)/m}(x)}\right) f(x)\,dx \geqslant \left(\int_a^b \frac{f(x)}{g(x)}\,dx\right)^{1/(m+1)} \left(\int_a^b f(x)\, g^{1/m}(x)\,dx\right)^{m/(m+1)},$$

where

$$\bar{X} = \int_a^b \frac{f(x)}{g(x)}\,dx, \quad \bar{Y} = \int_a^b f(x)\, g^{1/m}(x)\,dx.$$

Hence

$$\int_a^b \frac{f(x)}{g(x)}\,dx \leqslant \frac{\left(\int_a^b S\left(\frac{\bar{Y}}{\bar{X} g^{(m+1)/m}(x)}\right) f(x) dx\right)^{m+1}}{\left(\int_a^b f(x) g^{1/m}(x)\,dx\right)^m}.$$

Replacing $f(x)$ and $g(x)$ by $u(x)$ and $v(x)$, respectively, and leting $u(x) = f(x)$ and $v(x) = \left(\frac{g(x)}{f(x)}\right)^m$, we get

$$\int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx \leqslant \frac{\left(\int_a^b S\left(\frac{\tilde{Y} f^{m+1}(x)}{\tilde{X} g^{m+1}(x)}\right) f(x)\,dx\right)^{m+1}}{\left(\int_a^b g(x)\,dx\right)^m},$$

where

$$\tilde{X} = \int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx, \quad \tilde{Y} = \int_a^b g(x)\,dx. \qquad \square$$

REMARK 3.1. Let $f(x)$ and $g(x)$ reduce to positive real sequences a_i and b_i $(i = 1, \dots, n)$, respectively and with appropriate changes in the proof of (3.1), we have

$$\sum \frac{a_i^{m+1}}{b_i^m} \leqslant \frac{\left(\sum S\left(\frac{\tilde{Y}' a_i^{m+1}}{\tilde{X}' b_i^{m+1}}\right) a_i\right)^{m+1}}{\left(\sum b_i\right)^m},$$

where $\tilde{X}' = \sum a_i^{m+1}/b_i^m$, and $\tilde{Y}' = \sum b_i$.

This is just an inverse of following well-known the Radon inequality [**5**]

$$\sum \frac{a_i^{m+1}}{b_i^m} \geqslant \frac{\left(\sum a_i\right)^{m+1}}{\left(\sum b_i\right)^m}.$$

THEOREM 3.2 (Jensen's reverse integral inequality). *Let $f(x)$ and $p(x)$ be positive continuous functions and $\int_a^b p(x)\,dx = 1$. If $0 < s < t$, then*

$$\left(\int_a^b S\left(\frac{f^t(x)}{\check{X}}\right) f^s(x) p(x)\,dx\right)^{1/s} \geqslant \left(\int_a^b p(x) f^t(x)\,dx\right)^{1/t}, \tag{3.3}$$

where

$$\check{X} = \int_a^b p(x) f^t(x) t\,dx.$$

PROOF. From the hypotheses, we have

$$\begin{aligned} \int_a^b S\left(\frac{f^t(x)}{\check{X}}\right) f^s(x) p(x)\,dx \\ = \int_a^b S\left(\frac{\check{Y}[p^{s/t}(x) f^s(x)]^{t/s}}{\check{X}[p^{1-s/t}(x)]^{t/(t-s)}}\right) p^{s/t}(x) f^s(x) \cdot p^{1-s/t}(x)\,dx, \end{aligned} \tag{3.4}$$

where $\check{X} = \int_a^b p(x) f^t(x)\,dx$ and $\check{Y} = \int_a^b p(x)\,dx$. In view if $p = \frac{t}{s}$ and then $q = \frac{t}{t-s}$, and by using (2.2) on the right-hand side of (3.4), we have

$$\begin{aligned} \int_a^b S\left(\frac{\check{Y}[p^{s/t}(x) f^s(x)]^{t/s}}{\check{X}[p^{1-s/t}(x)]^{t/(t-s)}}\right) p^{s/t}(x) f^s(x) \cdot p^{1-s/t}(x)\,dx \\ \geqslant \left(\int_a^b (p^{s/t}(x) f^s(x))^{t/s} dx\right)^{s/t} \left(\int_a^b [(p(x))^{1-s/t}]^{t/(t-s)} dx\right)^{(t-s)/t}. \end{aligned} \tag{3.5}$$

From (3.4), (3.5) and in view of $\int_a^b p(x)\,dx = 1$, we obtain

$$\int_a^b S\left(\frac{f^t(x)}{\check{X}}\right) f^s(x)\, p(x)\,dx \geqslant \left(\int_a^b p(x) f^t(x)\,dx\right)^{s/t}.$$

Hence

$$\left(\int_a^b S\left(\frac{f^t(x)}{\check{X}}\right) f^s(x)\, p(x)\,dx\right)^{1/s} \geqslant \left(\int_a^b p(x) f^t(x)\,dx\right)^{1/t}. \qquad \square$$

REMARK 3.2. If $f(x)$ and $p(x)$ reduce to positive real sequences a_i and λ_i $(i = 1, \ldots, n)$, respectively and with appropriate changes in (3.3), we have

$$\Big(\sum S\Big(\frac{a_i^t}{\check{X}'}\Big)a_i^s\lambda_i\Big)^{1/s} \geqslant \Big(\sum \lambda_i a_i^t\Big)^{1/t},$$

where $\check{X}' = \sum \lambda_i a_i^t$.

This is just an inverse of the following well-known Jensen's inequality [3]

$$\Big(\sum a_i^s\lambda_i\Big)^{1/s} \leqslant \Big(\sum \lambda_i a_i^t\Big)^{1/t},$$

THEOREM 3.3 (Reverse weighted power mean integral inequality). *Let $f(x)$ and $p(x)$ be positive and continuous functions. If $r > 0$ and $\int_a^b p(x)\,dx = 1$, then*

$$\Big(\int_a^b S\Big(\frac{\int_a^b p(x)f^{2r}(x)\,dx}{f^{2r}(x)}\Big)p(x)f^r(x)\,dx\Big)^{1/r} \geqslant \Big(\int_a^b p(x)f^{2r}(x)\,dx\Big)^{1/2r}. \tag{3.6}$$

PROOF. From the hypotheses, we have

$$\begin{aligned}\int_a^b S\Big(\frac{\int_a^b p(x)f^{2r}(x)\,dx}{f^{2r}(x)}\Big)p(x)f^r(x)\,dx \\ = \int_a^b S\Big(\frac{p(x)\int_a^b p(x)f^{2r}(x)\,dx}{p(x)f^{2r}(x)\int_a^b p(x)\,dx}\Big)p^{1/2}(x)\cdot p^{1/2}(x)f^r(x)\,dx.\end{aligned} \tag{3.7}$$

From (2.2), we obtain

$$\Big(\int_a^b f(x)^2dx\Big)^{1/2}\Big(\int_a^b g(x)^2dx\Big)^{1/2} \leqslant \int_a^b S\Big(\frac{\check{Y}f^2(x)}{\check{X}g^2(x)}\Big)f(x)\,g(x)\,dx, \tag{3.8}$$

where $\check{X} = \int_a^b f^2(x)\,dx$, $\check{Y} = \int_a^b g^2(x)\,dx$. By using (3.8) on the right-hand side of (3.7), we have

$$\begin{aligned}\Big(\int_a^b S\Big(\frac{\int_a^b p(x)f^{2r}(x)\,dx}{f^{2r}(x)}\Big)p(x)f^r(x)\,dx\Big)^2 \\ \geqslant \int_a^b p(x)\,dx\cdot\int_a^b p(x)f^{2r}(x)\,dx = \int_a^b p(x)f^{2r}(x)\,dx.\end{aligned}$$

Hence

$$\Big(\int_a^b S\Big(\frac{\int_a^b p(x)f^{2r}(x)\,dx}{f^{2r}(x)}\Big)p(x)f^r(x)\,dx\Big)^{1/r} \geqslant \Big(\int_a^b p(x)f^{2r}(x)\,dx\Big)^{1/2r}. \qquad \square$$

REMARK 3.3. If $f(x)$ and $p(x)$ reduce to positive real sequences a_i and λ_i $(i = 1, \ldots, n)$, respectively and with appropriate changes in (3.6), we have

$$\Big(\sum S\Big(\frac{\sum\lambda_i a_i^{2r}}{a_i^{2r}}\Big)\lambda_i a_i^r\Big)^{1/r} \geqslant \Big(\sum \lambda_i a_i^{2r}\Big)^{1/2r}.$$

This is just an inverse of the following well-known weighted power mean inequality [**5**]

$$\Big(\sum \lambda_i a_i^r\Big)^{1/r} \leqslant \Big(\sum \lambda_i a_i^{2r}\Big)^{1/2r}.$$

It is worth noting that literature [**11**] is relevant in spirit and methodology to the present paper, but for the Hilbert-type integral inequality and its connection with analytic number theory.

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