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ON THE CONJUGATES OF CERTAIN ALGEBRAIC INTEGERS

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Abstract. A well-known theorem, due to C. J. Smyth, asserts that two conjugates of a Pisot number, having the same modulus are necessary complex conjugates. We show that this result remains true for K-Pisot numbers, where K is a real algebraic number field. Also, we prove that a j-Pisot number, where j is a natural number, can not have more than $2j$ conjugates with the same modulus.

1. Introduction

An algebraic integer α, with modulus greater than 1, is said to be a j-Pisot (resp. a j-Salem), where j is a natural number, if α has j conjugates with modulus greater than one and no conjugate (resp. and a conjugate) with modulus one. Up to a conjugation, roots of unity, j-Pisot numbers and j-Salem numbers, where j runs through $\mathbb{N}$, form a partition of the set of algebraic integers, since we have by Kronecker's theorem that an algebraic integer is a root of unity when all its conjugates belong to the unit circle. Recall also that the absolute value of a 1-Pisot (resp. a 1-Salem) is usually called a Pisot (resp. a Salem) number, and from its definition a Salem number, with degree d, has $(d-2)$ conjugates with the same modulus (one) [**2**]. In contrast with this fact, Smyth has shown [**7**] that two conjugates of a Pisot number, having the same modulus, are necessary complex conjugates. This theorem is somewhat surprising since there is no obvious reason for such a distribution of the conjugates of a Pisot number in the complex plane. The aim of this note is to extend this result for some classes of algebraic integers.

Theorem 1.1. *A j-Pisot number has at most $2j$ conjugates with the same modulus.*

Consider, for instance, the smallest Pisot number, usually denoted by θ_0. Then, the minimal polynomial of θ_0 is $x^3 - x - 1 = (x-\theta_0)(x-\alpha_0)(x-\overline{\alpha_0})$ [**2**], and an easy argument shows that the polynomial $x^{3j} - x^j - 1$ is the minimal polynomial

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of a j-Pisot number, having $2j$ conjugates with the same modulus; thus the upper bound $2j$, in Theorem 1.1, is the best possible for all $j \in \mathbb{N}$.

Also, we may determine j-Pisot numbers, having $2j$ conjugates with the same modulus and with minimal polynomials not of the form $P(x^j)$, where $j \in \mathbb{N}$ and $P \in \mathbb{Z}[x]$, by considering the products of Pisot numbers by certain primitive $n-$th roots of unity, where $\varphi(n) = j$ and φ is the Euler function. For example, the polynomial $x^6+x^4-2x^3+x^2-x+1$ is the minimal polynomial of the 2-Pisot number $\zeta\theta_0$, where $\zeta := e^{i2\pi/3}$, having 4 conjugates with the same modulus, namely $\zeta\alpha_0$, $\zeta\overline{\alpha_0}$, $\overline{\zeta}\alpha_0$, $\overline{\zeta\alpha_0}$. Similarly, if $\zeta := e^{i2\pi/5}$, then $x^{12}+x^{10}+x^9+x^8-3x^7+2x^6-2x^5-x^4+x^2-x+1$ is the minimal polynomial of the 4-Pisot number $\zeta\theta_0$, having 8 conjugates with the same modulus.

Other special j-Pisot numbers are contained in the set of K-Pisot numbers, where K is an algebraic number field. Recall that an algebraic integer α, with modulus greater than 1, is said to be a K-Pisot number, if for each embedding σ of K into $\mathbb{C}$, the polynomial $M^{(\sigma)}_{(\alpha,K)}$, whose coefficients are the conjugates by σ of the coefficients of the minimal polynomial, say $M_{(\alpha,K)}$, of α, over K, has a unique root with modulus greater than one and no root with modulus one. Clearly, the roots of $M_{(\alpha,K)}$ are the conjugates of α over K, the polynomial $M^{(\sigma)}_{(\alpha,K)}$ is irreducible over the field $\sigma(K)$, and so the K-Pisot number α is a j-Pisot number, when the degree of K is j, and $\gcd(M_{(\alpha,K)}, M^{(\sigma)}_{(\alpha,K)}) = 1$ for all embeddings σ of K into $\mathbb{C}$ other than the identity of K. Some results about K-Pisot numbers may be found in [**3**, **4**, **8**–**10**], and the set of such numbers has been defined in [**1**].

Theorem 1.2. *Two conjugates, over a real algebraic number field K, of a K-Pisot number, having the same modulus are complex conjugates.*

It is worth noting that Theorem 1.2 is not true when K is not real. Indeed, consider, for example, the polynomial $P(x) = x^3 + x + i \in \mathbb{Z}[i, x]$, where $i^2 = -1$. Then, $P(i\theta_0) = P(i\alpha_0) = P(i\overline{\alpha}_0) = 0$ and so the cubic polynomial $P(x)$ is irreducible over $\mathbb{Q}(i)$. It follows that $x^3+x-i = (x+ i\theta_0)(x+i\alpha_0)(x+i\overline{\alpha}_0)$ is also irreducible over $\mathbb{Q}(i)$, and so $i\theta_0$ is a $\mathbb{Q}(i)$-Pisot number, having two conjugates, over $\mathbb{Q}(i)$, with the same modulus and which are not complex conjugates, as $\overline{i\alpha_0} = -i\overline{\alpha}_0 \neq i\overline{\alpha_0}$. Notice also, in this case, that the 2-Pisot number $i\theta_0$ has 4 conjugates with the same modulus, namely $\pm i\alpha_0$ and $\pm i\overline{\alpha}_0$, the polynomial $Q(x) = (x^3+x+i)(x^3+x-i) = x^6 + 2x^4 + x^2 + 1$ is the minimal polynomial of $i\theta_0$ and $Q(x) = R(x^2)$, where $R(x) = x^3 + 2x^2 + x + 1$ is the minimal polynomial of the $\mathbb{Q}$-Pisot number $-\theta_0^2$.

Throughout, when we speak about conjugates and the degree of an algebraic number, or of an algebraic number field, without mentioning the basic field, this is meant over the field of the rationals $\mathbb{Q}$. Except contrary mention, the minimal polynomial of an algebraic number is also considered over $\mathbb{Q}$. As usually the notation $\mathbb{N}$ (resp. the notation $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{R}$) designates the set of positive rational integers (resp. the field of complex numbers, the ring of the rational integers and the field of real numbers). The proofs of the theorems above are presented in the next section. Recall, finally, that an interesting related result, due to Boyd [**5**] and

Ferguson [**6**], asserts that *a real and a non-real conjugates of an algebraic number β can not have the same modulus, provided the minimal polynomial of β is not of the form $P(x^j)$, where $P \in \mathbb{Z}[x]$ and $j \in \mathbb{N}$.* The analog of the above cited result of [**7**], for algebraic numbers is not always true. Indeed, consider, for instance, a real algebraic number $\beta > 1$ whose other conjugates are of modulus at most one and with a conjugate of modulus one (β is a Salem number when it is an algebraic integer), and a rational number $r \in (1/\beta, 1)$. Then, the real algebraic number $r\beta$ is greater than one, and has $(d-2)$ conjugates with the same modulus (less than one), where d is the degree of β. Similarly, we may use a j-Salem number, with degree d, to obtain algebraic numbers having j conjugates with modulus greater than one, and $(d-2j)$ conjugates with the same modulus, less than one.

2. The proofs

Proof of Theorem 1.1. Let $j \in \mathbb{N}$ and let α be a j-Pisot number whose conjugates $\alpha_1,\ \alpha_2, \dots,\ \alpha_d$ are ordered so that $1 < |\alpha_j| \leqslant \cdots \leqslant |\alpha_1|$. To show that α can not have more than j conjugates belonging to the upper half plane $U := \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geqslant 0\}$ with the same modulus, we consider, similarly as in the proof of Theorem 8.1.3 of [**2**], the polynomial

$$f(x) = \prod_{1 \leqslant k < l \leqslant d} (x - \alpha_k\alpha_l).$$

Because the product describing f is a symmetric function of the conjugates of α, we see that $f \in \mathbb{Z}[x]$ and so f is a multiple, in the ring $\mathbb{Z}[x]$, of the minimal polynomial of each $\alpha_k\alpha_l$, where $1 \leqslant k < l \leqslant d$. Now, assume on the contrary that α has $j+1$ conjugates, say $\alpha_{1+j}, \dots, \alpha_{2j+1}$, in U with the same modulus. We may also suppose, without loss of generality, that α_{1+j} is (resp. that α_{1+j} and α_{2+j} are) real when the set $\{\alpha_{1+j}, \dots, \alpha_{2j+1}\}$ contains a real number (resp. contains two real numbers). Notice that the above mentioned theorem of Boyd and Ferguson yields that this last supposition is effective, only when the minimal polynomial of α is of the form $P(x^k)$ for some $P \in \mathbb{Z}[x]$ and $k \in \mathbb{N}$; in fact, we prefer to continue the proof without using this result. Then,

$$|\alpha_{2j+1}| = \cdots = |\alpha_{1+j}| < 1 < |\alpha_j| \leqslant \cdots \leqslant |\alpha_1|, \tag{2.1}$$

$d \geqslant j + 2 + 2(j-1) = 3j$, and we easily obtain a contradiction by considering separately each of the following three cases.

- $\alpha_{1+j} \in \mathbb{R}$ and $j \geqslant 2$. Then, for every $s \in \{3, \dots, j+1\}$, the number α_{s+j} is not real and there is $t = t(s) \in \{2j+2, \dots, d\}$ such that $\alpha_{1+j}^2 = |\alpha_{1+j}|^2 = |\alpha_{s+j}|^2 = \alpha_{s+j}\overline{\alpha_{s+j}} = \alpha_{s+j}\alpha_t$. Hence, α_{1+j}^2 is a root of f with multiplicity greater than $(j-2)$, α_1^2 is a root of f with multiplicity at least $(j-1)$, and so the set $S := \{(k,l) \mid \alpha_1^2 = \alpha_k\alpha_l \text{ and } 1 \leqslant k < l \leqslant d\}$ contains more than $(j-2)$ elements. Notice, by (2.1), that any pair (k,l) in S must satisfy $k < l \leqslant j$, and $(1,l) \notin S$ for each $l \in \{2, \dots, j\}$, as $\alpha_1 \neq \alpha_l \Rightarrow \alpha_1^2 \neq \alpha_1\alpha_l$ (this ends the proof of this case for $j = 2$). In a similar manner, we obtain that each of the following $(j-2)$ sets: $\{(2,3), (2,4) \dots, (2,j)\}$, $\{(3,4), (3,5), \dots, (3,j)\}, \dots, \{(j-1,j)\}$

may not contain more than one element of S, and so the cardinality of S is at most $(j-2)$, since $S \subset \bigcup_{2\leqslant k\leqslant j-1}\big(\bigcup_{k+1\leqslant l\leqslant j}\{(k,l)\}\big)$.

- $\alpha_2 \in \mathbb{R}$ and $j=1$. If $\alpha_3 \in \mathbb{R}$, then $\alpha_3 = -\alpha_2 \Rightarrow -\alpha_1$ is a conjugate of α_1, and otherwise we obtain, similarly as in the case above, that $\alpha_2^2 = \alpha_3\overline{\alpha_3}$ is a root of $f \Rightarrow \alpha_1^2$ is a root of $f \Rightarrow \alpha_1^2 = \alpha_k\alpha_l$, for some (k,l) with $1 \leqslant k < l \leqslant d$.
- $\alpha_{1+j} \notin \mathbb{R}$. Consider the real algebraic integer $\varphi := |\alpha_{1+j}|^2 = \cdots = |\alpha_{1+2j}|^2$. Then, $0 < \varphi < 1$, and by the same way as in the first case, we have that φ is a root of f with multiplicity greater than j. Hence, φ has a conjugate, say φ', with modulus greater than one, which is also a root of f with multiplicity at least $(j+1)$, and so the cardinality of the set $T := \{(k,l) \mid \varphi' = \alpha_k\alpha_l$ and $1 \leqslant k < l \leqslant d\}$ is greater than j. Since $T \subset \bigcup_{1\leqslant k\leqslant j}\bigcup_{k+1\leqslant l\leqslant d}\{(k,l)\}$, and each of the following j sets: $\{(1,2),(1,3),\ldots,(1,d)\}$, $\{(2,3),(2,4),\ldots,(2,d)\},\ldots,$ $\{(j,j+1),\ldots,(j,d)\}$ may not contain more than one element of T, we see that the cardinality of T is at most j. □

Proof of Theorem 1.2. Let K be a real algebraic number field and let α be a K-Pisot number with degree d over K. For any element σ of the set, say E, of the embeddings of K into $\mathbb{C}$, let $\alpha_1^{(\sigma)},\alpha_2^{(\sigma)},\ldots,\alpha_d^{(\sigma)}$ designate the roots of the polynomial $M_{(\alpha,K)}^{(\sigma)}$, ordered so that

$$|\alpha_1^{(\sigma)}| > 1 > |\alpha_2^{(\sigma)}| \geqslant \cdots \geqslant |\alpha_d^{(\sigma)}|. \tag{2.2}$$

Then, $M_{(\alpha^{(\sigma)},\sigma(K))}(x) = M_{(\alpha,K)}^{(\sigma)}(x) = \prod_{1\leqslant k\leqslant d}\big(x-\alpha_k^{(\sigma)}\big)$ and the product $\prod_{\sigma\in E} M_{(\alpha^{(\sigma)},\sigma(K))}$ is a multiple, in the ring $\mathbb{Z}[x]$, of the minimal polynomial of α. The scheme of the proof is similar to the one of Theorem 1.1, by considering the polynomials

$$f_\sigma(x) := \prod_{1\leqslant k<l\leqslant d}\big(x-\alpha_k^{(\sigma)}\alpha_l^{(\sigma)}\big),$$

where σ runs through E. If $\mathbb{Z}_{\sigma(K)}$ designates the ring of the integers of the field $\sigma(K)$, then $f_\sigma(x) \in \mathbb{Z}_{\sigma(K)}[x]$, and the minimal polynomial, over K, of each $\alpha_k^{(\sigma)}\alpha_l^{(\sigma)}$, where $1 \leqslant k < l \leqslant d$, divides, in $\mathbb{Z}_{\sigma(K)}[x]$, the polynomial f_σ. Now, assume on the contrary that α has two conjugates, say α_2 and α_3, over K, having the same modulus and belonging to the half upper plane U. It is clear that $\{\alpha_2,\alpha_3\} \subsetneqq \mathbb{R}$, since otherwise $\alpha_3 = -\alpha_2$ and so $-\alpha$ is a conjugate, over K, of $\alpha = \alpha_1^{(I)}$, where I is the identity of K. Also, the case where $\alpha_2 \in \mathbb{R}$ and $\alpha_3 \notin \mathbb{R}$, can not hold, because the equation $\alpha_2^2 = \alpha_3\overline{\alpha_3}$ implies that α_2^2 is a root of the polynomial f_I, the conjugate α^2 of α_2^2, over K, is also a root of f_I, and this last assertion leads to a contradiction, as we have by (2.2), $|\alpha_1^{(I)}||\alpha_1^{(I)}| > |\alpha_k^{(I)}||\alpha_l^{(I)}|$ $\forall(k,l)$ with $1 \leqslant k < l \leqslant d$. Finally, suppose $\alpha_2 \notin \mathbb{R}$ and $\alpha_3 \notin \mathbb{R}$. Similarly as above, we have that the number $\varphi := \alpha_2\overline{\alpha_2} = \alpha_3\overline{\alpha_3}$ is a root of f_I with multiplicity at least 2, and consequently the square of the polynomial $M_{(\varphi,K)}$ divides, in $\mathbb{Z}_K[x]$, the polynomial f_I. Hence, for any $\sigma \in E$ the square of the polynomial $M_{(\varphi,K)}^{(\sigma)}$, whose coefficients are the conjugates by σ of the coefficients of $M_{(\varphi,K)}$, divides, in the ring $\mathbb{Z}_{\sigma(K)}$, the polynomial f_σ, and so the roots of $M_{(\varphi,K)}^{(\sigma)}$ are of modulus less than one, as (2.2) $\Rightarrow \left|\alpha_k^{(\sigma)}\alpha_l^{(\sigma)}\right| < 1$ for each (k,l) satisfying $2 \leqslant k < l \leqslant d$, and for every pair (l,m) with $2 \leqslant l \neq m \leqslant d$, we have $\alpha_l^{(\sigma)} \neq \alpha_m^{(\sigma)} \Rightarrow \alpha_1^{(\sigma)}\alpha_l^{(\sigma)} \neq \alpha_1^{(\sigma)}\alpha_m^{(\sigma)}$. It follows that the roots of the polynomial $\prod_{\sigma\in E} M_{(\varphi,K)}^{(\sigma)}(x) \in \mathbb{Z}[x]$ are all of modulus less than one, and this last assertion is not true because the algebraic integer φ, satisfying the relation $0 < \varphi < 1$, should have a conjugate with modulus greater than one. $\square$

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References

1. A. M. Bergé, J. Martinet, *Notions relatives de hauteurs et de régulateurs,* Acta Arith. **54** (1989), 155–170.
2. M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, J. P. Schreiber, *Pisot and Salem numbers,* Birkhäuser, Basel, 1992.
3. M. J. Bertin, *K-nombres de Pisot et de Salem*, F. Q. Gouvêa et N. Yui (eds.), *Advances in number theory*, Proceedings of the Third Canadian Number Theory Association, Oxford University Press, 1993, 391–397.
4. ______, *K-nombres de Pisot et de Salem,* Acta Arith. **68** (1994), 113–131.
5. D. W. Boyd, *Irreducible polynomials with many roots of maximal modulus*, Acta Arith. **68** (1994), 85–88.
6. R. Ferguson, *Irreducible polynomials with many roots of equal modulus*, Acta Arith. **78** (1997), 221–225.
7. C. J. Smyth, *The conjugates of algebraic integers*, Am. Math. Month. **82** (1975), 86.
8. T. Zaïmi, *Sur les K-nombres de Pisot de petite mesure*, Acta Arith. **77** (1996), 103–131.
9. ______, *Caractérisation d'un ensemble géné ralisant l'ensemble des nombres de Pisot*, Acta Arith. **87** (1998), 141–144.
10. ______, *Sur la fermeture de l'ensemble des K-nombres de Pisot*, Acta Arith. **88** (1998), 363–367.

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