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PURELY PERIODIC β-EXPANSIONS IN CUBIC SALEM BASE IN $\mathbb{F}_q((X^{-1}))$

Faïza Mahjoub

ABSTRACT. Let $\mathbb{F}_q$ be the finite field with q elements and β Salem series in $\mathbb{F}_q((X^{-1}))$. It is proved in [**15**] that, in this case, all elements in $\mathbb{F}_q(X,\beta)$ have purely periodic β-expansion. We characterize the formal power series f in $\mathbb{F}_q(X,\beta)$ with purely periodic β-expansions by the conjugate vector $\widetilde{f}$ when β is a cubic unit. No similar results exist in the real case.

1. Introduction

Let $\beta > 1$ be a real number. The β-expansion of a real number $x \in [0,1]$ is defined as the sequence $(x_i)_{i\geqslant 1}$ with values in $\{0,1,\cdots,[\beta]\}$ produced by the β-transformation $T_\beta : x \to \beta x \pmod 1$ as follows:

$$\forall i \geqslant 1, \quad x_i = [\beta T_\beta^{i-1}(x)], \quad \text{and thus} \quad x = \sum_{i\geqslant 1} \frac{x_i}{\beta^i}.$$

This expansion was first introduced by Rényi [**14**]. A β-expansion is periodic if there exists $p \geqslant 1$ and $m \geqslant 1$ such that $x_k = x_{k+p}$, holds for all $k \geqslant m$. When $x_k = x_{k+p}$ holds for all $k \geqslant 1$, then it is purely periodic. We denote by $Per(\beta)$ the numbers in [0,1) with periodic β-expansions, Pur(β) the numbers in [0,1) with purely periodic β-expansions and Fin(β) the numbers in [0,1) with finite β-expansions.

Let $\mathbb{Q}(\beta)$ be the smallest field containing $\mathbb{Q}$ and β. An easy argument shows that $\mathrm{Per}(\beta) \subseteq \mathbb{Q}(\beta) \cap [0,1)$ for every real number $\beta > 1$. In [**17**], Schmidt showed that if β is a Pisot number (an algebraic integer whose conjugates have modulus <1), then $\mathrm{Per}(\beta)=\mathbb{Q}(\beta) \cap [0,1)$.

Ito and Rao discussed the purely periodic β-expansions in the statement [**9**] and they characterized all reals in $[0,1[$ having purely periodic β-expansions with Pisot unit base. In [**6**], Berthé and Siegel completed the characterization in the Pisot non-unit base.

Set

$$\gamma(\beta) = \sup\{c \in [0,1) : \forall r \in \mathbb{Q} \cap [0,c],\ d_\beta(r) \text{ is purely periodic}\}.$$

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Akiyama proved in [**3**] that if β is a Pisot unit number satisfying the finiteness property ($\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$), then $\gamma(\beta) > 0$.

In the quadratic case, Schmidt [**17**] proved that if β satisfied $\beta^2 = n\beta + 1$ for some integer $n \geqslant 1$, then $\gamma(\beta) = 1$. Until now, it has been clear that for only known family is the $\gamma(\beta)$ of which equals 1. Authors of [**2**] have proved that if β is not a Pisot unit, then $\gamma(\beta) = 0$. They showed that if β is a cubic Pisot unit satisfying the finiteness property such that the number field $\mathbb{Q}(\beta)$ is not totally real, then $0 < \gamma(\beta) < 1$.

In 2006, Hbaib and Mkaour [**8**] introduced the β-expansion in the field of formal power series over a finite field $\mathbb{F}_q$. They developed some results concerning the β-expansion of unity. Later, Scheicher [**15**] proved that $Per(\beta) = \mathbb{F}_q(X, \beta)$ if and only if β is a Pisot or Salem series. In [**1**], Abbes and Hbaib gave families of Pisot and Salem elements β in $\mathbb{F}_q((X^{-1}))$ with the curious property that the β-expansion of any rational series in the unit disk $D(0, 1)$ is purely periodic. Ghorbel, Hbaib and Zouari showed in [**7**] that if β is a quadratic Pisot unit base, then every rational f in the unit disk has a purely periodic β-expansion.

In [**5**], the authors proved that the β-expansion of any rational element in the unit disk $D(0, 1)$ is purely periodic when β is a Pisot or Salem unit series in $\mathbb{F}_q((X^{-1}))$.

Here, we continue in the same context: we take β a cubic Salem unit series in $\mathbb{F}_q((X^{-1}))$ and characterize the formal power series $f \in \mathbb{F}_q(X, \beta)$ with purely periodic β-expansions by the norm of the conjugate vector $\widetilde{f}$.

Our work is organized as follows: In section 2, we introduce the field of formal power series over a finite field $\mathbb{F}_q$ and the β-expansion theory in this field. In Section 3, we give our main result with its proof.

2. β-expansions in $\mathbb{F}_q((X^{-1}))$

Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q[X]$ the ring of polynomials with coefficients in $\mathbb{F}_q$, $\mathbb{F}_q(X)$ the field of rational functions, $\mathbb{F}_q(X, \beta)$ the minimal extension of $\mathbb{F}_q$ containing X and β and $\mathbb{F}_q[X, \beta]$ the minimal ring containing X and β. Let $\mathbb{F}_q((X^{-1}))$ be the field of formal power series of the form:

$$f = \sum_{k=-\infty}^{l} f_k X^k, \quad f_k \in \mathbb{F}_q$$

where

$$l = \deg f := \begin{cases} \max\{k : f_k \neq 0\} & \text{for } f \neq 0; \\ -\infty & \text{for } f = 0. \end{cases}$$

We define the absolute value by

$$|f| = \begin{cases} q^{\deg f} & \text{for } f \neq 0; \\ 0 & \text{for } f = 0. \end{cases}$$

Since $|.|$ is not archimedean, $|.|$ fulfills the strict triangle inequality

$$|f+g| \leqslant \max(|f|,|g|) \quad \text{and}$$
$$|f+g| = \max(|f|,|g|) \quad \text{if} \ \ |f| \neq |g|.$$

Let $f \in \mathbb{F}_q((X^{-1}))$, define the integer (polynomial) part $[f] = \sum_{k=0}^{l} f_k X^k$ where the empty sum, as usual, is defined to be zero. Therefore $[f] \in \mathbb{F}_q[X]$ and $(f-[f])$ is in the unit disk $D(0,1)$ for all $f \in \mathbb{F}_q((X^{-1}))$.

LEMMA 2.1. *Let* $f = \sum_{i\geqslant 1} \frac{\alpha_i}{X^i} \in \mathbb{F}_q((X^{-1})) \cap D(0,1)$. *Then* $(\alpha_i)_{i\geqslant 1}$ *is periodic if and only if* $f \in \mathbb{F}_q(X)$.

LEMMA 2.2. *Let* $f = \sum_{i\geqslant 1} \frac{\alpha_i}{X^i} \in \mathbb{F}_q((X^{-1})) \cap D(0,1)$. *Then* $(\alpha_i)_{i\geqslant 1}$ *is purely periodic if and only if* $f \in \mathbb{F}_q(X)$ *and* 0 *is not a pole of* f.

PROPOSITION 2.1. [**13**] *Let K be complete field with respect to (a non archimedean absolute value $|.|$) and L/K ($K \subset L$) be an algebraic extension of degree m. Then $|.|$ has a unique extension to L defined by : $|a| = \sqrt[m]{|N_{L/K}(a)|}$ and L is complete with respect to this extension.*

We apply Proposition 2.1 to algebraic extensions of $\mathbb{F}_q((X^{-1}))$. Since $\mathbb{F}_q[X] \subset \mathbb{F}_q((X^{-1}))$, every algebraic element over $\mathbb{F}_q[X]$ can be evaluated. However, since $\mathbb{F}_q((X^{-1}))$ is not algebraically closed and such an element is not necessarily expressed as a power series over X^{-1}. For a full characterization of the algebraic closure of $\mathbb{F}_q[X]$, we refer to Kedlaya [**10**].

An element $\beta = \beta_1 \in \mathbb{F}_q((X^{-1}))$ is called a Pisot (respectively, Salem) element if it is an algebraic integer over $\mathbb{F}_q[X]$, $|\beta| > 1$ and $|\beta_j| < 1$ holds for all its conjugates β_j (respectively, $|\beta_j| \leqslant 1$ and there exists at least one conjugate β_k such that $|\beta_k| = 1$).

Bateman and Duquette [**4**] characterized the Pisot and Salem elements in $\mathbb{F}_q((X^{-1}))$:

THEOREM 2.1. *Let* $\beta \in \mathbb{F}_q((X^{-1}))$ *be an algebraic integer over* $\mathbb{F}_q[X]$ *and*

$$P(y) = y^n - A_1 y^{n-1} - \cdots - A_n, \quad A_i \in \mathbb{F}_q[X],$$

be its minimal polynomial. Then

(i) β *is a Pisot element if and only if* $|A_1| > \max_{2\leqslant i\leqslant n} |A_i|$
(ii) β *is a Salem element if and only if* $|A_1| = \max_{2\leqslant i\leqslant n} |A_i|$.

Let $\beta,\ f \in \mathbb{F}_q((X^{-1}))$ with $|\beta| > 1$. A representation in base β (or β-representation) of f is an infinite sequence $(d_i)_{i\geqslant 1}$, $d_i \in \mathbb{F}_q[X]$, such that

$$f = \sum_{i\geqslant 1} \frac{d_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f in base β, noted $d_\beta(f)$, which is obtained by using the β-transformation T_β in the unit disk which is given by $T_\beta(f) = \beta f - [\beta f]$. Then $d_\beta(f) = (a_i)_{i\geqslant 1}$ where $a_i = [\beta T_\beta^{i-1}(f)]$.

An equivalent definition of the β-expansion can be obtained by a greedy algorithm. This algorithm works as follows. We set $r_0 = f$, $a_i = [\beta r_{i-1}]$ and $r_i = \beta r_{i-1} - a_i$ for all $i \geqslant 1$. The β-expansion of f will be noted as $d_\beta(f) = (a_i)_{i\geqslant 1}$.

We note that $d_\beta(f)$ is finite if and only if there is a $k \geqslant 0$ such that $T^k(f) = 0$, $d_\beta(f)$ is ultimately periodic if and only if there is some smallest $p \geqslant 0$ (the preperiod length) and $s \geqslant 1$ (the period length) for which $T_\beta^{p+s}(f) = T_\beta^p(f)$.

Now, let $f \in \mathbb{F}_q((X^{-1}))$ be an element with $|f| \geqslant 1$. Then there is a unique $k \in \mathbb{N}$ such that $|\beta|^k \leqslant |f| < |\beta|^{k+1}$. Hence $|\frac{f}{\beta^{k+1}}| < 1$ and we can represent f by shifting $d_\beta(\frac{f}{\beta^{k+1}})$ by k digits to the left. Therefore, if $d_\beta(f) = 0.d_1d_2d_3\dots$, we obtain $d_\beta(\beta f) = d_1.d_2d_3\dots$. If we have $d_\beta(f) = d_l d_{l-1}\dots d_0.d_{-1}\dots d_m$, then we put $\deg_\beta(f) = l$ and $\mathrm{ord}_\beta(f) = m$. In the sequel, we will use the following notations:

- $\mathrm{Fin}(\beta) = \{f \in \mathbb{F}_q((X^{-1})) : d_\beta(f) \text{ is finite}\}$.
- $\mathrm{Per}(\beta) = \{f \in \mathbb{F}_q((X^{-1})) : d_\beta(f) \text{ is eventually periodic}\}$.
- $\mathrm{Pur}(\beta) = \{f \in \mathbb{F}_q((X^{-1})) \text{ and } |f| < 1 : d_\beta(f) \text{ is purely periodic}\}$.

REMARK 2.1. In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if $z, w \in \mathbb{F}_q((X^{-1}))$, we have $d_\beta(z+w) = d_\beta(z) + d_\beta(w)$ digitwise.

THEOREM 2.2. [**8**] *A β-representation $(d_j)_{j\geqslant 1}$ is the β-expansion of f in the unit disk if and only if $|d_j| < |\beta|$ for $j \geqslant 1$.*

In the field of formal series case, Scheicher, Jellali and Mkaouar [**16**] had studied the characterization of purely periodic β-expansions in the Pisot unit base. Later, Hbaib-Mkaouar and Scheicher proved independently the following:

THEOREM 2.3. [**15**] *β is a Pisot or Salem element if and only if* $\mathrm{Per}(\beta) = \mathbb{F}_q(X, \beta)$.

THEOREM 2.4. [**8**] *β is Pisot or Salem element if and only if $d_\beta(1)$ is periodic.*

The authors of [**5**] gave the following result.

THEOREM 2.5. [**5**] *Let β be a Pisot or Salem unit series in $\mathbb{F}_q((X^{-1}))$ and $r \in \mathbb{F}_q(X) \cap D(0,1)$. Then $d_\beta(r)$ is purely periodic.*

In [**11**] and [**12**], metric results were established and the relation to continued fractions was studied. Stating

$$\gamma(\beta) = \sup\{c \in [0,1) : \forall f \in \mathbb{F}_q(X) \cap D(0,c),\ d_\beta(f) \text{ is purely periodic}\}.$$

The study of the quality $\gamma(\beta)$ in $\mathbb{F}_q((X^{-1}))$ was interesting for some researchers in the last years. Specifically, we have the following theorems.

THEOREM 2.6. [**1**] *Let β be a Pisot or Salem unit series. Then $\gamma(\beta) > 0$.*

THEOREM 2.7. [**7**] *If β is a quadratic Pisot unit series, then $\gamma(\beta) = 1$*

3. Results

Let β be an algebraic unit series of minimal polynomial $\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_0$ where $A_i \in \mathbb{F}_q[X]$ for $i \in \{1, \ldots, d-1\}$ and $A_0 \in \mathbb{F}_q^*$. Let $\beta_2, \ldots, \beta_d$ be the conjugates of β. For $f = r_0 + r_1\beta + \cdots + r_{d-1}\beta^{d-1} \in \mathbb{F}_q(X, \beta)$, we define $f_i = r_0 + r_1\beta_i + \cdots + r_{d-1}\beta_i^{d-1} \in \mathbb{F}_q(X, \beta)$ with $2 \leqslant i \leqslant d$ and $\widetilde{f}$ the conjugate vector of f by $\widetilde{f} = \begin{pmatrix} f_2 \\ \vdots \\ f_d \end{pmatrix}$ and $\|\widetilde{f}\| = \sup_{2 \leqslant k \leqslant d} |f_k|$.

Theorem 3.1. *Let β be a Pisot unit series and $f \in \mathbb{F}_q(X, \beta) \cap D(0, 1)$. If $d_\beta(f)$ is purely periodic, then $\|\widetilde{f}\| < |\beta|$.*

Proof. We have $d_\beta(f) = \overline{a_1, \cdots, a_s}$ with $a_i \in \mathbb{F}_q[X]$. Then we can write

$$f = \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + \frac{1}{\beta^s}(f).$$

Hence,

$$f(1 - \frac{1}{\beta^s}) = \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s}.$$

Therefore, we obtain

$$(\beta^s - 1)f = a_1\beta^{s-1} + \cdots + a_s.$$

Let $(\beta_j)_{2 \leqslant j \leqslant d}$ the conjugates of β. Afterward $|\beta_j| < 1$ for all $2 \leqslant j \leqslant d$ which leads to

$$(\beta_j^s - 1)f_j = a_1\beta_j^{s-1} + \cdots + a_s.$$

Then, we get

$$|f_j| = |a_1\beta_j^{s-1} + \cdots + a_s| < |\beta|.$$

Finally, $\|\widetilde{f}\| < |\beta|$. □

Remark 3.1. The same arguments as in the proof of the last theorem, one can prove that if β is a Salem unit series and $f \in \mathbb{F}_q(X, \beta) \cap D(0, 1)$, and if $d_\beta(f)$ is purely periodic then $\|\widetilde{f}\| \leqslant |\beta|$.

Now, here is our main theorem.

Theorem 3.2. *Let β be a cubic Salem unit series in $\mathbb{F}_q((X^{-1}))$ and $f \in \mathbb{F}_q(X, \beta) \cap D(0, 1)$. $\|\widetilde{f}\| < |\beta|$ if and only if $d_\beta(f)$ is purely periodic.*

Proof. The sufficient condition is deduced by Theorem 3.1. As for the necessary condition, let $f = r_0 + r_1\beta + r_2\beta^2 \in \mathbb{F}_q(X, \beta) \cap D(0, 1)$ and f_1, f_2 the two conjugates of f in $\mathbb{F}_q(X, \beta)$. Let $f_i = r_0 + r_1\beta_i + r_2\beta_i^2$ such that $|f_i| = \|\widetilde{f}\|$. Since, we have $|r_0 + r_1\beta + r_2\beta^2| < 1$. Then we obtain two cases:

Case 1: $|r_0| < 1$ and $|r_1\beta + r_2\beta^2| < 1$. If $|r_0| < 1$ then thanks to Theorem 2.5 $d_\beta(r_0)$ is purely periodic. If $|r_1\beta + r_2\beta^2| < 1$ then $|r_1 + r_2\beta| < \frac{1}{|\beta|} < 1$. Hence, we obtain two subcases:

(i) $|r_1| < 1$ and $|r_2\beta| < 1$. If $|r_1| < 1$ then by Theorem 2.5 $d_\beta(r_1)$ is purely periodic. So, $d_\beta(r_1\beta)$ is purely periodic. If $|r_2\beta| < 1$ then $|r_2| < \frac{1}{|\beta|} < 1$. Therefore, from Theorem 2.5 $d_\beta(r_2)$ is purely periodic, consequently $d_\beta(r_2\beta^2)$ is purely periodic.

(ii) $|r_1| > 1$, $|r_2\beta| > 1$ and $|r_1| = |r_2\beta|$. If $|r_1| = |r_2\beta|$ then $|r_1\beta_i| = |r_2\beta\beta_i| > |r_2\beta_i^2|$. Reminding that we have $|r_0 + r_1\beta_i + r_2\beta_i^2| < |\beta|$ and $|r_0| < 1 < \beta$, we obtain $|r_1\beta_i + r_2\beta_i^2| < |\beta|$. However, when $|\beta_i| = 1$ we get $|r_1\beta_i + r_2\beta_i^2| = |r_1\beta_i| = |r_1|$. Thus, $|r_1| < |\beta|$ and $d_\beta(r_1)$ is purely periodic and it follows that $d_\beta(r_1\beta)$ is purely periodic. Moreover, we have $|r_1| = |r_2\beta|$ then $|r_2| < 1$. Using Theorem 2.5 we get $d_\beta(r_2)$ is purely periodic. Then $d_\beta(r_2\beta^2)$ is purely periodic.

Case 2: $|r_0| > 1$, $|r_1\beta + r_2\beta^2| > 1$ and $|r_0| = |r_1\beta + r_2\beta^2|$. Reminding that we have $|r_1\beta + r_2\beta^2| > |r_1\beta_i + r_2\beta_i^2|$, we get $|r_0| < |\beta|$. Therefore $d_\beta(r_0)$ is purely periodic. We have $|r_1\beta + r_2\beta^2| = |r_0| < |\beta|$ thereby $|r_1 + r_2\beta| < 1$. Thus we obtain two subcases.

(i) $|r_1| < 1$ and $|r_2\beta| < 1$. If $|r_1| < 1$ then from Theorem 2.5 $d_\beta(r_1)$ is purely periodic. Consequently $d_\beta(r_1\beta)$ is purely periodic. If $|r_2\beta| < 1$ then $|r_2| < \frac{1}{|\beta|} < 1$. Afterward, we use Theorem 2.5 which asserts that $d_\beta(r_2)$ is purely periodic, moreover $d_\beta(r_2\beta^2)$ is purely periodic.

(ii) $|r_1| > 1$, $|r_2\beta| > 1$ and $|r_1| = |r_2\beta|$. We have $|r_0 + r_1\beta_i + r_2\beta_i^2| < |\beta|$ and $|r_0| < |\beta|$, we get $|r_1\beta_i + r_2\beta_i^2| < |\beta|$ afterward $|r_1 + r_2\beta_i| < |\beta|$. Then we obtain two subsubcases.
   - $|r_1| < |\beta|$ and $|r_2\beta_i| < |\beta|$. If $|r_1| < |\beta|$ then $d_\beta(r_1)$ is purely periodic, consequently $d_\beta(r_1\beta)$ is purely periodic. If $|r_2\beta_i| < |\beta|$ then $|r_2| < |\beta|$ when $|\beta_i| = 1$, we get $d_\beta(r_2)$ is purely periodic and it follows that $d_\beta(r_2\beta^2)$ is purely periodic.
   - $|r_1| > |\beta|$, $|r_2\beta_i| > |\beta|$ and $|r_1| = |r_2\beta_i|$ we obtain $|r_1| = |r_2|$ when $|\beta_i| = 1$. However, $|r_1| = |r_2\beta|$ afterward $|\beta| = 1$ impossible.

Since, we obtained in the last cases that $d_\beta(r_0)$, $d_\beta(r_1\beta)$ and $d_\beta(r_2\beta^2)$ are purely periodic, one can deduce that the desired result ($d_\beta(f)$ is purely periodic). ☐

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Department of Mathematics
Higher institute of applied science and technology
University of Monastir
Mahdia
Tunisia
faiza.mahjoub@yahoo.fr

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