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DECOMPOSITIONS OF 2×2 MATRICES OVER LOCAL RINGS

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Abstract. An element a of a ring R is called perfectly clean if there exists an idempotent $e \in \mathrm{comm}^2(a)$ such that $a - e \in U(R)$. A ring R is perfectly clean in case every element in R is perfectly clean. In this paper, we completely determine when every 2×2 matrix and triangular matrix over local rings are perfectly clean. These give more explicit characterizations of strongly clean matrices over local rings. We also obtain several criteria for a triangular matrix to be perfectly J-clean. For instance, it is proved that for a commutative local ring R, every triangular matrix is perfectly J-clean in $T_n(R)$ if and only if R is strongly J-clean.

1. Introduction

The commutant and double commutant of an element a in a ring R are defined by $\mathrm{comm}(a) = \{x \in R \mid xa = ax\}$, $\mathrm{comm}^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in \mathrm{comm}(a)\}$, respectively. An element $a \in R$ is strongly clean provided that there exists an idempotent $e \in \mathrm{comm}(a)$ such that $a - e \in U(R)$. A ring R is called strongly clean in the case that every element in R is strongly clean. Strongly clean matrix rings and triangular matrix rings over local rings have been extensively studied by many authors (cf. [**1, 2, 5, 6**] and [**12, 13**]. An element $a \in R$ is quasipolar provided that there exists an idempotent $e \in \mathrm{comm}^2(a)$ such that $a+e \in U(R)$ and $ae \in R^{\mathrm{qnil}}$, where $R^{\mathrm{qnil}} = \{x \in R \mid 1 + xr \in U(R) \text{ for any } r \in \mathrm{comm}(x)\}$. A ring R is called quasipolar if every element in R is quasipolar. As is well known, a ring R is quasipolar if and only if for any $a \in R$ there exists a $b \in \mathrm{comm}^2(a)$ such that $b = bab$ and $b - b^2a \in R^{\mathrm{qnil}}$. This concept has evolved from Banach algebra. In fact, for a Banach algebra R,

$$a \in R^{\mathrm{qnil}} \Leftrightarrow \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0.$$

It is shown that every quasipolar ring is strongly clean. Recently, quasipolar 2×2 matrix rings and triangular matrix rings over local rings were also studied from different point of views (cf. [**7, 9, 11**]).

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The motivation for this article is to introduce a medium class between strongly clean rings and quasipolar rings, and then explore more explicit decompositions of 2×2 matrices over a local ring. An element a of a ring R is called perfectly clean if there exists an idempotent $e \in \text{comm}^2(a)$ such that $a - e \in U(R)$. A ring R is perfectly clean in the case every element in R is perfectly clean. We completely determine when every 2×2 matrix and triangular matrix over local rings are perfectly clean. These also give more explicit characterizations of strong clean matrices over local rings, and enhance many known results, e.g., [**5**, Theorem 8], [**11**, Theorem 2.8] and [**12**, Theorem 7]. Replaced $U(R)$ by $J(R)$, we introduce perfectly J-clean rings as a subclass of perfectly clean rings. Furthermore, we show that strong J-cleanness for triangular matrices over a local ring can be enhanced to such stronger properties. These also generalize the corresponding properties of J-quasipolarity, e.g., [**8**, Theorem 4.9].

We write $U(R)$ and $J(R)$ for the set of all invertible elements and the Jacobson radical of R; $M_n(R)$ and $T_n(R)$ stand for the rings of all $n \times n$ matrices and triangular matrices over a ring R.

2. Perfect rings

Clearly, an abelian exchange ring is perfectly clean. Every quasipolar ring is perfectly clean. For instance, every strongly π-regular ring. In fact, we have $\{$quasipolar rings$\} \subsetneq \{$perfectly clean rings$\} \subsetneq \{$strongly clean rings$\}$. In this section, we explore the properties of perfect rings, which will be used in the sequel. We begin with

THEOREM 2.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is perfectly clean.*
(2) *For any $a \in R$, there exists an $x \in \text{comm}^2(a)$ such that $x = xax$ and $1 - x \in (1 - a)R \cap R(1 - a)$.*

PROOF. (1) $\Rightarrow$ (2) For any $a \in R$, there exists an idempotent $e \in \text{comm}^2(a)$ such that $u := a - e \in U(R)$. Set $x = u^{-1}(1 - e)$. Let $y \in \text{comm}(a)$. Then $ay = ya$. As $uy = (a - e)y = y(a - e) = yu$, we get $u^{-1}y = yu^{-1}$. Thus, $xy = u^{-1}(1 - e)y = u^{-1}y(1 - e) = yu^{-1}(1 - e) = yx$. This implies that $x \in \text{comm}^2(a)$. Further, $xax = u^{-1}(1 - e)(u + e)u^{-1}(1 - e) = u^{-1}(1 - e) = x$. Clearly, $u = (1 - e) - (1 - a)$, and so $1 - u^{-1}(1 - e) = u^{-1}(1 - a)$. This implies that $1 - x \in R(1 - a)$. Likewise, $1 - x \in (1 - a)R$ as $(1 - e)u^{-1} = u^{-1}(1 - e)$. Therefore $1 - x \in (1 - a)R \cap R(1 - a)$, as required.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists an $x \in \text{comm}^2(a)$ such that $x = xax$ and $1 - x \in (1 - a)R \cap R(1 - a)$. Write $e = 1 - ax$. If $y \in \text{comm}(a)$, then $ay = ya$, and so $axy = ayx = yax$. This shows that $ey = ye$; hence, $e \in \text{comm}^2(a)$. In addition, $ex = xe = 0$. Write $1 - x = (1 - a)s = t(1 - a)$ for some $s, t \in R$. Then

$$\begin{aligned}(a - e)(x - es) &= ax - aes + es = ax + (1 - a)es \\ &= ax + e(1 - a)s = ax + e(1 - x) = ax + e = 1.\end{aligned}$$

Likewise, $(x - te)(a - e) = 1$. Therefore $a - e \in U(R)$, as desired. □

Corollary 2.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is perfectly clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in \mathrm{comm}^2(a)$ such that $eae \in U(eRe)$ and $(1-e)(1-a)(1-e) \in U\big((1-e)R(1-e)\big)$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, it follows from Theorem 2.1 that there exists an $x \in \mathrm{comm}^2(a)$ such that $x = xax$ and $1-x \in (1-a)R \cap R(1-a)$. Write $1-x = (1-a)s = t(1-a)$ for some $s, t \in R$. Set $e = ax$. For any $y \in \mathrm{comm}(a)$, we have $ay = ya$, and so $ey = (ax)y = a(yx) = (ay)x = y(ax) = ye$. Hence, $e^2 = e \in \mathrm{comm}^2(a)$. Clearly, $(eae)(exe) = (exe)(eae) = e$; hence, $eae \in U(eRe)$. Furthermore, $1-e = (1-x)+(1-a)x = (1-a)(s+x)$. This shows that $(1-e)(1-a)(1-e)(1-x)(1-e) = 1-e$. Likewise, $(1-e)(1-x)(1-e)(1-e)(1-a)(1-e) = 1-e$. Therefore $(1-e)(1-a)(1-e) \in U\big((1-e)R(1-e)\big)$.

(2) $\Rightarrow$ (1) For any $a \in R$, we have an idempotent $e \in \mathrm{comm}^2(a)$ such that $eae \in U(eRe)$ and $(1-e)(1-a)(1-e) \in U\big((1-e)R(1-e)\big)$. Hence, $a-(1-e) = \big(eae - (1-e)(1-a)(1-e)\big) \in U(R)$. Set $p = 1-e$. Then $a - p \in U(R)$ with $p \in \mathrm{comm}^2(a)$, as desired. □

Recall that a ring R is strongly nil clean provide that every element in R is the sum of an idempotent and a nilpotent element that commutate (cf. [**4**] and [**10**]).

Theorem 2.2. *Let R be a ring. Then R is strongly nil clean if and only if*
(1) *R is perfectly clean,* (2) *$N(R) = \{x \in R \mid 1-x \in U(R)\}$.*

Proof. Let R be strongly nil clean. For any $a \in R$, we see that $a - a^2 \in N(R)$. Write $(a-a^2)^n = 0$. Let $f(t) = \sum_{i=0}^{n} \binom{2n}{i} t^{2n-i}(1-t)^i \in \mathbb{Z}[t]$. Then we have $f(t) \equiv 0 \pmod{t^n}$. Clearly,

$$f(t) + \sum_{i=n+1}^{2n} \binom{2n}{i} t^{2n-i}(1-t)^i = (t+(1-t))^n = 1;$$

hence, $f(t) \equiv 1 \pmod{(1-t)^n}$. This shows that $f(t)(1-f(t)) \equiv 0 \pmod{t^n(1-t)^n}$. Let $e = f(a)$. Then $e \in R$ is an idempotent. For any $x \in \mathrm{comm}(a)$, we see that $xa = ax$, and so $xe = xf(a) = f(a)x = ex$. Thus, $e \in \mathrm{comm}^2(a)$. Furthermore, $a - e = a - a^{2n} + (a-a^2)g(a) = (a-a^2)\big(1 + a + a^2 + \cdots + a^{2n-2} + g(a)\big) \in N(R)$, where $g(t) \in \mathbb{Z}[t]$. Thus, $a = (1-e) + (2e-1+a-e)$ with $1-e \in \mathrm{comm}^2(a)$ and $2e-1+a-e \in U(R)$. Therefore, R is perfectly clean.

Clearly, $N(R) \subseteq \{x \in R \mid 1-x \in U(R)\}$. If $1-x \in U(R)$, then $x = e+w$ with $e \in \mathrm{comm}(x)$ and $w \in N(R)$. Hence, $1-e = (1-x)+w \in U(R)$. This implies that $1-e = 1$, and so $x = w \in N(R)$. Therefore $N(R) = \{x \in R \mid 1-x \in U(R)\}$.

Conversely, assume that (1) and (2) hold. For any $a \in R$, there exist an idempotent $e \in \mathrm{comm}^2(a)$ and a unit $u \in R$ such that $-a = e - u$. Hence, $a = -e + u = (1-e) - (1-u)$. By hypothesis, $1-u \in N(R)$. Accordingly, R is strongly nil clean. □

Corollary 2.2. *Let R be a ring. Then R is strongly nil clean if and only if*
(1) *R is quasipolar;* (2) *$N(R) = \{x \in R \mid 1-x \in U(R)\}$.*

Proof. Suppose that R is strongly nil clean. Then (2) holds by Theorem 2.2. For any $a \in R$, as in the proof of Theorem 2.2, $a = e + w$ with $e \in \mathrm{comm}^2(a)$ and $w \in N(R)$. Hence, $a = (1-e)+(2e-1+w)$ where $2e-1+w \in U(R)$. Furthermore, $(1-e)a = (1-e)w \in N(R) \subseteq R^{\mathrm{qnil}}$. Therefore R is quasipolar.

Conversely, assume that (1) and (2) hold. Then R is perfectly clean. Accordingly, we complete the proof by Theorem 2.2. □

Lemma 2.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is perfectly clean.*
(2) *For each $a \in R$ there exists an idempotent $e \in \mathrm{comm}^2(a)$ such that $a - e$ and $a + e$ are invertible.*

Proof. (1) ⇒ (2) Let $a \in R$. Then $a^2 \in R$ is perfectly clean. Thus, we can find an idempotent $e \in \mathrm{comm}^2(a^2)$ such that $a^2 - e \in U(R)$. Since $a \cdot a^2 = a^2 \cdot a$, we see that $ae = ea$. Hence, $a^2 - e = (a-e)(a+e)$, and therefore we conclude that $a - e, a + e \in U(R)$.

(2) ⇒ (1) is trivial. □

Theorem 2.3. *Let R be perfectly clean. Then for any $A \in M_n(R)$ there exist $U, V \in \mathrm{GL}_n(R)$ such that $2A = U + V$.*

Proof. We prove the result by induction on n. For any $a \in R$, there exists an idempotent $e \in \mathrm{comm}^2(a)$ such that $u := a - e, v := a + e \in U(R)$, by Lemma 2.1. Hence, $2a = u + v$, and so the result holds for $n = 1$. Assume that the result holds for $n \leqslant k$ $(k \geqslant 1)$. Let $n = k + 1$, and let $A \in M_n(R)$. Write $A = \left(\begin{smallmatrix} x & \alpha \\ \beta & X \end{smallmatrix}\right)$, where $x \in R, \alpha \in M_{1\times k}(R), \beta \in M_{k\times 1}(R)$ and $X \in M_k(R)$. In view of Lemma 2.1, we have a $u \in U(R)$ such that $2x - u = v \in U(R)$. By hypothesis, we have a $U \in \mathrm{GL}_k(R)$ such that $2\big(X - 2\beta v^{-1}\alpha\big) - U = V \in \mathrm{GL}_k(R)$. Hence

$$2A - \begin{pmatrix} u & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} v & 2\alpha \\ 2\beta & V + 4\beta v^{-1}\alpha \end{pmatrix}.$$

It is easy to verify that

$$\begin{pmatrix} v & 2\alpha \\ 2\beta & V + 4\beta v^{-1}\alpha \end{pmatrix} = \begin{pmatrix} 1 & \\ 2\beta v^{-1} & I_k \end{pmatrix} \begin{pmatrix} v & 2\alpha \\ 0 & V \end{pmatrix} \in \mathrm{GL}_n(R).$$

By induction, we complete the proof. □

Corollary 2.3. *Let R be a quasipolar ring. If $\frac{1}{2} \in R$, then every $n \times n$ matrix over R is the sum of two invertible matrices.*

Proof. As every quasipolar ring is perfectly clean, the proof follows by Theorem 2.3. □

As a consequence, we derive the following known fact: Let R be a strongly π-regular ring with $\frac{1}{2} \in R$. Then every $n \times n$ matrix over R is the sum of two invertible matrices.

3. Matrices and triangular matrices

Recall that a ring R is local if it has only one maximal right ideal. A ring R is local if and only if for any $a \in R$ either a or $1-a$ is invertible. Necessary and sufficient conditions under which 2×2 matrices over a local ring are attractive. In this section, we extend these known results on strongly clean matrices to perfect cleanness.

LEMMA 3.1. *Let R be a ring, and $u \in U(R)$. Then the following are equivalent:*
(1) *$a \in R$ is perfectly clean.* (2) *$uau^{-1} \in R$ is perfectly clean.*

PROOF. (1) $\Rightarrow$ (2) By hypothesis, there exists an idempotent $e \in \mathrm{comm}^2(a)$ such that $a-e \in U(R)$. Hence, $uau^{-1}-ueu^{-1} \in U(R)$. For any $x \in \mathrm{comm}(uau^{-1})$, we see that $x(uau^{-1}) = (uau^{-1})x$, and so $(u^{-1}xu)a = a(u^{-1}xu)$. Thus, $(u^{-1}xu)e = e(u^{-1}xu)$. Hence $x(ueu^{-1}) = (ueu^{-1})x$. We conclude that $ueu^{-1} \in \mathrm{comm}^2(uau^{-1})$, as required.

(2) $\Rightarrow$ (1) is symmetric. □

A ring is *weakly cobleached* provided that for any $a \in J(R)$, $b \in 1+J(R)$, $l_a - r_b$ and $l_b - r_a$ are both injective. For instance, every commutative local ring, every local ring with nil Jacobson radical.

THEOREM 3.1. *Let R be a weakly cobleached local ring. Then the following are equivalent:*
(1) *$M_2(R)$ is perfectly clean.* (2) *$M_2(R)$ is strongly clean.*
(3) *For any $A \in M_2(R)$, $A \in \mathrm{GL}_2(R)$, or $I_2 - A \in \mathrm{GL}_2(R)$, or A is similar to a diagonal matrix.*

PROOF. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) is obtained by [**13**, Theorem 7].

(3) $\Rightarrow$ (1) For any $A \in M_2(R)$, $A \in \mathrm{GL}_2(R)$, or $I_2 - A \in \mathrm{GL}_2(R)$, or A is similar to a diagonal matrix. If A or $I_2 - A \in \mathrm{GL}_2(R)$, then A is perfectly clean. Assume now that A is similar to a diagonal matrix with $A, I_2 - A \notin \mathrm{GL}_2(R)$. We may assume that A is similar to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, where $\lambda \in U(R)$, $\mu \in J(R)$. If $\lambda \in 1+U(R)$, then $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) - I_2 \in \mathrm{GL}_2(R)$; hence, it is perfectly clean. In view of Lemma 3.1, A is perfectly clean. Thus, we assume that $\lambda \in 1 + J(R)$. By Lemma 3.1, it will suffice to show that $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$ is perfectly clean. Clearly,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \mu - 1 \end{pmatrix},$$

where $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu-1 \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$.

We show that the idempotent $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{comm}^2\left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)\right)$. For any $\left(\begin{smallmatrix} x & s \\ t & y \end{smallmatrix}\right) \in \mathrm{comm}\left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)\right)$, one has $\lambda s = s\mu$ and $\mu t = t\lambda$; hence, $s = t = 0$. This implies

$$\begin{pmatrix} x & s \\ t & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & s \\ t & y \end{pmatrix}.$$

Therefore $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{comm}^2\left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)\right)$, hence the result. □

Corollary 3.1. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *$M_2(R)$ is perfectly clean.* (2) *$M_2(R)$ is strongly clean.*
(3) *For any $A \in M_2(R)$, $A \in \mathrm{GL}_2(R)$, or $I_2 - A \in \mathrm{GL}_2(R)$, or A is similar to a diagonal matrix.*

Proof. It is a consequence of Theorem 3.1 as every commutative local ring is weakly cobleached. □

Let p be a prime. We use $\widehat{\mathbb{Z}_p}$ to denote the ring of all p-adic integers. In view of [**6**, Theorem 2.4], $M_2\big(\widehat{\mathbb{Z}_p}\big)$ is strongly clean, and therefore $M_2\big(\widehat{\mathbb{Z}_p}\big)$ is perfectly clean, by Corollary 3.1.

Theorem 3.2. *Let R and S be local rings. Then the following are equivalent:*

(1) *$\left(\begin{smallmatrix} R & V \\ 0 & S \end{smallmatrix}\right)$ is perfectly clean.*
(2) *For any $a \in J(R)$, $b \in 1 + J(S)$, $v \in V$, there exists a unique $x \in V$ such that $ax - xb = v$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in 1 + J(R)$, $b \in J(S)$ and $v \in V$. Set $A = \left(\begin{smallmatrix} a & -v \\ 0 & b \end{smallmatrix}\right)$. By hypothesis, we can find an idempotent $E \in \mathrm{comm}^2(A)$ such that $A - E \in \left(\begin{smallmatrix} R & V \\ 0 & S \end{smallmatrix}\right)$ is invertible. Clearly, $E = \left(\begin{smallmatrix} 0 & x \\ 0 & 1 \end{smallmatrix}\right)$ for some $x \in V$. Thus, $ax - xb = v$. Suppose that $ay - yb = v$ for a $y \in V$. Then

$$A \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} A,$$

and so $\left(\begin{smallmatrix} 0 & y \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{comm}(A)$. This implies that

$$E \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} E;$$

hence, $x = y$. Therefore there exists a unique $x \in V$ such that $ax - xb = v$, as desired.

(2) $\Rightarrow$ (1) Let $T = \left(\begin{smallmatrix} R & V \\ 0 & S \end{smallmatrix}\right)$, and let $A = \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right) \in \left(\begin{smallmatrix} R & V \\ 0 & S \end{smallmatrix}\right)$.

Case I. $a \in J(R), b \in J(S)$. Then $A - \left(\begin{smallmatrix} 1_R & 0 \\ 0 & 1_S \end{smallmatrix}\right) \in U(T)$; hence, A is perfectly clean.

Case II. $a \in U(R), b \in U(S)$. Then $A - 0 \in U(T)$; hence, A is perfectly clean.

Case III. $a \in U(R)$, $b \in J(S)$. (i) $a \in 1 + U(R)$, $b \in J(S)$. Then $A - \left(\begin{smallmatrix} 1_R & 0 \\ 0 & 1_S \end{smallmatrix}\right) \in T$ is invertible; hence, $A \in T$ is perfectly clean. (ii) $a \in 1 + J(R)$, $b \in J(S)$. Then we can find a $t \in V$ such that $at - tb = -v$. Let $\left(\begin{smallmatrix} x & s \\ 0 & y \end{smallmatrix}\right) \in \mathrm{comm}(A)$. Then

$$A \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} A,$$

and so $ax = xa$, $by = yb$, and $as - sb = xv - vy$. Hence, we check that

$$\begin{aligned} a(xt - ty + s) - (xt - ty + s)b &= x(at - tb) - (at - tb)y + (as - sb) \\ &= -xv + vy + (as - sb) \\ &= 0. \end{aligned}$$

By hypothesis, $xt - ty = -s$, and so we get

$$\begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & ty \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & xt+s \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix}.$$

We infer that

$$\begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} \in \text{comm}^2(A).$$

Furthermore, $A - \left(\begin{smallmatrix} 0 & t \\ 0 & 1 \end{smallmatrix}\right) \in U(T)$. Therefore A is perfectly clean.

Case IV. $a \in J(R)$, $b \in U(S)$ Then A is perfectly clean, as in the preceding discussion. □

A ring R is *uniquely weakly bleached* provided that for any $a \in J(R)$, $b \in 1 + J(R)$, $l_a - r_b$ and $l_b - r_a$ are both isomorphisms.

Corollary 3.2. *Let R be local. Then the following are equivalent:*

(1) *$T_2(R)$ is perfectly clean.* (2) *R is uniquely weakly bleached.*

Proof. It is clear by Theorem 3.2. □

For instance, if R is a commutative local ring or a local ring with nil Jacobson radical, then $T_2(R)$ is perfectly clean.

4. Perfectly J-clean rings

An element $a \in R$ is said to be perfectly J-clean provided that there exists an idempotent $e \in \text{comm}^2(a)$ such that $a - e \in J(R)$. A ring R is perfectly J-clean if every element in R is perfectly J-clean.

Theorem 4.1. *Let R be a ring. Then R is perfectly J-clean if and only if*

(1) *R is quasipolar.* (2) *$R/J(R)$ is Boolean.*

Proof. Suppose that R is perfectly J-clean. Let $a \in R$ is perfectly J-clean. Then there exists an idempotent $e \in \text{comm}^2(a)$ such that $w := a - e \in J(R)$. Hence, $a - (1-e) = 2e - 1 + w \in U(R)$. Additionally, $(1-e)a = (1-e)w \in J(R) \subseteq R^{\text{qnil}}$. This implies that $a \in R$ is quasipolar. Furthermore, $a - a^2 = (e+w) - (e+w)^2 \in J(R)$, and then $R/J(R)$ is Boolean.

Conversely, assume that (1) and (2) hold. Let $a \in R$. Then there exists an idempotent $e \in \text{comm}^2(a)$ such that $u := a - e \in U(R)$. Moreover, $R/J(R)$ is Boolean, and so $a - a^2 = (e+u) - (e+u)^2 = u(1 - 2e - u) \in J(R)$. This shows that $1 - 2e - u \in J(R)$, whence $a - (1-e) = (e+u) - (1-e) = 2e - 1 + u \in J(R)$. Therefore R is perfectly J-clean. □

Corollary 4.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is perfectly J-clean.*
(2) *R is perfectly clean, and $R/J(R)$ is Boolean.*
(3) *R is quasipolar, and R is strongly J-clean.*

Proof. (1) $\Rightarrow$ (2) is obvious by Theorem 4.1, as every quasipolar ring is perfectly clean.

(2) $\Rightarrow$ (1) For any $a \in R$ there exists an idempotent $p \in \text{comm}^2(a)$ such that $u := a - p \in U(R)$. As $R/J(R)$ is Boolean, we have $\bar{u} = \bar{u}^2$; hence, $u \in 1 + J(R)$. Furthermore, $2 \in J(R)$. Accordingly, $a = p + u = (1 - p) + (2p - 1 + u)$ with $1 - p \in \text{comm}^2(a)$ and $2p - 1 + u \in J(R)$, as desired.

(1) $\Rightarrow$ (3) Suppose R is perfectly J-clean. Then R is strongly J-clean. By the preceding discussion, R is quasipolar.

(3) $\Rightarrow$ (1) Since R is strongly J-clean, $R/J(R)$ is Boolean. Therefore the proof is complete by the discussion above. □

Example 4.1. Let $R = T_2(\mathbb{Z}_{2^n})$ $(n \in \mathbb{N})$. Then $T_2(R)$ is perfectly J-clean.

Proof. As R is finite, it is periodic. This shows that R is strongly π-regular. Hence, $T_2(R)$ is quasipolar, by [**9**, Theorem 2.6]. As $J(\mathbb{Z}_{2^n}) = 2\mathbb{Z}_{2^n}$, we see that $R/J(R) \cong \mathbb{Z}_2$ is Boolean. Hence, $T_2(R)/J(T_2(R))$ is Boolean. Therefore the result follows by Theorem 4.1. □

Recall that a ring R is uniquely strongly clean provided that for any $a \in R$ there exists a unique idempotent $e \in \text{comm}(a)$ such that $a - e \in U(R)$.

Proposition 4.1. *Let R be a ring. Then R is perfectly J-clean if and only if*

(1) *R is perfectly clean,* (2) *R is uniquely strongly clean.*

Proof. Suppose R is perfectly J-clean. Then R is perfectly clean. Hence, R is strongly clean. Let $a \in R$. Write $a = e + u = f + v$ with $e = e^2 \in \text{comm}^2(a)$, $f = f^2 \in R$, $u \in J(R)$, $v \in U(R)$, $ea = ae$ and $fa = af$. Then $f \in \text{comm}(a)$, and so $ef = fe$. Thus, $e - f = v - u \in U(R)$ and $(e - f)(e + f - 1) = 0$. This implies that $f = 1 - e$, and therefore R is uniquely strongly clean.

Conversely, assume that (1) and (2) hold. Then $R/J(R)$ is Boolean. Therefore we complete the proof by Corollary 4.1. □

Corollary 4.2. *A ring R is uniquely clean if and only if R is abelian perfectly J-clean.*

Proof. As every uniquely clean ring is abelian (cf. [**4**, Corollary 16.4.16]), it is clear by Proposition 4.1. □

Theorem 4.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is perfectly J-clean.*
(2) *For any $a \in R$, there exists a unique idempotent $e \in \text{comm}^2(a)$ such that $a - e \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, there exists an idempotent $e \in \text{comm}^2(a)$ such that $a-e \in J(R)$. Assume that $a-f \in J(R)$ for an idempotent $f \in \text{comm}^2(a)$. Clearly, $e \in \text{comm}^2(a) \subseteq \text{comm}(a)$. As $f \in \text{comm}^2(a)$, we see that $ef = fe$. Thus, $(e-f)^3 = e-f$, and so $(e-f)(1-(e-f)^2) = 0$. But $e-f = (a-f)-(a-e) \in J(R)$, as $a - f, a - e \in J(R)$. Hence, $e = f$, as desired.

(2) $\Rightarrow$ (1) is trivial. □

Recall that a ring R is strongly J-clean provided that for any $a\in R$ there exists an idempotent $e\in \mathrm{comm}(a)$ such that $a-e\in J(R)$ (cf. [**3, 4**]).

Corollary 4.3. *A ring R is perfectly J-clean if and only if*

(1) *R is quasipolar,* (2) *R is strongly J-clean.*

Proof. Suppose R is perfectly J-clean. Then R is strongly J-clean. For any $a\in R$, there exists an idempotent $p\in \mathrm{comm}^2(a)$ such that $w:=a-p\in J(R)$. Hence, $a=(1-p)+(2p-1+w)$ with $1-p\in \mathrm{comm}^2(a)$ and $2p-1+w\in U(R)$. Furthermore, $(1-p)a=(1-p)w\in J(R)\subseteq R^{\mathrm{qnil}}$. Therefore, R is quasipolar.

Conversely, assume that (1) and (2) hold. Since R is quasipolar, it is perfectly clean. By virtue of [**4**, Proposition 16.4.15], $R/J(R)$ is Boolean. Therefore the proof is complete by Corollary 4.1. □

Following Cui and Chen [**8**], a ring R is called J-quasipolar provided that for any element $a\in R$ there exists an $e\in \mathrm{comm}^2(a)$ such that $a+e\in J(R)$. We further show that the two concepts coincide. But this is not the case for a single element. That is,

Proposition 4.2. *A ring R is perfectly J-clean if and only if for any element $a\in R$ there exists an $e\in \mathrm{comm}^2(a)$ such that $a+e\in J(R)$.*

Proof. Let R be perfectly J-clean. Then $R/J(R)$ is Boolean, by Theorem 4.1. Hence, $\overline{2}^2=\overline{2}$, i.e., $2\in J(R)$. For any $a\in R$, there exists an idempotent $e\in \mathrm{comm}^2(a)$ such that $a-e\in J(R)$. This implies that $a+e=(a-e)+2e\in J(R)$. The converse is similar by [**8**, Corollary 2.3]. □

Example 4.2. Let $R=\mathbb{Z}_3$. Note that $J(R)=0$. Since $\overline{1}-\overline{1}=\overline{0}\in J(R)$, $\overline{1}$ is perfectly J-clean, but we can not find an idempotent $e\in R$ such that $\overline{1}+e\in J(R)$, because $\overline{1}+\overline{0}\notin J(R)$ and $\overline{1}+\overline{1}=\overline{2}\notin J(R)$.

Further, though $\overline{2}+\overline{1}=\overline{0}\in J(R)$, we can not find an idempotent $e\in R$ such that $\overline{2}-e\in J(R)$, because $\overline{2}-\overline{0}=\overline{2}\notin J(R)$ and $\overline{2}-\overline{1}=\overline{1}\notin J(R)$.

Lemma 4.1. *Let R be a ring. Then $a\in R$ is perfectly J-clean if and only if*

(1) *$a\in R$ is quasipolar,* (2) *$a-a^2\in J(R)$.*

Proof. Suppose that $a\in R$ is perfectly J-clean. Then there exists an idempotent $e\in \mathrm{comm}^2(a)$ such that $w:=a-e\in J(R)$. Hence, $a-(1-e)=2e-1+w\in U(R)$. Additionally, $(1-e)a=(1-e)w\in J(R)\subseteq R^{\mathrm{qnil}}$. This implies that $a\in R$ is quasipolar. Furthermore, $(e+w)-(e+w)^2=-(2e-1+w)w\in J(R)$.

Conversely, assume that (1) and (2) hold. Then there exists an idempotent $e\in \mathrm{comm}^2(-a)$ such that $(-a)+e\in U(R)$. Set $u:=a-e$. Then $a-a^2=(e+u)-(e+u)^2=u(1-2e-u)\in J(R)$; hence, $1-2e-u\in J(R)$. This shows that $a-(1-e)=(e+u)-(1-e)=2e-1+u\in J(R)$. Therefore $a\in R$ is perfectly J-clean. □

Theorem 4.3. *Let R be a commutative ring, and let $A\in T_n(R)$. If $2\in J(R)$, then the following are equivalent:*

(1) *$A\in T_n(R)$ is perfectly J-clean.* (2) *Each $A_{ii}\in T_n(R)$ is perfectly J-clean.*

PROOF. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Clearly, the result holds for $n = 1$. Suppose that the result holds for $n-1$ ($n \geqslant 2$). Let $A = \left(\begin{smallmatrix} a_{11} & \alpha \\ 0 & A_1 \end{smallmatrix}\right) \in T_n(R)$ where $a_{11} \in R, \alpha \in M_{1\times(n-1)}(R)$ and $A_1 \in T_{n-1}(R)$. Then we have an idempotent $e_{11} \in R$ such that $w_{11} := a_{11} - e_{11} \in J(R)$. By hypothesis, we have an idempotent $E_1 \in T_{n-1}(R)$ such that $W_1 := A_1 - E_1 \in J\big(T_{n-1}(R)\big)$ and $E_1 \in \mathrm{comm}^2(A_1)$. As $2 \in J(R)$,

$$W_1 + \big(1 - 2e_{11} - w_{11}\big)I_{n-1} \in I_{n-1} + J\big(T_{n-1}(R)\big) \subseteq U\big(T_{n-1}(R)\big).$$

Let $E = \left(\begin{smallmatrix} e_{11} & \beta \\ 0 & E_1 \end{smallmatrix}\right)$, where $\beta = \alpha(E_1 - e_{11}I_{n-1})(W_1 + (1-2e_{11}-w_{11})I_{n-1})^{-1}$. Then $A - E \in J\big(T_n(R)\big)$. As

$$\begin{aligned} e_{11}\beta + \beta E_1 &= \beta(E_1 + e_{11}I_{n-1}) \\ &= \alpha(E_1 - e_{11}I_{n-1})(E_1 + e_{11}I_{n-1})(W_1 + (1-2e_{11}-w_{11})I_{n-1})^{-1} = \beta, \end{aligned}$$

we see that $E = E^2$.

For any $X = \left(\begin{smallmatrix} x_{11} & \gamma \\ 0 & X_1 \end{smallmatrix}\right) \in \mathrm{comm}(A)$, we have $x_{11}\alpha + \gamma A_1 = a_{11}\gamma + \alpha X_1$; hence,

$$\alpha(X_1 - x_{11}I_{n-1}) = \gamma(A_1 - a_{11}I_{n-1}).$$

As $E_1 \in \mathrm{comm}^2(A_1)$, we get

$$\begin{aligned} &\gamma(A_1 - a_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ &\quad = \alpha(X_1 - x_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ &\quad = \alpha(E_1 - e_{11}I_{n-1})(X_1 - x_{11}I_{n-1}) \\ &\quad = \beta(W_1 + (1-2e_{11}-w_{11})I_{n-1})(X_1 - x_{11}I_{n-1}) \\ &\quad = \beta(X_1 - x_{11}I_{n-1})(W_1 + (1-2e_{11}-w_{11})I_{n-1}). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\gamma(A_1 - a_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ &\quad = \gamma(E_1 - e_{11}I_{n-1})(E_1 + W_1 - (e_{11}+w_{11})I_{n-1}) \\ &\quad = \gamma(E_1 - e_{11}I_{n-1})(E_1 + e_{11}I_{n-1} + (W_1 - 2e_{11} - w_{11})I_{n-1}) \\ &\quad = \gamma\big(E_1 - e_{11}I_{n-1} + (E_1 - e_{11}I_{n-1})(W_1 - 2e_{11} - w_{11})I_{n-1}\big) \\ &\quad = \gamma(E_1 - e_{11}I_{n-1})(W_1 + (1-2e_{11}-w_{11})I_{n-1}). \end{aligned}$$

It follows from $W_1 + (1-2e_{11}-w_{11})I_{n-1} \in U\big(T_{n-1}(R)\big)$ that $\gamma(E_1 - e_{11}I_{n-1}) = \beta(X_1 - x_{11}I_{n-1})$. Hence, $e_{11}\gamma + \beta X_1 = x_{11}\beta + \gamma E_1$, and so $EX = XE$. This implies that $E \in \mathrm{comm}^2(A)$. By induction, $A \in T_n(R)$ is perfectly J-clean for all $n \in \mathbb{N}$. $\square$

COROLLARY 4.4. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is strongly J-clean.*
(2) *$T_n(R)$ is perfectly J-clean for all $n \in \mathbb{N}$.*
(3) *$T_n(R)$ is perfectly J-clean for some $n \in \mathbb{N}$.*

PROOF. (1) $\Rightarrow$ (2) As R is strongly J-clean, $R/J(R)$ is Boolean. Hence, $2 \in J(R)$. For any $n \in \mathbb{N}$, $T_n(R)$ is perfectly J-clean by Theorem 4.3.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) These are clear by Theorem 4.3. □

Let R be Boolean. As a consequence of Corollary 4.4, $T_n(R)$ is perfectly J-clean for all $n \in \mathbb{N}$.

LEMMA 4.2. *Let R be a ring, and $u \in U(R)$. Then the following are equivalent:*

(1) *$a \in R$ is perfectly J-clean.* (2) *$uau^{-1} \in R$ is perfectly J-clean.*

PROOF. (1) $\Rightarrow$ (2) As in the proof of Lemma 3.1, $uau^{-1} \in R$ is quasipolar. Furthermore, $uau^{-1} - \big(uau^{-1}\big)^2 = u(a - a^2)u^{-1} \in J(R)$. As in the proof of Theorem 4.1, $uau^{-1} \in R$ is perfectly J-clean.

(2) $\Rightarrow$ (1) is symmetric. □

We end this paper by showing that strong J-cleanness of 2×2 matrix ring over a commutative local ring can be enhanced to perfect J-cleanness.

THEOREM 4.4. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *A is perfectly J-clean.* (2) *A is strongly J-clean.*

(3) *$A \in J\big(M_2(R)\big)$, or $I_2 - A \in J\big(M_2(R)\big)$, or the equation $x^2 - tr(A)x + \det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.*

PROOF. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) is proved by [**4**, Theorem 16.4.31].

(3) $\Rightarrow$ (1) If $A \in J\big(M_2(R)\big)$ or $I_2 - A \in J(M_2(R))$, then A is perfectly J-clean. Otherwise, it follows from [**4**, Theorem 16.4.31 and Proposition 16.4.26] that there exists a $U \in \mathrm{GL}_2(R)$ such that

$$UAU^{-1} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \alpha & \\ & \beta - 1 \end{pmatrix},$$

where $\alpha \in J(R), \beta \in 1 + J(R)$. For any $X \in \mathrm{comm}(UAU^{-1})$, we have $X\left(\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right)X$; hence, $\beta X_{12} = \alpha X_{12}$. This implies that $X_{12} = 0$. Likewise, $X_{21} = 0$. Thus,

$$X \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} X,$$

and so $\left(\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix}\right) \in \mathrm{comm}^2(UAU^{-1})$. As a result, UAU^{-1} is perfectly J-clean, and then so is A by Lemma 4.2. □

COROLLARY 4.5. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *$M_2(R)$ is perfectly clean.*

(2) *For any $A \in M_2(R)$, $A \in \mathrm{GL}_2(R)$, or $I_2 - A \in \mathrm{GL}_2(R)$, or $A \in M_2(R)$ is perfectly J-clean.*

PROOF. (1) $\Rightarrow$ (2) is proved by Theorem 3.1, [**4**, Corollary 16.4.33] and Theorem 4.4.

(2) $\Rightarrow$ (1) is obvious. □

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