PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 101(115) (2017), 223–230 https://doi.org/10.2298/PIM1715223Y

# CLASSIFICATION OF PRODUCT SHAPED HYPERSURFACES IN LORENTZ SPACE FORMS

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Abstract. We define the product shaped hypersurfaces in Lorentz space forms by imposing the shape operator to be product type. Based on the classification of the isoparametric hypersurfaces, we obtain the whole families of the product shaped hypersurfaces in Minkowski, de Sitter and anti-de Sitter spaces.

## 1. Introduction

Let $\mathbb{R}^n_k$ be an $n$-dimensional real vector space together with an inner product given by

$$\langle x, x\rangle = -\sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{n} x_j^2,$$

where $x = (x_1, \ldots, x_n)$ is the natural coordinate of $\mathbb{R}^n_k$ and $\mathbb{R}^n_k$ is called an $n$-dimensional semi-Euclidean space. We recall that the semi-Riemannian manifolds $\mathbb{S}^n_k(c)$ and $\mathbb{H}^n_k(c)$ are as follows:

$$\mathbb{S}^n_k(c) = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_k \,\middle|\, -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n+1} x_i^2 = \frac{1}{c}\right\}, (c > 0),$$

$$\mathbb{H}^n_k(c) = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_{k+1} \,\middle|\, -\sum_{i=1}^{k+1} x_i^2 + \sum_{i=k+2}^{n+1} x_i^2 = \frac{1}{c}\right\}, (c < 0).$$

These spaces are complete ones with constant curvature $c$. $\mathbb{S}^n_k(c)$ and $\mathbb{H}^n_k(c)$ are called semi-sphere and semi-hyperbolic space, respectively. In general relativity, the Lorentz manifolds $\mathbb{R}^n_1$, $\mathbb{S}^n_1(c)$ and $\mathbb{H}^n_1(c)$ are respectively known as the Minkowski, de Sitter and anti-de Sitter space, which is called Lorentz space form and is denoted by $\mathbb{N}^n_1(c)$.

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2010 *Mathematics Subject Classification:* 53C42.
*Key words and phrases:* product shaped hypersurface, isoparametric hypersurface, Lorentz space form.
Communicated by Stevan Pilipović.

The notion of golden structure on a manifold was introduced in [**2, 3**] as a tensor field of (1,1)-type $J$ on $M$ which satisfies the same equation as the golden ratio: $J^2 = J + I$, where $I$ is the usual Kronecker tensor field of $M$. In real space form, Crasmareanu–Hretcanu–Munteanu [**1**] gave the definition of the golden shaped hypersurfaces and product shaped hypersurfaces, and obtained the classification of this two kinds of hypersurfaces. In Lorentz space form, there are two kinds of hypersurfaces according to the index of the hypersurfaces, that is, spacelike hypersurfaces and Lorentz hypersurfaces. A hypersurface in a Lorentz space form $\mathbb{N}_1^{n+1}(c)$ is said to be *spacelike* if the induced metric on the hypersurface from that of the Lorentz space is positive definite. A hypersurface in a Lorentz space form $\mathbb{N}_1^{n+1}(c)$ is said to be *Lorentz* if the induced metric on the hypersurface is indefinite and the index is one. In the Lorentz space form, the first author and Fu [**6**] defined the golden shaped hypersurfaces and gave the whole families of the golden shaped hypersurfaces. Since, in [**2**], there is a natural correspondence derived between golden structures and almost product structures, it is necessary to study the product shaped version in the Lorentz space form.

Let $M^n$ be a hypersurface in the Lorentz space form $\mathbb{N}_1^{n+1}(c)$ and for a certain normal vector field $N$, let $A = A_N$ be the associated shape operator. We firstly give the following definition.

Definition 1.1. A hypersurface $M^n$ in a Lorentz space form $\mathbb{N}_1^{n+1}(c)$ is called a *product shaped hypersurface* if $A^2 = I$.

In this paper, we will give the complete classification of the product shaped hypersurfaces in Lorentz space forms. Our results state that there are two kinds of hypersurfaces in the Minkowski space $\mathbb{R}_1^{n+1}$, three kinds of hypersurfaces in the de Sitter space $\mathbb{S}_1^{n+1}(1)$ and three kinds of hypersurfaces in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$, respectively. We find that there are large differences between the golden shaped hypersurface and the product shaped hypersurface.

## 2. The classification of product shaped hypersurfaces

In this section, we will give the complete classification of the product shaped hypersurfaces in the Minkowski space, the de Sitter space and the anti-de Sitter space, respectively. In order to give the proof of the theorems, we firstly give the following proposition.

Proposition 2.1. *The product shaped hypersurfaces in a Lorentz space form $\mathbb{N}_1^{n+1}(c)$ are isoparametric hypersurfaces with two distinct principle curvatures* 1 *and* $-1$*.*

Proof. If $M$ is a spacelike hypersurface in $\mathbb{N}_1^{n+1}(c)$, the normal vector is timelike and the shape operator $A$ can be diagonalized by choosing the orthogonal frame field on $M$. Denote the principal curvatures of the spacelike hypersurface by $\lambda_1, \ldots, \lambda_n$. By $A^2 = I$, the principal curvatures of the spacelike product shaped hypersurface are 1 or $-1$ respectively, and hence the hypersurface is isoparametric.

If $M$ is a Lorentz hypersurface in $\mathbb{N}_1^{n+1}(c)$, the normal vector is spacelike. By [**4**], the shape operator maybe have the following four forms

$$(1)\ A = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix} \qquad (2)\ A = \begin{pmatrix} a_0 & 0 & 0 & \dots & 0 \\ 1 & a_0 & 0 & \dots & 0 \\ 0 & 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-2} \end{pmatrix}$$

$$(3)\ A = \begin{pmatrix} a_0 & 0 & 0 & 0 & \dots & 0 \\ 0 & a_0 & 1 & 0 & \dots & 0 \\ -1 & 0 & a_0 & 0 & \dots & 0 \\ 0 & 0 & 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & a_{n-3} \end{pmatrix} \qquad (4)\ A = \begin{pmatrix} a_0 & b_0 & 0 & \dots & 0 \\ -b_0 & a_0 & 0 & \dots & 0 \\ 0 & 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-2} \end{pmatrix}.$$

We consider the four forms one by one for the Lorentz product shaped hypersurface.

(a) If the shape operator has the form (1), by $A^2 = I$, we get $a_i = 1$ or $-1$ for $i = 1, 2 \dots, n$. So the principal curvatures of the hypersurface are $\pm 1$ and the hypersurface is isoparametric.

(b) If the shape operator has the form (2), by $A^2 = I$, we get $a_0^2 = 1$ and $2a_0 = 0$, which is impossible.

(c) If the shape operator has the form (3), by $A^2 = I$, we get $-1 = 0$, which is impossible.

(d) If the shape operator has the form (4), by $A^2 = I$, we get $b_0 = 0$ and $a_i = 1$ or $-1$ for $i = 0, 1 \dots, n-2$. Therefore the principal curvatures of the product shaped hypersurface are $\pm 1$, and the hypersurface is isoparametric. □

Next we will give the proof of the main theorems.

THEOREM 2.1. *The only product shaped hypersurfaces in the Minkowski space* $\mathbb{R}_1^{n+1}$ *are*

(1) *Spacelike hypersurface* $H^n(-1) = \{x \in \mathbb{R}_1^{n+1} \big| -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -1\}$;

(2) *Lorentz hypersurface* $S_1^n(1) = \{x \in \mathbb{R}_1^{n+1} \big| -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = 1\}$.

PROOF. By [**5**], the isoparametric hypersurfaces in Minkowski space $\mathbb{R}_1^{n+1}$ have the following six cases:

$\mathrm{R}^{(1)}$: $R^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}_1^{n+1} \,\big|\, x_1 = 0\}$ with $A = 0$;

$$\mathrm{R}^{(2)}\colon H^n(c) = \left\{x \in \mathbb{R}_1^{n+1} \,\Big|\, -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}\right\} \ (c < 0) \text{ with } A = \pm\sqrt{-c}\,I;$$

$$\mathrm{R}^{(3)}\colon R^r \times H^{n-r} = \left\{x \in \mathbb{R}_1^{n+1} \,\Big|\, -x_1^2 + \sum_{i=2+r}^{n+1} x_i^2 = \frac{1}{c}\right\} \ (c < 0) \text{ with } A = \pm(0_r \oplus \sqrt{-c}\,I_{n-r});$$

$\mathrm{R}^{(4)}$: $R_1^n = \{x \in \mathbb{R}_1^{n+1} \mid x_{n+1} = 0\}$ with $A = 0$;

$\mathrm{R}^{(5)}$: $S_1^n(c) = \left\{x \in \mathbb{R}_1^{n+1} \,\middle|\, -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}\right\}$ $(c > 0)$ with $A = \pm\sqrt{c}\, I$;

$\mathrm{R}^{(6)}$: $R^r \times S_1^{n-r} = \left\{x \in \mathbb{R}_1^{n+1} \,\middle|\, -x_1^2 + \sum_{i=r+2}^{n+1} x_i^2 = \frac{1}{c}\right\}$ $(c > 0)$ with $A = \pm(0_r \oplus \sqrt{c}\, I_{n-r})$.

Here $\mathrm{R}^{(1)}$, $\mathrm{R}^{(2)}$, $\mathrm{R}^{(3)}$ are spacelike hypersurfaces and $\mathrm{R}^{(4)}$, $\mathrm{R}^{(5)}$, $\mathrm{R}^{(6)}$ are Lorentz hypersurfaces.

It follows from Proposition 2.1 that the product shaped hypersurfaces are isoparametric hypersurfaces with two distinct principal curvatures 1 and $-1$, which are nonzero constant. So the cases $\mathrm{R}^{(1)}$,$\mathrm{R}^{(3)}$, $\mathrm{R}^{(4)}$ and $\mathrm{R}^{(6)}$ are impossible.

For the case $\mathrm{R}^{(2)}$, since the eigenvalues of the shape operator are $\pm 1$, so $c = -1$ and the spacelike product shaped hypersurface is

$$H^n(-1) = \left\{x \in \mathbb{R}_1^{n+1} \,\middle|\, -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -1\right\}.$$

This gives case (1) of Theorem 2.1.

For the case $\mathrm{R}^{(5)}$, since the eigenvalues of the shape operator are $\pm 1$, then $c = 1$ and the Lorentz product shaped hypersurface is

$$S_1^n(1) = \left\{x \in \mathbb{R}_1^{n+1} \,\middle|\, -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = 1\right\}.$$

This gives case (2) of Theorem 2.1. □

For $n = 2$, we give the pictures of the product shaped surfaces (1) and (2) in Theorem 2.1, see Fig. 1.

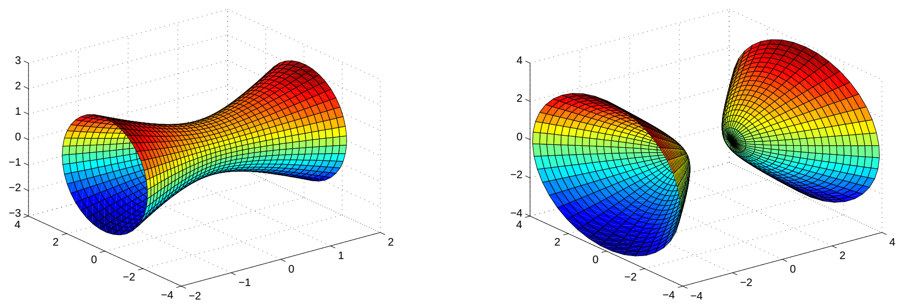


FIGURE 1.

Theorem 2.2. *The only product shaped hypersurfaces in the de Sitter space* $\mathbb{S}_1^{n+1}(1)$ *are*

(1) *Spacelike hypersurface* $R^n = \{x \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = x_{n+2} + t_0\} (t_0 > 0)$;
(2) *Lorentz hypersurface* $S_1^n(2) = \{x \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{\sqrt{2}}{2}\}$;
(3) *Lorentz hypersurface* $S^r(2) \times S_1^{n-r}(2) = \{x \in \mathbb{R}_1^{n+1} \mid \sum_{i=2}^{2+r} x_i^2 = \frac{1}{2}, -x_1^2 + \sum_{i=3+r}^{n+2} x_i^2 = \frac{1}{2}\}$, *where* $r = 1, \dots, n-1$.

Proof. The isoparametric hypersurfaces in the de Sitter space $\mathbb{S}_1^{n+1}(1)$ have the following cases [**5**]:

$\mathrm{S}^{(1)}$: $R^n = \{x \in \mathbb{S}_1^{n+1} \subset R_1^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ with $A = \pm I$;
$\mathrm{S}^{(2)}$: $S^n(c) = \{x \in \mathbb{S}_1^{n+1} \subset R_1^{n+2} \,|\, x_1 = \sqrt{1/c - 1}\}$ $(0 < c \leqslant 1)$ with $A = \pm\sqrt{1-c}\, I$;
$\mathrm{S}^{(3)}$: $H^n(c) = \{x \in \mathbb{S}_1^{n+1} \subset R_1^{n+2} \,|\, x_{n+2} = \sqrt{1 - 1/c}\}(c < 0)$ with $A = \pm\sqrt{1-c}\, I$;

$$\mathrm{S}^{(4)}\colon S^r(c_1) \times H^{n-r}(c_2) = \left\{ x \in \mathbb{S}_1^{n+1} \subset R_1^{n+2} \,\middle|\, \sum_{i=2}^{2+r} x_i^2 = \frac{1}{c_1}, -x_1^2 + \sum_{i=3+r}^{n+2} x_i^2 = \frac{1}{c_2} \right\},$$

with $A = \pm(\sqrt{1-c_1}\, I_r \oplus \sqrt{1-c_2}\, I_{n-r})$, and $\frac{1}{c_1} + \frac{1}{c_2} = 1$, $c_1 > 0$, $c_2 < 0$;

$\mathrm{S}^{(5)}$: $S_1^n(c) = \{x \in \mathbb{S}_1^{n+1} \subset R_1^{n+2} \,|\, x_{n+2} = \sqrt{1 - 1/c}\}$ $(c \geqslant 1)$ with $A = \pm\sqrt{c-1}\, I$;

$$\mathrm{S}^{(6)}\colon S^r(c_1) \times S_1^{n-r}(c_2) = \left\{ x \in \mathbb{S}_1^{n+1} \subset R_1^{n+2} \,\middle|\, \sum_{i=2}^{2+r} x_i^2 = \frac{1}{c_1}, -x_1^2 + \sum_{i=3+r}^{n+2} x_i^2 = \frac{1}{c_2} \right\},$$

with $A = \pm\big(\sqrt{c_1 - 1}\, I_r \oplus (-\sqrt{c_2 - 1}\, I_{n-r})\big)$, and $\frac{1}{c_1} + \frac{1}{c_2} = 1$, $c_1 > 0$, $c_! > 0$.

Here $\mathrm{S}^{(1)}$, $\mathrm{S}^{(2)}$, $\mathrm{S}^{(3)}$, $\mathrm{S}^{(4)}$ are spacelike hypersurfaces and $\mathrm{S}^{(5)}$, $\mathrm{S}^{(6)}$ are Lorentz hypersurfaces. It follows from Proposition 2.1 that the product shaped hypersurfaces are isoparametric hypersurfaces with two distinct principal curvatures 1 and $-1$. So case $\mathrm{S}^{(4)}$ cannot occur.

For the case $\mathrm{S}^{(1)}$, the eigenvalue of the shape operator is exactly 1 or $-1$, so

$$R^n = \{x \in \mathbb{S}_1^{n+1} \subset R_1^{n+2} \mid x_1 = x_{n+2} + t_0\} \ \ (t_0 > 0)$$

is the spacelike product shaped hypersurface in the de Sitter space. This gives case (1) of Theorem 2.2.

For the case $\mathrm{S}^{(2)}$, since the eigenvalue of the shape operator is $\pm 1$, then $c = 0$, which contradicts to $0 < c \leqslant 1$. This implies that case $\mathrm{S}^{(2)}$ is impossible. Similarly, case $\mathrm{S}^{(3)}$ is impossible as well.

For the case $\mathrm{S}^{(5)}$, since the eigenvalue of the shape operator is $\pm 1$, then $c = 2$, and the Lorentz product shaped hypersurface is

$$S_1^n(2) = \{x \in \mathbb{S}_1^{n+1} \subset R_1^{n+2} \,|\, x_{n+2} = \sqrt{2}/2\}.$$

This gives case (2) of Theorem 2.2.

For the case $S^{(6)}$. If $\sqrt{c_1-1}=1$, $-\sqrt{c_2-1}=-1$, then $c_1=c_2=2$, and the Lorentz product shaped hypersurface is

$$S^r(2)\times S_1^{n-r}(2)=\left\{x\in\mathbb{R}_1^{n+1}\,\Big|\,\sum_{i=2}^{2+r}x_i^2=\frac{1}{2},-x_1^2+\sum_{i=3+r}^{n+2}x_i^2=\frac{1}{2}\right\},$$

with $A=I_r\oplus(-I_{n-r})$. This gives case (3) of Theorem 2.2. Similarly, if $-\sqrt{c_1-1}=-1$, $\sqrt{c_2-1}=1$, we also get case (3) of Theorem 2.2. □

For $n=1$, we give the pictures of the product shaped curves (1) and (2) in Theorem 2.2

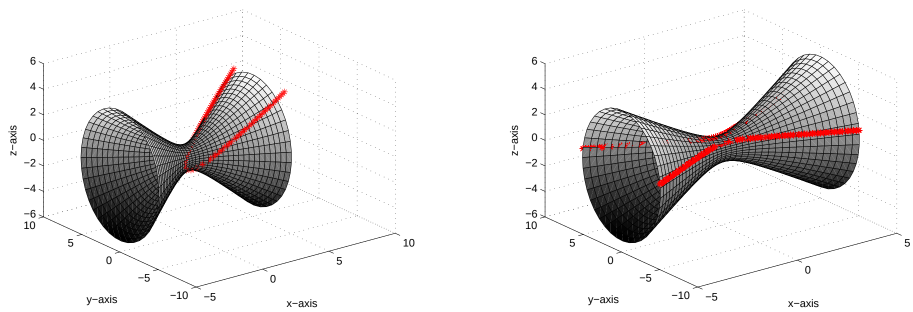


FIGURE 2.

THEOREM 2.3. *The only product shaped hypersurfaces in the anti-de Sitter space* $\mathbb{H}_1^{n+1}(-1)$ *are*

(1) *Spacelike hypersurface* $H^n(-2)=\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}\mid x_1=\frac{\sqrt{2}}{2}\}$;
(2) *Spacelike hypersurface* $H^r(-2)\times H^{n-r}(-2)=\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}\mid -x_1^2+\sum_{i=3}^{2+r}x_i^2=-\frac{1}{2},\ -x_2^2+\sum_{i=3+r}^{n+2}x_i^2=-\frac{1}{2}\}$;
(3) *Lorentz hypersurface* $R_1^n=\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}\mid x_1=x_{n+2}+t_0\}$ $(t_0>0)$.

PROOF. By [**5**], the isoparametric hypersurfaces in $\mathbb{H}_1^{n+1}(-1)$ have the following cases:

$$\mathrm{H}^{(1)}\colon H^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset R_2^{n+2}\,\Big|\,x_1=\sqrt{1/c+1}\right\}\ (c\leqslant-1)$$
$$\text{with } A=\pm\sqrt{-1-c}\,I;$$

$$\mathrm{H}^{(2)}\colon H^r(c_1)\times H^{n-r}(c_2)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset R_2^{n+2}\,\Big|\,-x_1^2+\sum_{i=3}^{2+r}x_i^2=\frac{1}{c_1},\right.$$
$$\left.-x_2^2+\sum_{i=3+r}^{n+2}x_i^2=\frac{1}{c_2}\right\}$$

$$\text{with } A=\sqrt{-1-c_1}\,I_r\oplus\sqrt{-1-c_2}\,I_{n-r},\ \text{and } \frac{1}{c_1}+\frac{1}{c_2}=-1,\ c_1<0,\ c_2<0;$$

$\mathrm{H}^{(3)}\colon R_1^n = \{x \in \mathbb{H}_1^{n+1}(-1) \subset R_2^{n+2} \mid x_1 = x_{n+2} + t_0\}$ $(t_0 > 0)$ with $A = \pm I$;

$\mathrm{H}^{(4)}\colon S_1^n(c) = \{x \in \mathbb{H}_1^{n+1}(-1) \subset R_2^{n+2} \mid x_1 = \sqrt{1/c+1}\}$ $(c > 0)$ with $A = \pm\sqrt{1+c}\, I$;

$\mathrm{H}^{(5)}\colon H_1^n(c) = \{x \in \mathbb{H}_1^{n+1}(-1) \subset R_2^{n+2} \mid x_{n+2} = \sqrt{-1-1/c}\}$ $(-1 \leqslant c < 0)$ with $A = \pm\sqrt{1+c}\, I$;

$$\mathrm{H}^{(6)}\colon S_1^r(c_1) \times H^{n-r}(c_2) = \Big\{x \in \mathbb{H}_1^{n+1}(-1) \subset R_2^{n+2} \,\Big|\, -x_1^2 + \sum_{i=3}^{r+2} x_i^2 = \frac{1}{c_1}, \; -x_2^2 + \sum_{i=3+r}^{n+2} x_i^2 = \frac{1}{c_2}\Big\},$$

with $A = \pm(\sqrt{1+c_1} I_r \oplus \sqrt{1+c_2} I_{n-r})$, and $\dfrac{1}{c_1} + \dfrac{1}{c_2} = -1$, $c_1 > 0$, $c_2 < 0$,

see Fig. 2. Here $\mathrm{H}^{(1)}$, $\mathrm{H}^{(2)}$ are spacelike hypersurfaces and $\mathrm{H}^{(3)}$, $\mathrm{H}^{(4)}$, $\mathrm{H}^{(5)}$, $\mathrm{H}^{(6)}$ are Lorentz hypersurfaces, respectively.

Since the eigenvalues of the shape operator for the hypersurface in $\mathbb{H}_1^{n+1}(-1)$ are 1 and $-1$, so $\mathrm{H}^{(6)}$ cannot occur.

For the case $\mathrm{H}^{(1)}$, the eigenvalue of the shape operator is $\pm 1$, then $c = -2$ and the spacelike product shaped hypersurface is

$$H^n(-2) = \{x \in \mathbb{H}_1^{n+1}(-1) \subset R_2^{n+2} \mid x_1 = \sqrt{2}/2\}.$$

This gives case (1) of Theorem 2.3.

For the case $\mathrm{H}^{(2)}$, since the eigenvalues of the shape operator are $\pm 1$. If $\sqrt{-1-c_1} = 1$, $-\sqrt{-1-c_2} = -1$, then $c_1 = c_2 = -2$, and the spacelike hypersurface is

$$H^r(-2) \times H^{n-r}(-2) = \Big\{x \in \mathbb{H}_1^{n+1}(-1) \subset R_2^{n+2} \,\Big|\, -x_1^2 + \sum_{i=3}^{2+r} x_i^2 = -\frac{1}{2}, -x_2^2 + \sum_{i=3+r}^{n+2} x_i^2 = -\frac{1}{2}\Big\}.$$

with $A = I_r \oplus (-I_{n-r})$. This gives case (2) of Theorem 2.3.

For the case $\mathrm{H}^{(3)}$, since the eigenvalue of the shape operator is just $\pm 1$, so

$$R_1^n = \{x \in \mathbb{H}_1^{n+1}(-1) \subset R_2^{n+2} \mid x_1 = x_{n+2} + t_0\} \quad (t_0 > 0)$$

is the product shaped hypersurface. This gives case (3) of Theorem 2.3.

For the case $\mathrm{H}^{(4)}$, since the eigenvalue of the shape operator is $\pm 1$, then $c = 0$, which contradicts to $c > 0$. So case $\mathrm{H}^{(4)}$ does not exist. Similarly, case $\mathrm{H}^{(5)}$ does not exist, too. □

Remark 2.1. For $n = 1$, we can draw the pictures for the product shaped curve in Theorem 2.3 similar to Theorem 2.2.

Remark 2.2. We conclude that there are only two product shaped hypersurfaces in the Minkowski space $\mathbb{R}_1^{n+1}$, while there are $n$ such hypersurfaces in the de Sitter space $\mathbb{S}_1^{n+1}(1)$ and $[\frac{n}{2}]$ such hypersurfaces in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$ since $r = 1, \ldots, n-1$ in Theorem 2.2 and Theorem 2.3, where $[x]$ denotes the integral part of $x$.

**Acknowledgments.** This work was supported by the Liaoning Science and Technology Project No. 201602528, the General Project for Department of Liaoning Education No. L2014482.

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(Received 03 12 2015)
(Revised 29 09 2016)

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