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TRANSFORMS FOR MINIMAL SURFACES IN 5-DIMENSIONAL SPACE FORMS

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Abstract. For a minimal surface in a 5-dimensional space form, we give transforms to get another minimal surface in another 5-or 4-dimensional space form.

1. Introduction

For a minimal surface in the 3-sphere S^3, the unit normal vector field, that is, the Gauss map gives another minimal surface in S^3 possibly with singularities (cf. [**5**]). It is generalized by Bolton, Pedit and Woodward [**2**] for superconformal minimal surfaces in odd-dimensional spheres. On the other hand, Bolton and Vrancken [**3**] discovered new transforms from a minimal surface with non-circular ellipse of curvature in the 5-sphere S^5, to another minimal surface in S^5, which are called ($\pm$)transforms (see also [**1, 4**]).

In this paper, generalizing them, we give transforms from a minimal surface in a 5-dimensional space form, to another minimal surface in another 5-or 4-dimensional space form.

Let $N^n(c)$ be the n-dimensional Riemannian space form of constant curvature c, where c is either 1, 0 or -1. In particular, let $N^n(1) = S^n$, $N^n(0) = R^n$ and $N^n(-1) = H^n$. Let R_1^{n+1} be the $(n+1)$-dimensional Minkowski space with standard coordinate system $(x_1, \cdots, x_n, x_{n+1})$ of signature $(+, \cdots, +, -)$. Then

$$H^n = \{(x_1, \cdots, x_n, x_{n+1}) \in R_1^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\},$$

and

$$S_1^n = \{(x_1, \cdots, x_n, x_{n+1}) \in R_1^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\},$$

where S_1^n is the n-dimensional de Sitter space.

Let $f\colon M \to N^5(c)$ be an immersion of a 2-dimensional manifold M into $N^5(c)$. We denote by h the second fundamental form of f. The first normal space $T_1^\perp(x)$ at $x \in M$ is defined by

$$T_1^\perp(x) = \{h(X, Y) \mid X, Y \in T_xM\}.$$

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The ellipse of curvature $E(x)$ at $x \in M$ is defined by

$$E(x) = \{h(X,X) \mid X \in T_xM,\ |X| = 1\}.$$

We assume that $f\colon M \to N^5(c)$ is a minimal immersion. Suppose that the ellipse of curvature is non-degenerate at any point. Then the dimension of the first normal space is 2 at any point. Let e_5 be the unit normal vector to $f(M)$ which is orthogonal to the first normal space. Then we can regard $G = e_5$ as a map to either S^5, S^4 or S^5_1, according to when $c = 1, 0$ or -1. It is the Gauss-like map.

THEOREM 1.1. *Let $f\colon M \to N^5(c)$ be a minimal surface. Suppose that the ellipse of curvature is a non-degenerate circle at any point. If the Gauss-like map G is non-degenerate, then it gives a minimal surface in either S^5, S^4 or S^5_1.*

REMARK 1.1. The case $c = 1$ can be seen in [**2**].

Next we consider the case where the ellipse of curvature is not a circle. For a minimal surface $f\colon M \to N^5(c)$, suppose that the ellipse of curvature is non-degenerate and non-circular at any point. Let a and b be the semi-minor and semi-major axes of the ellipse of curvature, respectively. We choose the local normal orthonormal frame field $\{e_\alpha\}_{3\leqslant\alpha\leqslant5}$ so that e_3 is in the direction of the semi-minor axis and e_4 is in the direction of the semi-major axis. Now, for $\varepsilon = +1$ or -1, let

$$f^\varepsilon = \varepsilon\sqrt{1-\left(\frac{a}{b}\right)^2}\,e_4 + \frac{a}{b}e_5.$$

Then f^ε is a map to either S^5, S^4 or S^5_1, according to when $c = 1, 0$ or -1.

THEOREM 1.2. *Let $f\colon M \to N^5(c)$ be a minimal surface. Suppose that the ellipse of curvature is non-degenerate and non-circular at any point. Then f^ε gives a minimal surface in either S^5, S^4 or S^5_1.*

REMARK 1.2. It is a generalization of [**3**] for S^5.

2. Preliminaries

In this section, we recall the method of moving frames for surfaces in 5-dimensional space forms. We shall use the following convention on the ranges of indices:

$$1 \leqslant A, B, \ldots \leqslant 5, \quad 1 \leqslant i, j, \ldots \leqslant 2, \quad 3 \leqslant \alpha, \beta, \ldots \leqslant 5.$$

Let $\{e_A\}$ be a local orthonormal frame field in $N^5(c)$, and $\{\omega^A\}$ be the dual coframe field. Let ω^A_B denote the connection forms which satisfy $\omega^A_B = -\omega^B_A$. The structure equations are given by

$$d\omega^A = -\sum_B \omega^A_B \wedge \omega^B, \tag{2.1}$$

$$d\omega^A_B = -\sum_C \omega^A_C \wedge \omega^C_B + \frac{1}{2}\sum_{C,D} R^A_{BCD}\omega^C \wedge \omega^D, \quad R^A_{BCD} = c(\delta^A_C\delta_{BD} - \delta^A_D\delta_{BC}).$$

Let $f\colon M \to N^5(c)$ be a surface in $N^5(c)$. When $c = 1$, f is an R^6-valued map with $\langle f, f\rangle = 1$. When $c = -1$, f is an R^6_1-valued map with $\langle f, f\rangle = -1$.

We choose the frame $\{e_A\}$ so that $\{e_i\}$ are tangent to $f(M)$. In the following, the argument will be restricted to $f(M)$. Then $\omega^\alpha = 0$ along $f(M)$, and by (2.1), we have

$$0 = -\sum_i \omega_i^\alpha \wedge \omega^i.$$

So there exists a symmetric tensor $\{h_{ij}^\alpha\}$ so that

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j,$$

where h_{ij}^α are the components of the second fundamental form h of f.

In the ambient $R^6(\supset S^5)$, R^5 or $R_1^6(\supset H^5)$, according to when $c = 1, 0$ or -1, we have

$$de_j = \sum_i e_i \omega_j^i + \sum_\alpha e_\alpha \omega_j^\alpha - cf\omega^j,$$

and

$$de_\beta = \sum_i e_i \omega_\beta^i + \sum_\alpha e_\alpha \omega_\beta^\alpha.$$

The mean curvature vector H of f is given by

$$H = \frac{1}{2} \sum_\alpha (h_{11}^\alpha + h_{22}^\alpha) e_\alpha.$$

We say that f is minimal if $H = 0$ identically.

3. Proof of Theorem 1.1

Proof. Since the ellipse of curvature is a non-degenerate circle at any point, we can choose the local orthonormal frame field $\{e_A\}$ so that

$$(h_{ij}^3) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad (h_{ij}^5) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $a > 0$. Then

$$\omega_1^3 = a\omega^1, \quad \omega_2^3 = -a\omega^2, \quad \omega_1^4 = a\omega^2, \quad \omega_2^4 = a\omega^1, \quad \omega_1^5 = \omega_2^5 = 0.$$

We compute that

$$0 = d\omega_1^5 = -\omega_3^5 \wedge \omega_1^3 - \omega_4^5 \wedge \omega_1^4 = a(\omega^1 \wedge \omega_3^5 - \omega_4^5 \wedge \omega^2)$$

and

$$0 = d\omega_2^5 = -\omega_3^5 \wedge \omega_2^3 - \omega_4^5 \wedge \omega_2^4 = a(\omega_3^5 \wedge \omega^2 + \omega^1 \wedge \omega_4^5).$$

Then, using the notation like

$$\omega_3^5 = (\omega_3^5)_1 \omega^1 + (\omega_3^5)_2 \omega^2, \quad \omega_4^5 = (\omega_4^5)_1 \omega^1 + (\omega_4^5)_2 \omega^2,$$

we have

$$(\omega_3^5)_2 - (\omega_4^5)_1 = 0, \quad (\omega_3^5)_1 + (\omega_4^5)_2 = 0.$$

So we can write

$$\omega_3^5 = p\omega^1 + q\omega^2, \quad \omega_4^5 = q\omega^1 - p\omega^2$$

for some functions p and q.

For the Gauss-like map $G = e_5$, we have

$$dG(e_1) = de_5(e_1) = (\omega_5^3)_1\ e_3 + (\omega_5^4)_1\ e_4 = -pe_3 - qe_4,$$
$$dG(e_2) = de_5(e_2) = (\omega_5^3)_2\ e_3 + (\omega_5^4)_2\ e_4 = -qe_3 + pe_4,$$

and

$$\langle dG(e_1), dG(e_1)\rangle = \langle dG(e_2), dG(e_2)\rangle = p^2 + q^2, \quad \langle dG(e_1), dG(e_2)\rangle = 0.$$

Assume that G is non-degenerate in the following. Then $p^2 + q^2 > 0$, and G is conformal to f.

Now we have

$$dG = -e_3(p\omega^1 + q\omega^2) - e_4(q\omega^1 - p\omega^2).$$

Let $*$ denote the Hodge star operator so that $*\omega^1 = \omega^2$ and $*\omega^2 = -\omega^1$. Then

$$*dG = e_3(q\omega^1 - p\omega^2) - e_4(p\omega^1 + q\omega^2) = e_3\omega_4^5 - e_4\omega_3^5.$$

We can compute that

$$d(*dG) = -2(p^2 + q^2)e_5\omega^1 \wedge \omega^2.$$

Denoting the Laplacian by Δ, we get $\Delta G = -2(p^2 + q^2)G$. So the Gauss-like map G is a conformal harmonic map to either S^5, S^4 or S_1^5, according to when $c = 1, 0$ or -1. Thus G gives a minimal surface in either S^5, S^4 or S_1^5. □

4. Proof of Theorem 1.2

PROOF. Since the ellipse of curvature is non-degenerate and non-circular at any point, we can choose the local orthonormal frame field $\{e_A\}$ so that

$$(h_{ij}^3) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad (h_{ij}^5) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $0 < a < b$. We note that a and b are the semi-minor and semi-major axes of the ellipse of curvature, respectively. Then we have

$$\omega_1^3 = a\omega^1, \quad \omega_2^3 = -a\omega^2, \quad \omega_1^4 = b\omega^2, \quad \omega_2^4 = b\omega^1, \quad \omega_1^5 = \omega_2^5 = 0.$$

We compute that

$$d\omega_1^3 = da \wedge \omega^1 - a\omega_2^1 \wedge \omega^2 = -\omega_2^3 \wedge \omega_1^2 - \omega_4^3 \wedge \omega_1^4 = a\omega_2^1 \wedge \omega^2 - b\omega_4^3 \wedge \omega^2.$$

Using the notation like

$$\omega_2^1 = (\omega_2^1)_1\omega^1 + (\omega_2^1)_2\omega^2, \quad \omega_4^3 = (\omega_4^3)_1\omega^1 + (\omega_4^3)_2\omega^2,$$
$$da = a_1\omega^1 + a_2\omega^2, \quad db = b_1\omega^1 + b_2\omega^2,$$

we have

$$2a(\omega_2^1)_1 - b(\omega_4^3)_1 = -a_2.$$

Similarly, from $d\omega_2^3$, $d\omega_1^4$ and $d\omega_2^4$,

$$2a(\omega_2^1)_2 - b(\omega_4^3)_2 = a_1, \quad 2b(\omega_2^1)_2 - a(\omega_4^3)_2 = b_1, \quad 2b(\omega_2^1)_1 - a(\omega_4^3)_1 = -b_2.$$

Thus we get

$$2a\omega_2^1 - b\omega_4^3 = *da, \quad 2b\omega_2^1 - a\omega_4^3 = *db,$$

and

$$\omega_2^1 = \frac{1}{4}(*d\log(b^2-a^2)), \quad \omega_4^3 = \frac{a(*db)-b(*da)}{b^2-a^2} = -\frac{*d(a/b)}{1-(a/b)^2}.$$

Next we compute that

$$0 = d\omega_1^5 = -\omega_3^5\wedge\omega_1^3 - \omega_4^5\wedge\omega_1^4 = a\omega^1\wedge\omega_3^5 - b\omega_4^5\wedge\omega^2$$

and

$$0 = d\omega_2^5 = -\omega_3^5\wedge\omega_2^3 - \omega_4^5\wedge\omega_2^4 = a\omega_3^5\wedge\omega^2 + b\omega^1\wedge\omega_4^5.$$

Then we can write

$$\omega_3^5 = b(p\omega^1+q\omega^2), \quad \omega_4^5 = a(q\omega^1-p\omega^2)$$

for some functions p and q.

From $d\omega_4^3 = -\omega_1^3\wedge\omega_4^1 - \omega_2^3\wedge\omega_4^2 - \omega_5^3\wedge\omega_4^5$, we obtain

$$-\frac{\Delta(a/b)}{1-(a/b)^2} - \frac{2(a/b)|d(a/b)|^2}{(1-(a/b)^2)^2} = ab(2-p^2-q^2). \tag{4.1}$$

Set $r=a/b$. Then $f^\varepsilon = \varepsilon\sqrt{1-r^2}e_4 + re_5$. We can compute that

$$\begin{aligned} df^\varepsilon(e_1) = &-\varepsilon b\sqrt{1-r^2}e_2 + \Big(\varepsilon\frac{r_2}{\sqrt{1-r^2}} - ap\Big)e_3 \\ &- \Big(\varepsilon\frac{r_1}{\sqrt{1-r^2}} + aq\Big)(re_4 - \varepsilon\sqrt{1-r^2}e_5), \end{aligned}$$

and

$$\begin{aligned} df^\varepsilon(e_2) = &-\varepsilon b\sqrt{1-r^2}e_1 - \Big(\varepsilon\frac{r_1}{\sqrt{1-r^2}} + aq\Big)e_3 \\ &- \Big(\varepsilon\frac{r_2}{\sqrt{1-r^2}} - ap\Big)(re_4 - \varepsilon\sqrt{1-r^2}e_5). \end{aligned}$$

Set

$$A = \varepsilon\frac{r_1}{\sqrt{1-r^2}} + aq, \quad B = \varepsilon\frac{r_2}{\sqrt{1-r^2}} - ap.$$

Then we have

$$\begin{aligned} \langle df^\varepsilon(e_1), df^\varepsilon(e_1)\rangle &= \langle df^\varepsilon(e_2), df^\varepsilon(e_2)\rangle = b^2-a^2+A^2+B^2 (>0) \\ &= b^2-a^2+\frac{|dr|^2}{1-r^2} + \frac{2\varepsilon a(qr_1-pr_2)}{\sqrt{1-r^2}} + a^2(p^2+q^2), \end{aligned}$$

and $\langle df^\varepsilon(e_1), df^\varepsilon(e_2)\rangle = 0$. So f^ε is conformal to f.

Now we have

$$\begin{aligned} df^\varepsilon = &-\varepsilon b\sqrt{1-r^2}(e_2\omega^1+e_1\omega^2) - \varepsilon e_3(*d(\sin^{-1}r)) - ae_3(p\omega^1+q\omega^2) \\ &+ \varepsilon e_4 d(\sqrt{1-r^2}) + e_5 dr - are_4(q\omega^1-p\omega^2) + \varepsilon a\sqrt{1-r^2}e_5(q\omega^1-p\omega^2), \end{aligned}$$

and

$$\begin{aligned} *df^\varepsilon = &\,\varepsilon\sqrt{1-r^2}(e_1\omega_2^4 - e_2\omega_1^4) + \varepsilon e_3 d(\sin^{-1}r) + e_3\omega_4^5 \\ &+ \varepsilon e_4(*d(\sqrt{1-r^2})) + e_5(*dr) - r^2e_4\omega_3^5 + \varepsilon r\sqrt{1-r^2}e_5\omega_3^5. \end{aligned}$$

We need to compute $d(*df^\varepsilon)$ to get Δf^ε. We note that

$$\Delta(\sqrt{1-r^2}) = -\frac{r\Delta r}{\sqrt{1-r^2}} - \frac{|dr|^2}{(1-r^2)^{3/2}},$$

and by (4.1),

$$\Delta r = ab(p^2+q^2-2)(1-r^2) - \frac{2r|dr|^2}{1-r^2}.$$

By a little long but straight computation, we can show that

$$\Delta f^\varepsilon = -2\Big(b^2 - a^2 + \frac{|dr|^2}{1-r^2} + \frac{2\varepsilon a(qr_1 - pr_2)}{\sqrt{1-r^2}} + a^2(p^2+q^2)\Big) f^\varepsilon.$$

Hence, the map f^ε is a conformal harmonic map to either S^5, S^4 or S^5_1, according to when $c = 1, 0$ or -1. Thus f^ε gives a minimal surface in either S^5, S^4 or S^5_1. □

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