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ON WARPED PRODUCT MANIFOLDS SATISFYING RICCI–HESSIAN CLASS TYPE EQUATIONS

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ABSTRACT. We deal with a study of warped product manifold which is also a generalized quasi Einstein manifold. Then, we investigate the relationships between such warped products and certain manifolds that provide some Ricci–Hessian type equations, such as $\mathrm{Ric}_f^m = \lambda g$ for some smooth function λ, where Ric_f^m denotes the m-Bakery–Emery Ricci tensor. Finally, we obtain some rigidity conditions for such manifolds.

1. Introduction

Let (M^n, g) be a Riemannian manifold of dimension n with $n > 2$ and let $f : M \to \mathbb{R}$ be a smooth function. A natural extension of the Ricci tensor is the m-Bakry–Emery–Ricci tensor [**3**] defined by $\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{m} df \otimes df$, for some positive integer m. According to [**5**], a smooth manifold (M^n, g) is said to be generalized quasi Einstein manifold if there exist three smooth functions f, α and λ satisfying the equation

$$\mathrm{Ric} + \mathrm{Hess}\, f - \alpha df \otimes df = \lambda g \tag{1.1}$$

In such a manifold, f is called the potential function. This generalized class reduces to a *gradient Ricci soliton* [**2**] when $\alpha = 0$ and $\lambda \in \mathbb{R}$; *m-quasi Einstein manifold* [**4**] when $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ and also *m-generalized quasi Einstein manifold* whenever $\lambda \in C^\infty(M)$.

1.1. Backgrounds and Basic Formulas. In this section, we recall some basic definitions and formulas about warped products. Let (M^q, g) and $(N^{n-q}, \bar{g})$; $(1 \leqslant \dim M = q \leqslant n$, $\dim N = n - q)$ be two Riemannian manifolds and let f be a positive, smooth function of M. The *warped product* $M \times_f N$ of (M, g) and $(N, \bar{g})$ is the product manifold $M \times N$ with the metric $\tilde{g} = g \times_f \bar{g}$, [**1, 12**]. Here, (M, g) is base, $(N, \bar{g})$ is fiber, f is the warping function and any geometric objects denoted by "bar" (resp. "tilde") symbol will be assumed to lie on the fiber N (resp. on the

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product $M \times N$). Also, for such a warped product, any geometric objects without symbol will be assumed to lie on the base M. The local components of the metric $\tilde{g}$ can be expressed as

$$\tilde{g}_{ij} = \begin{cases} g_{ab}, & \text{if } i = a,\ j = b \\ f\bar{g}_{\alpha\beta}, & \text{if } i = \alpha,\ j = \beta \\ 0, & \text{otherwise} \end{cases}$$

where $a, b, c \cdots \in \{1, \cdots, q\}$, $\alpha, \beta, \gamma \cdots \in \{q+1, \cdots, n\}$ and $i, j, k \cdots \in \{1, \cdots, n\}$. The warped product manifold with constant warping function is simply called Riemannian product, (for more details, see [**9, 12, 14**]). Then, all components of the Ricci tensor of $M \times_f N$ are given by:

LEMMA 1.1. [**12**] *On $M^q \times_f N^{(n-q)}$, for all $X, Y \in \chi(M)$ and $V, W \in \chi(N)$,*

$$(1)\ \tilde{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y) - \frac{n-q}{f}\,\mathrm{Hess}(f)(X,Y), \quad (2)\ \tilde{\mathrm{Ric}}(X,V) = 0$$

$$(3)\ \tilde{\mathrm{Ric}}(V,W) = \bar{\mathrm{Ric}}(V,W) - \Big[\frac{\Delta f}{f} + (n-q-1)\frac{\|\nabla f\|^2}{f^2}\Big]\tilde{g}(V,W)$$

DEFINITION 1.1. A Riemannian manifold (M^n, g) $(n > 2)$ is called a mixed super quasi Einstein manifold [**13**] (briefly $(\mathrm{MSQE})_n$) if its Ricci tensor is nonzero and satisfies the following condition

$$\begin{aligned} \mathrm{Ric}(X,Y) = {} & ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ & + d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y) \end{aligned}$$

where a, b, c, d, e are real valued, nonzero scalar functions on (M^n, g), A, B are dual 1-forms of the vector fields U and V (which are called the generators), i.e., for all $X \in \chi(M)$, $A(X) = g(X,U)$ and $B(X) = g(X,V)$, D is a symmetric $(0,2)$-tensor field such that $D(X,U) = 0$ and $tr(D) = 0$. If $e = c = 0$, then the manifold is said to be a generalized quasi Einstein manifold $(G(QE)_n)$ [**6**]; if $e = d = c = 0$, then the manifold reduces to a quasi Einstein manifold $((QE)_n)$ [**7**]. These manifolds have been studied extensively by several authors (see [**8, 10**]).

REMARK 1.1. These two definitions of generalized quasi Einstein manifolds given by [**5**] and [**6**] are not equivalent, but our main goal in this work is to establish a link between these two concepts.

2. Main Results

In this section, we express the Ricci tensor of warped product according to its base and fiber when the warped product is also a generalized quasi Einstein manifold in the sense of Chaki.

THEOREM 2.1. *Let $M^q \times_f N^{n-q}$, ($q = \dim M$, $n - q = \dim N > 1$) be a $G(QE)_n$ warped product manifold whose generators are ξ_1 and ξ_2. Then for all $X, Y \in \chi(M)$ and $U, V \in \chi(N)$, the following statements hold:*

(i) *If* $\xi_1, \xi_2 \in \chi(M)$, *then*

$$\operatorname{Ric}(X,Y) = ag(X,Y) + \frac{n-q}{f}\operatorname{Hess} f(X,Y) + bg(X,\xi_1)g(Y,\xi_1) + c[g(X,\xi_1)g(Y,\xi_2) + g(X,\xi_2)g(Y,\xi_1)] \tag{2.1}$$

$$\bar{\operatorname{Ric}}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{n-q-1}{f^2}\|\operatorname{grad} f\|^2\Big)\bar{g}(U,V) \tag{2.2}$$

(ii) *If* $\xi_1, \xi_2 \in \chi(N)$, *then*

$$\operatorname{Ric}(X,Y) = ag(X,Y) + \frac{n-q}{f}\operatorname{Hess} f(X,Y)$$

$$\bar{\operatorname{Ric}}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{n-q-1}{f^2}\|\operatorname{grad} f\|^2\Big)\bar{g}(U,V) + bf^4\bar{g}(U,\xi_1)\bar{g}(V,\xi_1) + cf^4[\bar{g}(U,\xi_1)\bar{g}(V,\xi_2) + \bar{g}(U,\xi_2)\bar{g}(V,\xi_1)]$$

(iii) *If* $\xi_1 \in \chi(M)$ *and* $\xi_2 \in \chi(N)$, *then*

$$\operatorname{Ric}(X,Y) = ag(X,Y) + \frac{n-q}{f}\operatorname{Hess} f(X,Y) + bg(X,\xi_1)g(Y,\xi_1)$$

$$\bar{\operatorname{Ric}}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{n-q-1}{f^2}\|\operatorname{grad} f\|^2\Big)\bar{g}(U,V)$$

(iv) *If* $\xi_1 \in \chi(N)$ *and* $\xi_2 \in \chi(M)$, *then*

$$\operatorname{Ric}(X,Y) = ag(X,Y) + \frac{n-q}{f}\operatorname{Hess} f(X,Y)$$

$$\bar{\operatorname{Ric}}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{n-q-1}{f^2}\|\operatorname{grad} f\|^2\Big)\bar{g}(U,V) + bf^4\bar{g}(U,\xi_1)\bar{g}(V,\xi_1)$$

PROOF. Since $M \times_f N$ is a $G(QE)_n$, we have

$$\widetilde{\operatorname{Ric}} = a\tilde{g} + b\,\tilde{A} \otimes A + c[\tilde{A} \otimes \tilde{B} + \tilde{B} \otimes \tilde{A}] \tag{2.3}$$

(i) Let $\xi_1, \xi_2 \in \chi(M)$. Then for all $X \in \chi(M)$, we get $\tilde{A}(X) = \tilde{g}(X,\xi_1) = g(X,\xi_1)$ and $\tilde{B}(X) = \tilde{g}(X,\xi_2) = g(X,\xi_2)$. Thus, equation (2.3) yields

$$\tilde{\operatorname{Ric}}(X,Y) = ag(X,Y) + bg(X,\xi_1)g(Y,\xi_1) + c[g(X,\xi_1)g(Y,\xi_2) + g(X,\xi_2)g(Y,\xi_1)] \tag{2.4}$$

Using (2.4) in Lemma 1.1(1), the Ricci tensor of the base manifold is of the form (2.1). Similarly, for all $U \in \chi(N)$, we get $\tilde{A}(U) = \tilde{g}(U,\xi_1) = 0$ and $\tilde{B}(U) = \tilde{g}(U,\xi_2) = 0$ and so again (2.3) yields

$$\tilde{\operatorname{Ric}}(U,V) = af^2\bar{g}(U,V) \tag{2.5}$$

Using (2.5) in Lemma 1.1(3), the Ricci tensor of the fiber is of the form (2.2).
Analogously, (ii)–(iv) can be proved in a similar way with the case (i). □

THEOREM 2.2. *Let* $M^q \times_f N^{n-q}$ ($q = \dim M$, $n-q = \dim N > 1$) *be a* $G(QE)_n$ *warped product manifold with generators* ξ_1, ξ_2 *and let* $h = -(n-q)\ln f$. *Then:*

(i) *If* $\xi_1, \xi_2 \in \chi(M)$ *such that* $\xi_2 = \nabla h$, *then* M *is* $(MSQE)_q$ *with generators* $\xi_1, \nabla h$ *and associated tensor* $\mathrm{Hess}(h)$, *provided that* $g(\nabla_X \xi_1, \nabla f) = 0$, *for any* $X \in \chi(M)$ *and* $\Delta f = \frac{1}{f}\|\nabla f\|^2$. *Also,* N *is Einstein.*
(ii) *If* $\xi_1, \xi_2 \in \chi(N)$, *then* (M, g, h, a) *is a* $(n-q)$*-quasi Einstein manifold in the sense of Case and* N *is a* $G(QE)_{n-q}$ *in the sense of Chaki with generators* $\bar{\xi} = f\xi_1$ *and* $\bar{\mu} = f\xi_2$.
(iii) *If* $\xi_1 \in \chi(M)$ *and* $\xi_2 \in \chi(N)$ *such that* $\xi_1 = \nabla h$, *then* (M, g, h, a) *is a* $G(QE)_q$ *in the sense of Catino and* N *is Einstein.*
(iv) *If* $\xi_1 \in \chi(N)$ *and* $\xi_2 \in \chi(M)$, *then* (M, g, h, a) *is an* $(n-q)$*-quasi Einstein manifold in the sense of Case and* N *is a* $(QE)_{n-q}$ *in the sense of Chaki with generator* $\bar{\xi} = f\xi_1$.

PROOF. (i) Suppose that $\xi_1, \xi_2 \in \chi(M)$. Now, let us define

$$h = -(n-q)\ln f \quad \textit{and} \quad \xi_2 = \nabla h \tag{2.6}$$

Then we calculate

$$\mathrm{Hess}(h)(X,Y) = -\frac{n-q}{f}\,\mathrm{Hess}(f)(X,Y) + \frac{n-q}{f^2}\,df(X)\,df(Y) \tag{2.7}$$

In view of (2.6) and (2.7), (2.1) yields

$$\begin{aligned}\mathrm{Ric}(X,Y) = ag(X,Y) - \mathrm{Hess}(h)(X,Y) + \frac{1}{n-q}\,dh(X)\,dh(Y)\\ + bA(X)A(Y) + [A(X)\,dh(Y) + A(Y)\,dh(X)]\end{aligned}$$

Moreover, since ξ_1 and ξ_2 are orthonormal, $g(\xi_1, \xi_2) = g(\xi, \nabla h) = 0$ so $\xi_1(h) = 0$ and $\xi_1(f) = 0$. Thus, if $g(\nabla_X \xi_1, \nabla f) = 0$ holds for any $X \in \chi(M)$, we get $\mathrm{Hess}(f)(X, \xi) = 0$. That is, choosing $D =: \mathrm{Hess}(h)$, we get $D(X, \xi_1) = 0$, for all $X \in \chi(M)$. Also, we have $\Delta h = \frac{n-q}{f}[-\Delta f + \frac{1}{f}\|\nabla f\|^2]$. Thus, if we assume that the relation $\Delta f = \frac{1}{f}\|\nabla f\|^2$ holds, then the $(0,2)$-tensor field D also satisfies $tr(D) = 0$. Therefore, the base manifold becomes $(MSQE)_q$. On the other hand, from (2.2) the fiber manifold becomes Einstein.

Analogously, (ii)–(iv) can be proved in a similar way with the case (i). □

EXAMPLE 2.1. Let us consider the function $f : \mathbb{R}^3 \to \mathbb{R}^+$ defined by $f(t) = e^t$ and $M = \{(t,x,y,z) \in \mathbb{R}^4 : \ t \in (-1-\sqrt{7}, 1-\sqrt{3}) \cup (-1+\sqrt{7}, 1+\sqrt{3})\}$ be an open subset of $\mathbb{R}^4$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = (dt)^2 + t^4(dx)^2 + t^4(dy)^2 + e^{2t}(dz)^2$$

In [**11**], the authors proved that M is a $G(QE)_4$ with a nonconstant scalar curvature. This line element on M also provides the warped product structure $\mathbb{R}^3 \times_{e^t} \mathbb{R}$, where $\mathbb{R}^3$ and $\mathbb{R}$ denote the base and respectively fiber.

Now, recall that a smooth vector field ϕ on a Riemannian manifold (M^n, g) is called *conformal* if it satisfies

$$\mathcal{L}_\phi g = 2\Omega g \tag{2.8}$$

where $\mathcal{L}_\phi$ denotes the Lie derivative in the direction of ϕ and Ω is some smooth function on M, which is called a conformal factor. Particularly, if the vector field ϕ

satisfies the condition $\nabla_X \phi = \Omega X$, for all $X \in \chi(M)$ and a smooth function Ω, then it is called *closed conformal vector field*. Moreover, if $\Omega = 0$, it is called parallel. It can be easily proved that the Ricci tensor of a Riemannian manifold admitting a nontrivial closed and conformal vector field ϕ is of the form

$$\operatorname{Ric}(\phi, X) = -(n-1)g(\nabla\Omega, X). \tag{2.9}$$

for any $X \in \chi(M)$ where Ω denotes the conformal factor of ϕ. The following algebraic characterization of a Riemannian manifold admitting closed and conformal vector fields will be used in the proof of our main theorem.

Lemma 2.1. *Let (M^n, g, f, λ) be an m-generalized quasi Einstein manifold admitting a closed conformal vector field ϕ with conformal factor Ω. Then the following statements hold:*

(1) $\nabla\phi(f) = \frac{1}{m}\phi(f)\nabla f + (n-1)\nabla\Omega + \lambda\phi + \Omega\nabla f.$
(2) $\operatorname{Hess}(\phi(f)) = \frac{1}{m}[d(\phi(f)) \otimes df + \phi(f)\operatorname{Hess} f] + (n-1)\operatorname{Hess}\Omega$
$\qquad + (d\lambda \otimes \phi^\flat) + \lambda\Omega g + \Omega\operatorname{Hess} f + d\Omega \otimes df.$
(3) $\frac{1}{m}d(\phi(f)) \otimes df + (d\lambda \otimes \phi^\flat) + d\Omega \otimes df$ *is symmetric.*

Proof. In view of (2.9), the fundamental equation m-generalized quasi Einstein manifold yields

$$\operatorname{Hess} f(\phi, X) = \frac{1}{m}df(\phi)df(X) + (n-1)d\Omega(X) + \lambda g(\phi, X) \tag{2.10}$$

By using $\operatorname{Hess}(f)(\phi, X) = X(\phi(f)) - (\nabla_X\phi)f$ and (2.9) in (2.10), we get

$$X(\phi(f)) = \frac{1}{m}df(\phi)df(X) + (n-1)d\Omega(X) + \lambda g(\phi, X) + \Omega df(X) \tag{2.11}$$

If we express the last equation without the inner product of X, we obtain the first assertion. Taking the second covariant derivative of (2.11) in the direction of arbitrary vector field Z, we get

$$\begin{aligned}\nabla_Z X(\phi(f)) = \frac{1}{m}&[Z(\phi(f))df(X) + \phi(f)ZX(f)] + (n-1)\nabla_Z(X(\Omega))\\ &+ d\lambda(Z)\phi^\flat(X) + \lambda(\nabla_Z\phi^\flat)(X) + d\Omega(Z)df(X) + \Omega\nabla_Z(X(f))\end{aligned} \tag{2.12}$$

Again using the definition of Hessian and equation (2.9) in (2.12), we obtain

$$\begin{aligned}\operatorname{Hess}(\phi(f))(Z, X) = \frac{1}{m}&[d(\phi(f))(Z)df(X) + \phi(f)\operatorname{Hess}(f)(Z, X)]\\ &+ (n-1)\operatorname{Hess}(\Omega)(Z, X) + d\lambda \otimes \phi^\flat(Z, X) + \lambda\Omega g(Z, X)\\ &+ d\Omega \otimes df(Z, X) + \Omega\operatorname{Hess}(f)(Z, X)), \quad \forall X, Z \in \chi(M)\end{aligned}$$

which proves the second assertion. Since the Hessian and the metric tensor are symmetric, we can easily verify the third assertion. □

Now, by virtue of Lemma 2.1, we shall establish the following results.

THEOREM 2.3. *Let (M^n, g, f, λ) be an m-generalized quasi Einstein manifold admitting a nonzero closed conformal vector field ϕ with conformal factor Ω. Then, its Ricci tensor is of the form*

$$\mathrm{Ric} = \lambda g + \Big(\frac{1-n}{\|\phi\|} d\Omega(U) - \lambda\Big) U^\flat \otimes U^\flat \tag{2.13}$$

where $U^\flat$ is a 1-form corresponding to the unit vector field U in the direction of ϕ. If, in particular, ϕ is parallel, then the Ricci tensor reduces to the form $\mathrm{Ric} = \lambda(g - U^\flat \otimes U^\flat)$.

PROOF. From Lemma 2.1(1)(2), we get for any $X, Y \in \chi(M)$

$$\begin{aligned} \Big[\frac{1}{m}\lambda df(X) - d\lambda(X)\Big] g(\phi, Y) + \frac{n+m-1}{m}[d\Omega(Y) df(X) - d\Omega(X) df(Y)] \\ = \Big[\frac{1}{m}\lambda df(Y) - d\lambda(Y)\Big] g(\phi, X) \end{aligned} \tag{2.14}$$

Putting $X = \phi$ in (2.14), we get

$$\begin{aligned} \Big[\frac{1}{m}\lambda df(\phi) - d\lambda(\phi)\Big] g(\phi, Y) + \frac{n+m-1}{m}[d\Omega(Y) df(\phi) - d\Omega(\phi) df(Y)] \\ = \Big[\frac{1}{m}\lambda df(Y) - d\lambda(Y)\Big] \|\phi\|^2 \end{aligned} \tag{2.15}$$

Let U be a unit vector field in the direction of ϕ. Then by (2.15), we obtain

$$\nabla f\Big(1 + \frac{n+m-1}{\lambda\|\phi\|} U(\Omega)\Big) = U(f)U + \frac{m}{\lambda}[\nabla\lambda - U(\lambda)U] + \frac{n+m-1}{\lambda\|\phi\|} \nabla\Omega U(f) \tag{2.16}$$

and so by virtue of (2.16), (2.10) yields

$$\mathrm{Hess}\, f = \frac{1}{m} df \otimes df + \lambda U^\flat \otimes U^\flat + \frac{n-1}{\|\phi\|} d\Omega \otimes U^\flat \tag{2.17}$$

Next, substituting (2.17) in the fundamental equation of m-generalized quasi Einstein manifold, we obtain the Ricci tensor as in (2.13). If we take $\Omega = 0$, the rest of the proof is completed. □

Also from (2.8), the unit vector field U in the direction of the closed conformal vector field ϕ satisfies $\nabla_X U = \frac{\Omega}{\|\phi\|}[X - U^\flat(X)U]$, for any $X \in \chi(M)$. This means that U is a unit concircular vector field. According to K. Yano [**15**], a Riemannian manifold admits a concircular vector field if and only if there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form $ds^2 = (dx^1)^2 + e^q g^*_{\alpha\beta} dx^\alpha dx^\beta$, where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are the functions of x^γ only $(\alpha, \beta, \gamma, \delta = 2, 3, \ldots, n)$ and $q = q(x^1) \neq$ *constant* is a function of x^1 only. Hence we can state the following:

THEOREM 2.4. *Let (M^n, g, f, λ) be an m-generalized quasi Einstein manifold admitting a nonzero closed conformal vector field ϕ. Then (M^n, g) is a locally warped product manifold $I \times_{e^{q/2}} M^*$, where I is a real interval, (M^*, g^*) is an $(n-1)$-dimensional Riemannian manifold and q is a smooth function on I.*

EXAMPLE 2.2. Here, we construct an example for the particular case of Theorem 2.3 and 2.4: Let $M = I \times_f M^*$ be the warped product with the warping function $f = f(t)$. Then there exists a closed conformal vector field $\phi = f\partial_t$ on M with conformal factor $\Omega = f'$. By using the fundamental equation (1.1), it can be shown that M is an m-generalized quasi Einstein manifold with the potential function $h = h(t)$, provided that the fiber manifold M^* is Einstein and the warping function f and the potential funtion h are related to each other with the equation $(1-n)\frac{f''}{f} + h'' - \frac{1}{m}(h')^2 = \lambda$, where "$'$" denotes the ordinary differentiation with respect to t. This shows the existence of the closed conformal vector field on an m-generalized quasi Einstein manifold. Moreover, if the conformal factor $\Omega = f'$ vanishes (i.e., ϕ is a parallel vector field and so M is a simply direct product), then equation (2.13) also holds on M, where $U = \partial_t$.

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