PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 1 – 1994

QUASI-ORTHOGONALITY ON THE UNIT CIRCLE AND SEMI-CLASSICAL FORMS (*)

Manuel Alfaro and Leandro Moral

Abstract. In this paper we study a new concept of quasi-orthogonality on the unit circle, depending of the structure of the orthogonal polynomials on the unit circle, and we consider its relation with the semi-classical linear forms.

1 – Introduction

In several topics concerning orthogonal polynomials (O.P.) it is more convenient to use a weaker substitute of the concept of the orthogonality. One of the possible substitutes is the notion of quasi-orthogonality:

Let u be a linear form on the linear space of all real polynomials and let (P_n) be a sequence of polynomials with $\deg P_n = n$, (P_n) *is quasi-orthogonal of order k with respect to u if*

$$u(P_n(x)\,x^m) = 0 \ ,$$
$$u(P_n(x)\,x^{n-k}) \neq 0 \ ,$$

whenever $0 \leq m \leq n-k-1$ *and* $n \geq k+1$.

This concept was introduced by M. Riesz for $k = 1$ in relation to the moment problem ([20]). Subsequently, in papers concerning the formulas of mechanical quadrature, it was considered by Fejér ([8]) for $k = 2$ and by Shohat ([22]) for any $k \in \mathbb{N}$. Several questions on quasi-orthogonal polynomials have been studied, for instance, in [4], [7], [1], [21], [3], [13], [2], [18] and [19].

Received: June 14, 1991.
AMS Subject Classification (1991): 42C05.
Key words: Orthogonal polynomials, quasi-orthogonal polynomials, semi-classical linear forms.
(*) This research was partially supported by P.A.I. 1990 (Univ. de Zaragoza) nº 227-36.

The above definition can be formally generalized to the case of the orthogonality on the unit circle $\mathbb{T}$, as follows

Definition. Let u be an Hermitian and regular linear functional on the linear space of Laurent polynomials and let (P_n) be a sequence of complex polynomials with $\deg P_n = n$. The sequence (P_n) is called quasi-orthogonal of order k with respect to u if

$$u(P_n(z)\, z^{-m}) = 0 \;,$$

$$u(P_n(z)\, z^{-(n-k)}) \neq 0 \;,$$

whenever $0 \leq m \leq n-k-1$ and $n \geq k+1$.

However, this concept is not so appropriate as in the real case and only the Bernstein–Szegö polynomials satisfy the above definition ([17] and [10]).

In [14], sequences of polynomials, called para-orthogonal because their orthogonality properties, have been considered. These polynomials turn out to be adequate for some applications in quadrature formulas on $\mathbb{T}$ as well as in the trigonometric moment problem, but they are not adequate in order to develop other topics concerning the O.P. on $\mathbb{T}$.

Then, it seems convenient to introduce a new concept of quasi-orthogonality more depending of the structure of the O.P. on $\mathbb{T}$. How to do this can be derived by pointing out the relation between the orthogonal polynomials on $\mathbb{T}$ and the orthogonal polynomials on $[-1,1]$ ([23], §11.4) or how the trigonometric moments on $\mathbb{T}$ may be transformed in moments on $[-1,1]$ ([1], p. 30 and ff.).

The aim of this paper is to study this kind of quasi-orthogonality and its relation with the semi-classical forms in a parallel way to the one developed by Maroni in the real case, as a first step to establish a classification of the O.P. on $\mathbb{T}$ in terms of ordinary differential equations.

This paper is organized as follows. In section 2, we define this new notion and we prove that a sequence of monic orthogonal polynomials on $\mathbb{T}$ associated with a regular linear form u is quasi-orthogonal on $\mathbb{T}$ of order s with respect to a regular linear form v, $v \neq 0$, if and only if there exists only one polynomial A with $\deg A = s$, such that $v = [A(z) + \overline{A}(z^{-1})]\, u$. In section 3, we consider semi-classical forms on the unit circle and we show a characterization of these forms by using the derivation operator. In section 4, we study the relation between sequences of quasi-orthogonal polynomials on $\mathbb{T}$ and semi-classical forms and we find a necessary and sufficient condition for a sequence of polynomials to be quasi-orthogonal with respect to a semi-classical form.

2 – Quasi-orthogonal polynomials on $\mathbb{T}$

Let Λ be the linear space of Laurent polynomials

$$L(z) = \sum_{n=p}^{q} c_n\, z^n \ ,$$

with $c_n \in \mathbb{C}$ and p, q integers, where $p \le q$, $\mathcal{P}$ is the space of all complex polynomials and we denote by Λ' the dual algebraic space of Λ and by $\mathcal{H}$ the subspace of Λ' of all Hermitian linear forms.

Let $u \in \mathcal{H}$. Then, the Toeplitz Hermitian matrix associated with u is

$$M = \begin{pmatrix} c_0 & c_1 & c_2 & \cdot \\ c_{-1} & c_0 & c_1 & \cdot \\ c_{-2} & c_{-1} & c_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (c_{i-j})_{i,j\in\mathbb{N}} \ ,$$

where $c_n = u(z^n)$ for every $n \in \mathbf{Z}$ and $c_{-n} = \overline{c_n}$. (Here, $\mathbb{N}$ denotes the set of non-negative integer numbers $\{0, 1, 2, ...\}$ and $\mathbf{Z}$ denotes the set of integers $\{0, \pm 1, \pm 2, ...\}$).

Definition 2.1. A linear form u is called regular or quasi-definite if and only if $\Delta_n \neq 0$ for every $n \ge 0$, where Δ_n denotes the $(n+1) \times (n+1)$ principal minor of M (see [5] or [19]).

It is well known (see [11] or [5]) that the regularity of u is a necessary and sufficient condition for the existence of a sequence of orthogonal polynomials on $\mathbb{T}$. In this case, if we suppose that $(\phi_n(z))$ is the sequence of monic orthogonal polynomials on $\mathbb{T}$ (SMOP), then

$$u[\phi_n(z) \cdot z^{-k}] = 0 \ ,$$

for every $k = 0, 1, ..., n-1$ and

$$u[\phi_n(z) \cdot z^{-n}] = e_n = \frac{\Delta_n}{\Delta_{n-1}} \neq 0 \ .$$

In the other hand, the polynomials ϕ_n satisfy the so-called Szegö recurrence relations

$$\phi_{n+1}(z) = z\, \phi_n(z) + a_{n+1}\, \phi_n^*(z) \ , \tag{2.1}$$

$$\phi_{n+1}(z) = (1 - |a_{n+1}|^2)\, z\, \phi_n(z) + a_{n+1}\, \phi_{n+1}^*(z) \ , \tag{2.2}$$

with $a_n = \phi_n(0)$, $|a_n| \neq 1$ for every $n \geq 1$ and $\phi_n^*(z) = z^n\, \overline{\phi_n}(z^{-1})$. Conversely, given a sequence of monic polynomials (ϕ_n), with $\deg \phi_n = n$, satisfying (2.1) or (2.2) there exists only one $u \in \mathcal{H}$ (up to constant real factors) such that

$$u[\phi_n(z) \cdot z^{-k}] = e_n\, \delta_{nk}$$

with $e_n \neq 0$, for every $k = 0, 1, ..., n$.

Definition 2.2. Let $v \in \mathcal{H}$, $s \in \mathbb{N}$ and let (ϕ_n) be a sequence of monic polynomials, $\phi_n(z) = z^n + \ldots$. We say (ϕ_n) is $\mathbb{T}$-quasi-orthogonal of order s with respect to v provided

i) $v[\phi_n(z) \cdot z^{-k}] = 0$, for every k with $s \leq k \leq n-s-1$ and for every $n \geq 2s+1$;

ii) There exists $n_0 \geq 2s$ such that $v[\phi_{n_0}(z) \cdot z^{-n_0+s}] \neq 0$.

With the above conditions,

Definition 2.3. The sequence (ϕ_n) is strictly $\mathbb{T}$-quasi-orthogonal of order s with respect to v if (ϕ_n) is $\mathbb{T}$-quasi-orthogonal of order s and besides

iii) For every $n \geq 2s$, $v[\phi_n(z) \cdot z^{-n+s}] \neq 0$.

Remark. When $s = 0$, the usual definition of orthogonality on $\mathbb{T}$ appears.

The above concepts are related by

Proposition 2.4. *Let $u, v \in \mathcal{H}$ be with u regular and let (ϕ_n) be the SMOP associated with u. Then, (ϕ_n) is $\mathbb{T}$-quasi-orthogonal of order s with respect to v if and only if it is strictly $\mathbb{T}$-quasi-orthogonal of order s with respect to v.*

Proof: Because of the $\mathbb{T}$-quasi-orthogonality of (ϕ_n), from (2.1) and taking into account

$$v[\phi_n^*(z) \cdot z^{-n+s}] = v[\overline{\phi_n}(z^{-1}) \cdot z^s] = \overline{v[\phi_n(z) \cdot z^{-s}]} = 0$$

we get

$$v[\phi_n(z) \cdot z^{-n+s}] = \Big(\prod_{j=2s+1}^{n} (1 - |a_j|^2) \Big)\, v[\phi_{2s}(z) \cdot z^{-s}]$$

for every $n \geq 2s+1$. From the last relation the result follows directly. ∎

An easy consequence is the following

Corollary 2.5. *Let u, v and (ϕ_n) be as in the above proposition, then (ϕ_n) is strictly $\mathbb{T}$-quasi-orthogonal of order s with respect to v if and only if there exists $n_0 \geq 2s$ such that*

a) $v[\phi_{n_0+1}(z)\cdot z^{-k}]=0$, *for every* k *with* $s\le k\le n_0-s$,

b) $v[\phi_{n_0}(z)\cdot z^{-n_0+s}]\ne 0$,

c) $v[\phi_n(z)\cdot z^{-s}]=0$, *for every* $n\ge n_0+1$,

holds.

If $u\in\mathcal{H}$, by using a standard argument, it is easy to show that there exists a sequence (ϕ_n) of $\mathbb{T}$-quasi-orthogonal polynomials of order s with respect to u if and only if $\Delta_n\ne 0$ for every $n\ge 2s+1$. In this case, there exist infinitely many sequences of monic polynomials $\mathbb{T}$-quasi-orthogonal with respect to u.

Proposition 2.6. *Let* $w\in\mathcal{H}$, *then* $w=0$ *if and only if there exists a SMOP* (ϕ_n) *and* $n_0\in\mathbb{N}$ *such that* $w[\phi_n(z)\cdot z^{-k}]=0$ *for every* $n\ge n_0$ *and* $k=0,1,...,n$.

Proof: If $w=0$, the result is trivial. Conversely, from (2.1) it follows that $w[\phi_n(z)\cdot z^{-k}]=0$ for every $n\ge 0$ and $k=0,1,...,n$. As Λ is generated by the family

$$\left\{\phi_n(z)\cdot z^{-k};\ n\in\mathbb{N} \text{ and } k=0,1,...,n\right\},$$

we have $w(P)=0$ for every $P\in\Lambda$. ∎

Let $u\in\Lambda'$ and $f\in\Lambda$. We define the form $fu\in\Lambda'$ as

$$(fu)[g(z)]=u[f(z)\,g(z)]$$

for every $g\in\Lambda$.

Now we are going to characterize the forms u such that a given SMOP (ϕ_n) is $\mathbb{T}$-quasi-orthogonal with respect to u.

Theorem 2.7. *Let* $u\in\mathcal{H}$ *be regular and let* (ϕ_n) *be the SMOP associated with* u. *Then,* (ϕ_n) *is* $\mathbb{T}$*-quasi-orthogonal of order* s *with respect to* $v\in\mathcal{H}-\{0\}$ *if and only if there exists only one polynomial* A $(A\ne 0)$, *with* $\deg A=s$, *such that*

$$v=\left[A(z)+\overline{A}(z^{-1})\right]u\,. \tag{2.3}$$

Proof: *Uniqueness.* Let A be a polynomic solution of (2.3) with $\deg A=s$ and let us suppose that the polynomial A_1, with $\deg A_1=s_1$, is a solution too. If we define $A_2=A-A_1$, we can write $A_2=\frac{\mu_0}{2}+\sum_{j=1}^{r}\mu_j\,z^j$, where $r=\max\{s,s_1\}$. Then the formula

$$\left[A_2(z)+\overline{A_2}(z^{-1})\right]u=\sum_{j=-r}^{r}\mu_j\,z^j\,u=0$$

holds. So, for $n \geq 2r+1$ and $k \geq 0$, we have

$$\sum_{j=-r}^{r} \mu_j \, u\Big[\phi_n(z) \cdot z^{-k+j}\Big] = 0 \ .$$

Taking $k = n-r, ..., n$, we obtain a system of equations in the unknowns μ_j whose unique solution is $\mu_0 = ... = \mu_r = 0$. Hence, $A_2 = A - A_1 = 0$.

Existence. If there exists a polynomial A satisfying (2.3) it is easy to verify that (ϕ_n) is $\mathbb{T}$-quasi-orthogonal of order $\deg A$ with respect to v. Conversely, let (ϕ_n) be as in the hypothesis. We define $w = v - \sum_{j=-s}^{s} \alpha_j \, z^j \, u$ with $\alpha_j \in \mathbb{C}$. By the orthogonality and the $\mathbb{T}$-quasi-orthogonality of the SMOP (ϕ_n) with respect to u and v, respectively, the relation

$$w[\phi_n(z) \cdot z^{-k}] = 0$$

holds for every $\alpha_j \in \mathbb{C}$ whenever $n \geq 2s+1$ and $k = s, ..., n-s-1$.

If $k = n-s, ..., n$, then $w[\phi_n(z) \cdot z^{-k}] = 0$, whenever the coefficients $(\alpha_j^{(n)})_{j=-s}^{0}$ are the solutions of the system

$$(2.4)_n \qquad \begin{cases} v[\phi_n(z) \cdot z^{-n+s}] = \alpha_{-s}^{(n)} \, u[\phi_n(z) \cdot z^{-n}], \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ v[\phi_n(z) \cdot z^{-n}] = \alpha_{-s}^{(n)} \, u[\phi_n(z) \cdot z^{-n-s}] + ... + \alpha_0^{(n)} \, u[\phi_n(z) \cdot z^{-n}] \ , \end{cases}$$

which has a unique solution with $\alpha_s^{(n)} \neq 0$.

Now, let us suppose $a_{n+1} \neq 0$. Then, if $k = 0, ..., s-1$, $w[\phi_n(z) \cdot z^{-k}] = 0$ whenever the coefficients $(\alpha_j^{(n)})_{j=1}^{s}$ are the solutions of the system

$$(2.5)_n \qquad \begin{cases} v[\phi_n(z) \cdot z^{-s+1}] = \alpha_s^{(n)} \, u[\phi_n(z) \cdot z], \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ v[\phi_n(z)] = \alpha_s^{(n)} \, u[\phi_n(z) \cdot z^s] + ... + \alpha_1^{(n)} \, u[\phi_n(z) \cdot z] \ . \end{cases}$$

As $u[\phi_n(z) \cdot z] = -e_n \, a_{n+1} \neq 0$, the system $(2.5)_n$ has a unique solution.

Let us write $(\alpha_j^{(n)})_{j=-s}^{s}$, $(\alpha_j^{(n+1)})_{j=-s}^{s}$ the solutions of the systems $(2.4)_n$, $(2.5)_n$ and $(2.4)_{n+1}$, $(2.5)_{n+1}$, respectively. Using the recurrence relations (2.1), (2.2) and an induction on j, after straightforward computations, we obtain

$$\alpha_j^{(n)} = \alpha_j^{(n+1)} = \alpha_j \ ,$$

whenever $-s \leq j \leq s$; and

$$\alpha_{-j} = \overline{\alpha_j} \ ,$$

for $1 \le j \le s$.

So, if $a_{n+1} \ne 0$, we have $w[\phi_m(z) \cdot z^{-k}] = 0$ for every $m \ge n \ge 2s+1$ and $k = 0, ..., m$. Hence, from Proposition 2.6 it follows that $w = 0$.

Otherwise, $w[1] = 0$ and thus $v[1] = \sum_{j=-s}^{s} \alpha_j \, u[z^j] \in \mathbb{R}$; and consequently $\alpha_0 \in \mathbb{R}$.

Therefore, there exists one and only one $A(z) = \frac{\alpha_0}{2} + \sum_{j=1}^{s} \alpha_j \, z^j$, with $\deg A = s$, such that $v = [A(z) + \overline{A}(z^{-1})] \, u$.

Finally, if $a_{n+1} = ... = a_{n+l-1} = 0$ and $a_{n+l} \ne 0$ for some $l \ge 2$, then $\phi_{n+l}(z) = z^l \, \phi_n(z) + a_{n+l} \, \phi_n^*(z)$ and using the systems $(2.4)_{n+l-1}$ and $(2.5)_{n+l-1}$ the above situation becomes. If $a_{n+l} = 0$ for every $l \ge 1$, the coefficients in $(2.5)_n$ vanish and this system is verified by $\alpha_{-j} = \overline{\alpha_j}$, when $1 \le j \le s$. Because of the uniqueness of the polynomial A the result follows. ∎

3 – Semi-classical forms

Definition 3.1. For $v \in \Lambda'$, we define the form $\mathcal{D}v \in \Lambda'$ as

$$(\mathcal{D}v)[f] = -i(zv)[f'] = -iv[z \, f'(z)]$$

for every $f \in \Lambda$.

Then, if $v \in \mathcal{H}$, $\mathcal{D}v \in \mathcal{H}$. Besides, if $v \in \Lambda'$ and $f, g \in \Lambda$, then

$$[\mathcal{D}(gv)][f] = -i[z \, g(z) \, v][f'] = -iv[z \, g(z) \, f'(z)] \ ,$$

that is, $\mathcal{D}$ is the derivation operator with respect to θ, where $z = r \, e^{i\theta}$. (See [24]).

Definition 3.2. If $u \in \mathcal{H}$ is a regular form, we say that u is semi-classical if and only if there are polynomials $A \ne 0$ and B such that $\mathcal{D}(Au) = Bu$.

Proposition 3.3. *Let $u \in \mathcal{H}$ be a regular form. Then, u is semi-classical if and only if there are polynomials $A \ne 0$ and B such that*

$$\mathcal{D}[\overline{A}(z^{-1}) \, u] = \overline{B}(z^{-1}) \, u \ .$$

Proof: For every $k \in \mathbf{Z}$, we have

$$\overline{[\mathcal{D}(A(z) \, u)][z^k]} = [\mathcal{D}(\overline{A}(z^{-1}) \, u)][z^{-k}] \ ,$$

because $u \in \mathcal{H}$. Similarly,

$$\overline{[B(z) \, u][z^k]} = [\overline{B}(z^{-1}) \, u][z^{-k}] \ .$$

Thus, the characteristic condition for a semi-classical form

$$[\mathcal{D}(Au)][z^k] = (Bu)[z^k]$$

is verified if and only if

$$[\mathcal{D}(\overline{A}(z^{-1})\,u)][z^j] = [\overline{B}(z^{-1})\,u][z^j]\ ,$$

holds for every $j \in \mathbf{Z}$. ∎

If $v \in \Lambda'$ and $P \in \mathcal{P}$ let us write $v^P = [P(z) + \overline{P}(z^{-1})]\,v$. Note that, if $v \in \mathcal{H}$, then $v^P \in \mathcal{H}$.

Theorem 3.4. *Let $u \in \mathcal{H}$ be a regular form. Then, u is semi-classical if and only if there exist polynomials $A \neq 0$ and B such that*

$$\mathcal{D}[u^A] = u^B\ . \tag{3.1}$$

Proof: **(⇒)** It is straightforward from Proposition 3.3.

(⇐) From (3.1), the k-th moments corresponding to the forms $\mathcal{D}[u^A]$ and u^B are:

$$(\mathcal{D}[u^A])[z^k] = -iku\Big[(A(z) + \overline{A}(z^{-1}))\,z^k\Big] = -iku\Big[z^s(A(z) + \overline{A}(z^{-1}))z^{k-s}\Big]\ ,$$

$$u^B[z^k] = u\Big[(B(z) + \overline{B}(z^{-1}))\,z^k\Big] = u\Big[z^s(B(z) + \overline{B}(z^{-1}))\,z^{k-s}\Big]\ ,$$

where $s = \max\{\deg A, \deg B\}$ (if $B = 0$, then $s = \deg A$). As

$$A_1(z) = z^s(A(z) + \overline{A}(z^{-1})) \quad \text{and} \quad B_1(z) = z^s(B(z) + \overline{B}(z^{-1}))$$

belong to $\mathcal{P}$, and

$$[\mathcal{D}(A_1(z))\,u][z^j] = \Big[(B_1(z) + isA_1(z))\,u\Big][z^j]$$

holds for every $j \in \mathbf{Z}$, with $A_1 \neq 0$ and $B_1 + isA_1 \in \mathcal{P}$, the result holds. ∎

4 – Semi-classical forms and $\mathbb{T}$-quasi-orthogonality

The main aim of this paragraph is to prove the following:

Theorem 4.1. *Let $u \in \mathcal{H}$ be regular and let (ϕ_n) be the SMOP associated to u. Let us write*

$$\begin{cases} \psi_n(z) = \dfrac{1}{n}\,z\,\phi_n'(z) \quad (n \geq 1), \\ \psi_0(z) = 1\ . \end{cases}$$

The following assertions are equivalent:

i) *u is a semi-classical form;*

ii) *There exists $\widehat{u} \in \mathcal{H} - \{0\}$ such that the sequences (ϕ_n) and (ψ_n) are $\mathbb{T}$-quasi-orthogonal with respect to $\widehat{u}$;*

iii) *There exists $\widehat{u} \in \mathcal{H} - \{0\}$ such that the sequence (ψ_n) is $\mathbb{T}$-quasi-orthogonal with respect to $\widehat{u}$.*

First of all, let us remember that to give a regular form $u \in \mathcal{H}$ is equivalent to known any of the following data:

1) A sequence of monic polynomials (ϕ_n), orthogonal with respect to u;

2) A sequence of complex numbers $(\phi_n(0))$ with $|\phi_n(0)| \neq 1$ for every $n \geq 1$ (Schur parameters);

3) A quasi-definite sequence of moments $(c_n)_{n\in\mathbf{Z}}$, with $c_n = u(z^n)$ and $c_{-n} = \overline{c_n}$;

4) A formal series $F(z) = c_0 + 2\sum_{n=1}^{+\infty} c_{-n}\, z^n$, with $c_n = u(z^n)$. (If u is positive definite, $F(z)$ is a Carathéodory function);

5) A formal Laurent series $G(z) = \sum_{n=-\infty}^{+\infty} c_{-n}\, z^n$, with $c_n = u(z^n)$.

(For the positive definite case see [25], [12]; for the regular case see [12], [15] and [24]).

Before to prove the above theorem we need to establish some previous lemmas.

Lemma 4.2. *A regular form $u \in \mathcal{H}$ is semi-classical if and only if there exist two polynomials C and D ($C \neq 0$) such that*

$$i\, z\, C(z)\, G'(z) = D(z)\, G(z)\ ,$$

where $G(z)$ is the formal Laurent series associated to u.

Proof: See [24]. ∎

As an immediate consequence we obtain

Corollary 4.3. *If $F(z)$ or $G(z)$ are rational functions, the form u is semi-classical.* ∎

Lemma 4.4. *The SMOP (φ_n) and (χ_n) such that*

$$\varphi_n(0) = \frac{e^{in\alpha}}{n+1}\ , \qquad \chi_n(0) = -\frac{e^{in\alpha}}{n+1}\ ,$$

with $n \geq 1$ and $\alpha \in [0, 2\pi)$, are semi-classical.

Proof: From induction arguments the following relations:

$$\varphi_n(z) = z^n + \frac{1}{n+1} \sum_{k=0}^{n-1} (k+1)\, e^{i(n-k)\alpha}\, z^k \ ,$$

$$\chi_n(z) = z^n - \frac{1}{n+1} \sum_{k=0}^{n-1} e^{i(n-k)\alpha}\, z^k$$

hold and hence,

$$\varphi_n^*(z) = 1 + \frac{1}{n+1} \sum_{k=0}^{n-1} (k+1)\, (e^{-i\alpha}\, z)^k \ ,$$

$$\chi_n^*(z) = 1 - \frac{1}{n+1} \sum_{k=0}^{n-1} (e^{-i\alpha}\, z)^k \ .$$

Since (χ_n) is the SMOP of the second kind with respect to (φ_n), the Carathéodory functions $F_1(z)$ and $F_2(z)$, associated to (φ_n) and (χ_n) respectively, satisfy

$$F_1(z) = \frac{\chi_n^*(z)}{\varphi_n^*(z)} + O(z^{n+1})\,, \qquad F_2(z) = \frac{\varphi_n^*(z)}{\chi_n^*(z)} + O(z^{n+1}) \tag{4.1}$$

(see [12], p. 11).

Thus, we get $F_1(z) = 1 - e^{-i\alpha} z$ and $F_2(z) = \frac{1}{1-e^{-i\alpha} z}$.

By Corollary 4.3, the SMOP (φ_n) and (χ_n) are semi-classical. ∎

Remark. We want point out that $\varphi_n(z) = e^{in\alpha}\, \Phi_n(e^{-i\alpha} z)$ where (Φ_n) is the SMOP satisfying $\Phi_n(0) = \frac{1}{n+1}$, for every $n \in \mathbb{N}$.

Lemma 4.5. *Let $\{a_j;\, j = 1, ..., n_0\} \subset \mathbb{C}$ be with $|a_j| \neq 1$ and $\alpha, \beta \in [0, 2\pi)$. Let us consider the SMOP (Φ_n) defined by*

$$\Phi_j(0) = a_j, \quad \text{if } \ j = 1, ..., n_0 \ ,$$

$$\Phi_{n+n_0}(0) = \frac{e^{i(n\alpha+\beta)}}{n+n_1}, \quad \text{if } \ n \geq 1 \ ,$$

where $n_1 \in \mathbb{N}$ is fixed. Then, (Φ_n) is associated to a semi-classical form.

Proof: The difference equation of second order

$$\frac{1}{n+1}\, y_{n+1} = \left[\frac{e^{i\alpha}}{n+n_1+1} + \frac{z}{n+n_1}\right] y_n - \frac{e^{i\alpha}}{n+n_1+1} \left[1 - \frac{1}{(n+n_1)^2}\right] y_{n-1}$$

has the polynomic solutions $(\varphi_n)_{n>n_1}$, $(\chi_n)_{n>n_1}$, $(\Phi_n)_{n\geq n_0}$ and $(\Psi_n)_{n\geq n_0}$, where (φ_n), (χ_n) are as in the above lemma and (Ψ_n) is the SMOP of the second kind associated to (Φ_n). Since the two first solutions are linearly independent, there exist unique polynomials P_1, P_2, Q_1, Q_2 such that

$$\begin{aligned}\Phi_{n+n_0}(z) &= P_1(z)\,\varphi_{n+n_1-1}(z) + P_2(z)\,\chi_{n+n_1-1}(z)\ ,\\ \Psi_{n+n_0}(z) &= Q_1(z)\,\varphi_{n+n_1-1}(z) + Q_2(z)\,\chi_{n+n_1-1}(z)\ ,\end{aligned} \tag{4.2}$$

for every $n \geq 1$. So, the generating function $F(z)$ associated to (Φ_n) satisfy

$$F(z) = \frac{\Psi^*_{n+n_0}(z)}{\Phi^*_{n+n_0}(z)} + O(z^{n+n_0+1})\ .$$

By substituting the values of $\Psi^*_{n+n_0}(z)$ and $\Phi^*_{n+n_0}(z)$ derived from (4.2) and taking into account (4.1) we have

$$F(z) = \frac{Q_1^{*k}(z) + Q_2^{*k}(z)\,F_1(z)}{P_1^{*k}(z) + P_2^{*k}(z)\,F_1(z)}\ ,$$

where $k = \max\{\deg P_1, \deg P_2, \deg Q_1, \deg Q_2\}$ and $P^{*k}(z) = z^k\,\overline{P}(z^{-1})$ with $k \geq \deg P$. Since, $F_1(z) = 1 - e^{-i\alpha}z$, it follows that $F(z)$ is a rational function. ∎

Remark. Let us note that the SMOP (Φ_n) is a modified of the SMOP (φ_n) in the sense used in ([6]). So Lemma 4.5 gives an improvement of Proposition 3.1 in [9].

Proof of the theorem: The implication ii)⇒iii) is obvious. We will prove iii)⇒ii)⇔i).

i)⇒ii) Let u be a semi-classical form in $\mathcal{H}$. Then there exist $A, B \in \mathcal{P}$ with $A \neq 0$ such that $\mathcal{D}[u^A] = u^B$.

If $B \neq 0$, from Theorem 2.7, (ϕ_n) is (strictly) $\mathbb{T}$-quasi-orthogonal of order $p = \deg A$ with respect to u^A and (strictly) $\mathbb{T}$-quasi-orthogonal of order $p' = \deg B$ with respect to u^B. Thus, we can deduce

$$u^A[\psi_n(z)\cdot z^{-k}] = \frac{i}{n}\,u^B[\phi_n(z)\cdot z^{-k}] + \frac{k}{n}\,u^A[\phi_n(z)\cdot z^{-k}]$$

for every $n \geq 1$. From the $\mathbb{T}$-quasi-orthogonality for the SMOP (ϕ_n), $u^A[\psi_n(z)\cdot z^{-k}] = 0$ if $r \leq k \leq n-r-1$, $(n \geq 2r+1)$, where $r = \max\{p, p'\}$.

If $B = 0$, the above expression remains as $u^A[\psi_n(z)\cdot z^{-k}] = \frac{k}{n}\,u^A[\phi_n(z)\cdot z^{-k}]$, which vanishes for $p \leq k \leq n-p-1$ $(n \geq 2p+1)$. Now, we are going to show that $u^A[\psi_n(z)\cdot z^{-n+r}] \neq 0$ for some $n \geq 2r$. If $p \neq p'$ or $p' = 0$ or $B = 0$ the proof is trivial. Let $p = p' = r$ and let us suppose that $u^A[\psi_n(z)\cdot z^{-n+r}] = 0$ for some $n \geq 2r$. By using the recurrence relation (2.1) and by the strict $\mathbb{T}$-quasi-orthogonality of (ϕ_n) with respect to u^A we get

$$u^A[\psi_{n+1}(z) \cdot z^{-n-1+r}] = \frac{1-|a_{n+1}|^2}{n+1}\, u^A[\phi_n(z) \cdot z^{-n+r}] \neq 0 \,.$$

ii)⇒i) Let (ϕ_n) and (ψ_n) be $\mathbb{T}$-quasi-orthogonal of orders p and r, respectively, with respect to $\widehat{u} \in \mathcal{H} - \{0\}$. By Theorem 2.7, there exists $A \in \mathcal{P} - \{0\}$ such that $\widehat{u} = u^A$. Let $\widetilde{u} \in \mathcal{H}$ be the form defined by $\widetilde{u} = \mathcal{D}(u^A)$. For every $n \geq 1$ and $k \in \mathbf{Z}$ we get

$$\begin{aligned} \widehat{u}[\phi_n(z) \cdot z^{-k}] &= -i\, u^A[z\, \phi_n'(z) \cdot z^{-k}] + i\, k\, u^A[\phi_n(z) \cdot z^{-k}] \\ &= -i\, n\, u^A[\psi_n(z) \cdot z^{-k}] + i\, k\, u^A[\phi_n(z) \cdot z^{-k}] \,, \end{aligned} \tag{4.3}$$

which vanishes if $s \leq k \leq n-s-1$, $(n \geq 2s+1)$, with $s = \max\{p, r\}$.

We distinguish two possible situations:

a) If $p \neq r$, writing the relation (4.3) for $k = n-s$ and $n \geq 2s$ we have

$$\widehat{u}[\phi_n(z) \cdot z^{-n+s}] = -i\, n\, u^A[\psi_n(z) \cdot z^{-n+s}] + i(n-s)\, u^A[\phi_n(z) \cdot z^{-n+s}] \,.$$

Since in the above relation, at least for some $n \geq 2s$, the right member has a term equal zero and the other term different zero, the SMOP (ϕ_n) is $\mathbb{T}$-quasi-orthogonal of order s with respect to $\widehat{u}$ and there exists a polynomial B of degree s such that $\widehat{u} = u^B$. Therefore, u is a semi-classical form.

b) If $p = r = s$, let us suppose there exists $t \in \mathbb{N}$ such that

$$\begin{aligned} &\widehat{u}[\phi_n(z) \cdot z^{-n+t}] = 0\,, \qquad n \geq 2t \,, \\ &\widehat{u}[\phi_n(z) \cdot z^{-k}] = 0\,, \qquad t \leq k \leq n-t-1, \quad n \geq 2t+1 \,. \end{aligned}$$

From (4.3) it follows that, if there exists a non-negative integer t verifying the above conditions, then $t \leq s$ is true. Now, using (2.2), an induction on t implies that either there exists q with $0 \leq q \leq s$ such that

$$\widehat{u}[\phi_n(z) \cdot z^{-n+q}] \neq 0$$

holds for every $n \geq 2q$, and

$$\widehat{u}[\phi_n(z) \cdot z^{-k}] = 0$$

holds for every $n \geq 2q+1$ with $q \leq k \leq n-q-1$, or either

$$\widehat{u}[\phi_n(z) \cdot z^{-n+t}] = 0$$

holds for every $n \geq 2t$ and for every $t \in \mathbb{N}$. In the first case, the SMOP (ϕ_n) is $\mathbb{T}$-quasi-orthogonal of order q with respect to $\widehat{u}$ and, from Theorem 2.7, there exists a polynomial B of degree s such that $\widehat{u} = u^B$; in the second one, $\widehat{u} = 0$ and $B = 0$. In both cases, $\mathcal{D}[u^A] = u^B$ with B different zero or not.

iii)⇒ii) In [16], it has been proved that

$$(\phi_n^*(z))' = \frac{n}{z}\,[\phi_n^*(z) - \psi_n^*(z)]\ . \tag{4.4}$$

Derivating the recurrence relations (2.1) and (2.2) and taking into account (4.4) we obtain

$$(n+1)\,\psi_{n+1}(z) = z[\phi_n(z) + n\,\psi_n(z)] + n\,a_{n+1}[\phi_n^*(z) - \psi_n^*(z)]\ ,$$

$$\begin{aligned}(n+1)\,\psi_{n+1}(z) = {} & (1-|a_{n+1}|^2)\,z[\phi_n(z) + n\,\psi_n(z)] \\ & + (n+1)\,a_{n+1}[\phi_{n+1}^*(z) - \psi_{n+1}^*(z)]\ .\end{aligned}$$

Let us suppose the SMOP (ψ_n) is $\mathbb{T}$-quasi-orthogonal of order r with respect to $\widehat{u}$. If $r = 0$, the only SMOP such that (ψ_n) is orthogonal with respect to any $\widetilde{u} \in \mathcal{H}$ is $\phi_n(z) = z^n$ (see [16]). Thus, we suppose $r \geq 1$ and we do not consider the trivial case $r = 0$. Then, $\widetilde{u}[\psi_n(z) \cdot z^{-k}] = 0$ is true for every $r \leq k \leq n-r-1$ and $n \geq 2r+1$, and besides the following

$$\begin{aligned}&\widetilde{u}[\phi_n(z) \cdot z^{-k}] + n\,a_{n+1}\,\widetilde{u}[\phi_n^*(z) \cdot z^{-k-1}] = 0\ , \\ &(1-|a_{n+1}|^2)\,\widetilde{u}[\phi_n(z) \cdot z^{-k}] + (n+1)\,a_{n+1}\,\widetilde{u}[\phi_{n+1}^*(z) \cdot z^{-k-1}] = 0\ ,\end{aligned} \tag{4.5}$$

holds when $r \leq k \leq n-r-1$ and $n \geq 2r+1$. By substituting in (4.5) the values of $\phi_n^*(z)$ and $\phi_{n+1}^*(z)$ derived from the relations (2.1) and (2.2), we have the system

$$\begin{aligned}&\widetilde{u}[\phi_n(z) \cdot z^{-k}] - \frac{n}{n-1}\,\widetilde{u}[\phi_{n+1}(z) \cdot z^{-k-1}] = 0\ , \\ &(1-|a_{n+1}|^2)\,\widetilde{u}[\phi_n(z) \cdot z^{-k}] - \frac{n+1}{n}\,\widetilde{u}[\phi_{n+1}(z) \cdot z^{-k-1}] = 0\ ,\end{aligned} \tag{4.6}$$

with $r \leq k \leq n-r-1$ and $n \geq 2r+1$, whose determinant is

$$M_n = \frac{1}{n(n-1)}\,[1 - (n|a_{n+1}|)^2]\ .$$

If $M_n \neq 0$ for some $n \geq 2r+1$, it follows directly that $\widetilde{u}[\phi_m(z) \cdot z^{-k}] = 0$ whenever $2r+1 \leq m \leq n$ and $r \leq k \leq m-r-1$.

Let us suppose that every $n_0 \geq 2r+1$ there exists $n \geq n_0$ such that $|a_{n+1}| \neq \frac{1}{n}$. Then, by the above argument, the relation

$$\widetilde{u}[\phi_n(z) \cdot z^{-k}] = 0$$

holds for every $n \geq 2r+1$ with $r \leq k \leq n-r-1$. Now, by using the same argument as in the first part of this proof, we can conclude that the SMOP (ϕ_n) is $\mathbb{T}$-quasi-orthogonal of order s (with $s \leq r$) with respect to $\widetilde{u}$.

Finally, let us suppose $|a_{n+1}| = \frac{1}{n}$ is true for every $n \geq n_0 \geq 2r+1$. Then the determinant M_n vanishes and the system (4.5) reduces to

$$\widetilde{u}[\phi_n(z) \cdot z^{-k}] + e^{i\theta_n}\, \overline{\widetilde{u}[\phi_n(z) \cdot z^{-n+k-1}]} = 0 \;, \tag{4.7}$$

and the system (4.6) becomes

$$\widetilde{u}[\phi_{n+1}(z) \cdot z^{-k}] - \frac{n}{n-1}\, \widetilde{u}[\phi_n(z) \cdot z^{-k}] = 0 \;, \tag{4.8}$$

where $\theta_n = \arg a_{n+1}$, $n \geq n_0$ and $r \leq k \leq n-r-1$.

Let us denote $m_{nk} = |\widetilde{u}[\phi_n(z) \cdot z^{-k}]|$ and $\omega_{nk} = \arg(\widetilde{u}[\phi_n(z) \cdot z^{-k}])$. From (4.7) and (4.8), we get

$$m_{nk}\, e^{i\omega_{nk}} = m_{n,n-k-1}\, e^{i(\theta_n - \omega_{n,n-k-1} + \pi)} \;,$$
$$m_{n+1,k+1}\, e^{i\omega_{n+1,k+1}} = \frac{n}{n-1}\, m_{nk}\, e^{i\omega_{nk}} \;.$$

Therefore, $m_{nk} = m_{nj} = m_n$ whenever $k, j \in \{r, ..., n-r-1\}$, and $m_{n+1} = \frac{n-1}{n}\, m_n$. Moreover, $\omega_{nk} = \omega_{n+1,k+1}$ and $\omega_{nk} + \omega_{n,n-k-1} = \theta_n + \pi$ for every $n \geq n_0$, which implies the relation $\theta_{n+2} = 2\theta_{n+1} - \theta_n$ is true for every $n \geq n_0$. It follows easily that $\theta_{n_0+l} = l\,\alpha + \beta$ is true with $\alpha = \theta_{n_0+1} - \theta_{n_0}$ and $\beta = \theta_{n_0}$, and thus

$$\phi_{n_0+l}(0) = \frac{e^{i(l\alpha+\beta)}}{n_0+l-1}$$

for every $l \geq 0$. From Lemma 4.5, the SMOP (ϕ_n) is associated to a semi-classical form. ∎

Corollary 4.6. *Let $u \in \mathcal{H}$ be semi-classical and let $A, B \in \mathcal{P}$ (with $A \neq 0$) such that $\mathcal{D}[u^A] = u^B$. Then, (ψ_n) is $\mathbb{T}$-quasi-orthogonal of order r with respect to u^A, where $r = \max\{\deg A, \deg B\}$.* ∎

Corollary 4.7. *Let $u \in \mathcal{H}$ be semi-classical and let us suppose that $|a_{n+1}| = \frac{1}{n}$ for some $n \geq r+1$. Then*

$$a_{n+l} = \frac{e^{i(l\alpha+\beta)}}{n+l-1}$$

for every $l \geq 1$. ∎

ACKNOWLEDGEMENTS – We would like to thank Professors Francisco Marcellán and Pascal Maroni for their remarks and useful suggestions.

REFERENCES

[1] Akhiezer, N.I. and Krein, M.G. – *Some problems of the theory of moments*, Amer. Math. Soc., Translations Math. Monograph 2, Providence, Rhode Island, 1968.

[2] Bonan, S., Lubinsky, D.S. and Nevai, P. – Orthogonal polynomials and their derivatives, *SIAM J. Math. Anal.*, 18 (1987), 1163–1176.

[3] Brezinski, C. – *Padé-type Approximation and General Orthogonal Polynomials*, I.S.N.M., vol. 50, Birkhäuser-Verlag, Basel, 1980.

[4] Chihara, T.S. – On quasi-orthogonal polynomials, *Proc. Amer. Math. Soc.*, 8 (1957), 765–767.

[5] Chihara, T.S. – *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[6] Dehesa, J.S., Marcellán, F. and Ronveaux, A. – On orthogonal polynomials with perturbed recurrence relations, *J. Comput. Appl. Math.*, 30 (1990), 203–212.

[7] Dickinson, D.J. – On quasi-orthogonal polynomials, *Proc. Amer. Math. Soc.*, 12 (1961), 185–194.

[8] Fejér, L. – Mechanischen quadraturen mit positiven Cotesschen Zahlen, *Math. Zeitschrift*, 37 (1933), 287–309.

[9] García-Lázaro, P. and Moral, L. – *On a family of orthogonal polynomials on the unit circle*, in Proceedings of the "VI Simposio Nacional sobre Polinomios Ortogonales y Aplicaciones (Gijón, 1989)" (Arias de Velasco and al., Eds.), Gijón, 1990, 115–125.

[10] García-Lázaro, P. and Moral, L. – Quasi-orthogonality on Laurent polynomials, in *IMACS Annals*, Vol. 9 (1991), 267–274.

[11] Geronimus, Ya.L. – *Polynomials orthogonal on a circle and their applications*, in Amer. Math. Soc. Translations, Series 1, Vol. 3, Providence, Rhode Island, 1962, 1–78.

[12] Geronimus, Ya.L. – *Orthogonal polynomials*, Consultants Bureau, New York, 1961.

[13] Hendriksen, E. and van Rossum, H. – *Semiclassical orthogonal polynomials*, in (C. Brezinski and al., Eds.) Lecture Notes in Math., 1171, Springer-Verlag, Berlin, 1985, 354–361.

[14] Jones, W.B., Njastad, O. and Thron, W.J. – Moment theory, ortogonal polynomials, quadrature and continued fractions associated with the unit circle, *Bull. London Math. Soc.*, 21 (1989), 113–152.

[15] Marcellán, F. – *Orthogonal polynomials and Toeplitz matrices: some applications*, in "Rational Approximation and Orthogonal Polynomials" (M. Alfaro, Ed.), Seminario Matemático "García de Galdeano", Zaragoza, 1989, 31–57.

[16] Marcellán, F. and Maroni, P. – Orthogonal polynomials on the unit circle and their derivatives, *Constr. Approx.*, 7 (1991), 341–348.

[17] Marcellán, F. and Tasis, C. – *Sobre cuasi-ortogonalidad en la circunferencia unidad. Aplicaciones*, Communication to J.M.H.L., 1988, Valladolid.

[18] Maroni, P. – Une caractérisation des polynômes orthogonaux semi-classiques, *C.R. Acad. Sci. Paris Sér. I Math.*, 301 (1985), 269–272.

[19] Maroni, P. – Prolégomènes à l'étude des polynômes orthogonaux semi-classiques, *Ann. Mat. Pura ed Appl.*, 149 (1987), 165–184.

[20] Riesz, M. – Sur le problème des moments, III, *Ark. Mat. Astr. Fys.*, 17(16) (1923), 1–52.

[21] Ronveaux, A. – Polynômes orthogonaux dont les polynômes dérivés sont quasiorthogonaux, *C.R. Acad. Sci. Paris Sér. A*, 289 (1979), 433–436.

[22] Shohat, J.A. – On mechanical quadratures, in particular, with positive coefficients, *Trans. Amer. Math. Soc.*, 42 (1937), 461–496.

[23] Szegö, G. – *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Pub., 23, Providence, Rhode Island, 1975.

[24] Tasis, C. – *Propiedades diferenciales de los polinomios ortogonales relativos a la circunferencia unidad*, Doctoral Dissertation, Gijón, 1989.

[25] Vigil, L. – Polinomios ortogonales sobre la circunferencia unidad. El camino de ida y vuelta, *Rev. Acad. Ci. Madrid*, 78 (1984), 587–589.

Manuel Alfaro,
Dpto. de Matemáticas, Universidad de Zaragoza,
50009 Zaragoza – SPAIN

and

Leandro Moral,
Dpto. de Matemática Aplicada, Universidad de Zaragoza,
50009 Zaragoza – SPAIN