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A MULTIPLICITY RESULT FOR A CLASS OF SUPERLINEAR ELLIPTIC PROBLEMS

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Abstract: We prove the existence of at least two solutions for a superlinear problem $-\Delta u = \Phi(x,u) + \tau\, e_1$ ($u \in H^1_0(\Omega)$) and e_1 is the first eigenvector of $(-\Delta, H^1_0(\Omega))$, when τ is large enough, if $\Phi \in C(\mathbb{R}, \mathbb{R})$ and $\Phi(x,s) = g(x,s) + h(x,s)$ where h is a superlinear nonlinearity with a suitable growth at $+\infty$ and g is asymptotically linear.

0 – Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$. We study the solvability of the Dirichlet problem:

$$\text{(1)} \qquad \begin{cases} -\Delta u = \phi(x,u) + y & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \,, \end{cases}$$

where $\phi\colon\ \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory's function and $y = y(x)$ is a given function in $L^2(\Omega)$. The basic assumption on ϕ concern its behaviour at both $+\infty$ and $-\infty$, namely:

$$\text{(2)} \qquad \lim_{s\to-\infty} \frac{\phi(x,s)}{s} = \beta \quad \text{and} \quad \lim_{s\to+\infty} \frac{\phi(x,s)}{s} = +\infty \,.$$

It is clear that the role of the parameter β is important. If $\beta < \lambda_1$ (where λ_j denotes the j-th eigenvalue of $(-\Delta, H^1_0(\Omega))$), the problem (1) is of the Ambrosetti–Prodi type. From the result of Amann–Hess [1], Dancer [6], De Figueiredo [7], De Figueiredo–Solimini [9], it follows that (1) admits at least two solutions for certain y and no solutions for others.

Here we suppose that $\lambda_j < \beta < \lambda_{j+1}$ for some $j \geq 1$. When $\phi(x,s) = \beta\, s + (s^+)^p$ where $2 < p < (n+2)/(n-2)$ for $n \geq 3$, Ruf–Srikanth in [12] and [13]

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have obtained that for $y = \tau e_1$, with τ large enough ($e_1 > 0$ is an eigenfunction associated with λ_1) (1) has at least two solutions. A solution is found directly, the second one is found by an application of the Generalized Mountain Pass Theorem due to Rabinowitz [11]. De Figueiredo in [8] obtains a result similiar for a very large class of nonlinearities. The conditions required in [8] in order to apply the Generalized Mountain Pass theorem are:

$$(\phi) \qquad \phi \in C^1\,, \quad \mu < \phi'_s(x,s) \text{ with } \lambda_j < \mu < \lambda_{j+1}$$

and all the assumptions which are needed to get the Palais–Smale condition. It is therefore natural to ask if conditions (ϕ) are indeed necessary (see remark 8 in [8]).

In this work we give an answer finding another class of nonlinearities (which do not satisfy (ϕ)) for which the result remains valid under the weaker assumption $\phi \in C(\mathbb{R},\mathbb{R})$. We use a slight different variational arguments (see §2) to obtain directly the existence of two different critical values for the Euler–Lagrange functional associated with (1).

In theorem (1.6) we obtain that if (as it is usual) the Palais–Smale condition holds, there exist at least two solutions (for $y = \tau e_1$ with τ large enough), when $\phi(x,s) = g(x,s) + h(x,s)$, where h is the superlinear nonlinearity with a suitable growth assumption at $+\infty$ (see $H_{+\infty}$ of 1.4) and g is asymptotically linear (see 1.4). Substantially for $G(x,s) = \int_0^s g(x,\sigma)\,d\sigma$, we require that $\lambda_j < \lim_{s\to-\infty} \frac{2G(x,s)}{s^2} = \beta < \lambda_{j+1} \le \liminf_{s\to+\infty} \frac{2G(x,s)}{s^2}$ for some $j \ge 1$ and the the quantity $\liminf_{s\to+\infty} \frac{2G(x,s)}{s^2} - \lambda_{j+1}$ is suitable large with respect to $\lambda_{j+1} - \beta$.

1 – Functional setting and statement

Let Ω be an open bounded domain in $\mathbb{R}^n$. We consider the superlinear elliptic boundary problem:

$$(1.1) \qquad \begin{cases} -\Delta u = g(x,u) + h(x,u) - t\,e_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $g, h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Caratheodory's functions, which satisfy:

(1.2) There exist $a_1, a_2 \in L^2(\Omega)$ and $b_1, b_2 \in \mathbb{R}$ such that for $s \in \mathbb{R}$ and x a.e. in Ω, $|g(x,s)| \le a_1(x) + b_1\,|s|$ and $|h(x,s)| \le a_2(x) + b_2\,|s|^{2^*-1}$ where $2^* = \frac{2n}{n-2}$ for $n \ge 3$.

Let λ_i and e_i for $i \in \mathbb{N}$ denote, respectively, the eigenvalues and the associated normalized eigenfunctions of $-\Delta v = \lambda\,v$ in Ω with $v = 0$ on $\partial\Omega$. Recall that λ_1 is a simple eigenvalue and that e_1 can be choosen positive.

Consider $H_0^1(\Omega)$ with the norm $\|u\|^2 = \int_\Omega |Du|^2$. We consider the following C^1 functional $f_t\colon H_0^1(\Omega) \to \mathbb{R}$:

$$f_t(u) = \frac{1}{2}\int_\Omega |Du|^2 - \int_\Omega G(x,u) - \int_\Omega H(x,u) + t\int_\Omega e_1\, u\ , \tag{1.3}$$

where $G(x,s) = \int_0^s g(x,\sigma)\,d\sigma$ and $H(x,s) = \int_0^s h(x,\sigma)\,d\sigma$.

Note that the critical points of f_t are weak solutions of the problem (1.1); hence we will study the behaviour of the functional f_t, in order to look for critical points of f_t.

In the following we assume these conditions on G and H:

(1.4)

- $(G_{-\infty},\beta)$ There exists $k > 0$ such that for $s \le -k$ and x a.e. in $\Omega:$ $G(x,s) = \frac{1}{2}\beta\, s^2 + G_0(x,s)$, where $\beta \in \mathbb{R}$ and $|G_0(x,s)| \le c_0(x)$ with $c_0 \in L^1(\Omega)$;
- $(G_{+\infty},\alpha)$ For $s \ge 0$ and x a.e. in $\Omega:$ $G(x,s) \ge \frac{1}{2}\alpha\, s^2 + G_1(x)$, where $\alpha \in \mathbb{R}$ and $G_1 \in L^1(\Omega)$;
- $(H_{+\infty})$ There exists $k > 0$ such that for $s \ge k$ and x a.e. in Ω: $H(x,s) = \frac{1}{p+1}s^{p+1} + H_0(x,s)$ where $|H_0(x,s)| \le \overline{c}(x)$ with $\overline{c} \in L^1(\Omega)$ and $2 < p+1 < 2^*$;
- (H_-) There exists $k > 0$ such that for $s \le -k$ and x a.e. in $\Omega |H(x,s)| \le H_1(x)$ with $H_1 \in L^1(\Omega)$.

We recall the well known Palais-Smale condition (in short (P.S.) condition).

1.5 Definition. A C^1 function defined on a Hilbert space H satisfies (P.S.) condition if for every sequence $\{u_n\}_{n\ge1}$ in H with $f(u_n)_{n\ge1}$ bounded and $\{\nabla f(u_n)\}_{n\ge1}$ converging to zero, there exists a convergent subsequence.

In the following theorem we give an example of sufficient conditions for the existence of at least two different critical values of the functional (1.3).

1.6 Theorem. *Assume hypotheses (1.2) and (1.4) with $\beta \le \alpha$ and $\lambda_j < \beta < \lambda_{j+1}$ for some $j \ge 1$. Moreover suppose that the (P.S.) condition holds. Then there exists a number $m \in]0,1[$ such that if:*

$$\lambda_{j+1} - \frac{m}{m+1}(\lambda_{j+1} - \lambda_j) < \beta < \lambda_{j+1} \quad \text{and} \quad \frac{\lambda_{j+1}}{m} - \frac{1-m}{m}\beta < \alpha\ ,$$

the functions f_t admits, for $t < 0$ and $|t|$ large enough, at least two different critical values; hence (1.1) has at least two distinct solutions.

1.7 Remark. To find an example of assumptions for $\phi = g + h$, which imply the Palais–Smale condition for the functional f_t see [8] page 291, where De Figueiredo describes a large class of superlinear problems for which Palais–Smale condition holds for the associated Euler–Lagrange functional.

2 – The variational setting

In order to study the behaviour of the functional f_t of (1.3), we start recalling some abstract arguments about the generalizations of the "mountain pass" theorem due to Rabinowitz (see [11]).

Let H be an Hilbert space and $f: H \to \mathbb{R}$ a C^1 function. Let H be the topological direct sum of two subspaces H_1 and H_2 and let $u_0 \in H$.

2.1 Definition. The function f satisfies "linking condition", with respect to u_0, H_1, H_2 if there exist $\rho_1 > 0$, $\rho_2 > 0$ and $e \in H$ such that:

$$|\rho_2 - \rho_1| < \|e\| < \rho_2 + \rho_1$$

and denoted by B_1 the ball in H_1 centered at 0 with radius ρ_1 and B_2 the ball in $\operatorname{span}[e] \oplus H_2$ centered at e with radius ρ_2, it holds:

$$\sup_{u_0+\partial B_1} f < \inf_{u_0+\partial B_2} f \ .$$

We will use the following result (see 10).

2.2 Theorem. ("Linking condition" and existence of two critical values.) *If the function f satisfies "linking condition" with respect to u_0, H_1, H_2 where $\dim H_1 < +\infty$ and the Palais–Smale condition holds, then there exist two critical values c_0 and c_1 for f such that:*

$$\inf_{u_0+B_2} f \le c_1 \le \sup_{u_0+\partial B_1} f < \inf_{u_0+\partial B_2} f \le c_0 \le \sup_{u_0+B_1} f \ .$$

3 – Proof of theorem 1.6

We premise some technical lemmas.

3.1 Lemma. *Assume (1.2) and $(G_{-\infty},\beta)$, $(H,+\infty)$ and (H_-) of (1.4) with $\lambda_j < \beta < \lambda_{j+1}$ for some $j \ge 1$. If $z \in \operatorname{span}[e_{j+1},\ldots]$ and $s \le 0$, then:*

$$(3.2)\quad f_t(s\,e_1+z)-f_t(s\,e_1) \ge \frac{\lambda_{j+1}-\beta}{2\lambda_{j+1}}\,\|z\|^2-c_1-\omega(meas(\Omega\backslash\tilde\Omega))\,(\|z\|^2+\|z\|^{p+1})\ ,$$

where c_1 *is a positive constant,* $\omega\colon \mathbb{R} \to \mathbb{R}$ *is such that* $\lim_{t\to 0}\omega(t) = 0$ *and:*

$$\tilde\Omega = \tilde\Omega(s,z) = \Big\{x\in\Omega\colon\ s\,e_1+z\le -k \ \text{and}\ s\,e_1\le -k\Big\}\ . \tag{3.3}$$

Moreover:

$$\limsup_{s\to-\infty} \mathit{meas}(\Omega\backslash\tilde\Omega) = 0 \qquad \textit{uniformly for}\ \ \|z\|\le \textit{const.}. \tag{3.4}$$

Proof: By definition of f_t and by $(G_{-\infty},\beta)$ we obtain:

$$\begin{aligned} f_t(e_1+z)-f_t(s\,e_1) = \frac{1}{2}\|z\|^2 &- \int_\Omega\Big(G(x,se_1+z)-G(x,se_1)\Big)\\ &- \int_\Omega\Big(H(x,se_1+z)-H(x,se_1)\Big) = \end{aligned} \tag{1.5}$$

$$\begin{aligned}
&= \frac{1}{2}\|z\|^2 - \frac{1}{2}\beta\int_{\tilde\Omega}\Big((se_1+z)^2-(se_1)^2\Big) - \int_{\tilde\Omega}\Big(G_0(x,se_1+z)-G_0(x,se_1)\Big)\\
&\quad - \int_{\Omega\backslash\tilde\Omega}\Big(G(x,se_1+z)-G(x,se_1)\Big) - \int_\Omega\Big(H(x,se_1+z)-H(x,se_1)\Big) =\\
&= \frac{1}{2}\|z\|^2 - \frac{1}{2}\beta\int_\Omega z^2 + \frac{1}{2}\beta\int_{\Omega\backslash\tilde\Omega}(z^2+2s\,e_1\,z)\\
&\quad - \int_{\tilde\Omega}\Big(G_0(x,se_1+z)-G_0(x,se_1)\Big) - \int_{\Omega\backslash\tilde\Omega}\Big(G(x,se_1+z)-G(x,se_1)\Big)\\
&\quad - \int_\Omega\Big(H(x,se_1+z)-H(x,se_1)\Big)\\
&\ge \frac{\lambda_{j+1}-\beta}{2\lambda_{j+1}}\,\|z\|^2 - \frac{1}{2}\beta\int_{\Omega\backslash\tilde\Omega}(z^2+2s\,e_1\,z)\\
&\quad - \int_{\tilde\Omega}\Big(G_0(x,se_1+z)-G_0(x,se_1)\Big) - \int_{\Omega\backslash\tilde\Omega}\Big(G(x,se_1+z)-G(x,se_1)\Big)\\
&\quad - \int_\Omega\Big(H(x,se_1+z)-H(x,se_1)\Big)\ .
\end{aligned}$$

The definition of $\tilde\Omega$ and (1.2) imply that:

$$\begin{aligned}
\int_{\Omega\backslash\tilde\Omega}\Big|G(x,se_1+z)-G(x,se_1)\Big| &\le \int_{\Omega\backslash\tilde\Omega}\Big(2a(x)+b\Big((s\,e_1+z)^2+(s\,e_1)^2\Big)\Big)\\
&\le \int_{\Omega\backslash\tilde\Omega}\Big(2a(x)+2b(k^2+|z|^2)\Big)\\
&\le c\,\|z\|^2_{2^*}\,(\mathrm{meas}(\Omega\backslash\tilde\Omega))^{\varepsilon/\varepsilon+1}+c\ ,
\end{aligned}$$

where $\varepsilon > 0$ is such that $2^* = 2(1+\varepsilon)$.

Furthermore the definition of $\tilde\omega$ implies:

$$\text{(3.7)}\quad \frac{1}{2}\beta\int_{\Omega\setminus\tilde\Omega}|z^2+2s\,e_1\,z| \le 2\beta\int_{\Omega\setminus\tilde\Omega}|z|\,(k+|z|)\le$$
$$\le c\,\|z\|_2\,(\text{meas}(\Omega\setminus\tilde\Omega))^{1/2}+c\,\|z\|^2_{2^*}\,(\text{meas}(\Omega\setminus\tilde\Omega))^{\varepsilon/\varepsilon+1}\ .$$

Finally by $(G_{-\infty},\beta)$ it follows:

$$\text{(3.8)}\qquad \int_{\tilde\Omega}\Big|G_0(x,se_1+z)-G_0(x,se_1)\Big|\le c\ .$$

At this point by $(H,+\infty)$ and (H_-) of 1.4 we get:

$$\text{(3.9)}\quad -\int_\Omega H(x,se_1+z)+\int_\Omega H(x,se_1)=$$
$$=-\int_{\{se_1+z\ge k\}}\Big(\frac{1}{p+1}\,(se_1+z)^{p+1}+H_0(x,se_1+z)\Big)$$
$$-\int_{\{-k\le se_1+z\le k\}}H(x,se_1+z)-\int_{\{se_1+z\le -k\}}H(x,se_1+z)$$
$$+\int_{\{se_1\le -k\}}H(x,se_1)-\int_{\{-k\le se_1\le 0\}}H(x,se_1)\ .$$

Since $s\ge 0$, by definition of $\tilde\Omega$ we have:

$$0\le\int_{\{se_1+z\ge k\}}\frac{1}{p+1}\,(s\,e_1+z)^{p+1}\le\int_{\{se_1+z\ge k\}}\frac{1}{p+1}\,|z|^{p+1}$$
$$\le\|z\|^{p+1}_{2^*}\,(\text{meas}(\Omega\setminus\tilde\Omega))^{(2^*-(p+1))/2^*}\ .$$

Taking in account the hypotheses (1.2) and $(H,+\infty)$, (H_-) of (1.4), we estimate the other terms of (3.9); hence from (3.9) and (3.10) we obtain:

$$\text{(3.11)}\quad -\int_\Omega H(x,se_1+z)+\int_\Omega H(x,se_1)\ge-\|z\|^{p+1}\,(\text{meas}(\Omega\setminus\tilde\Omega))^{(2^*-(p+1))/2^*}-c\,.$$

Finally (3.5), (3.6), (3.7), (3.8) and (3.11) imply (3.2).

We claim that $\limsup_{s\to-\infty}\text{meas}(\Omega\setminus\tilde\Omega)=0$ uniformely for $\|z\|\le c$. For the sake of contradiction assume that there exists $(s_n)_{n\ge1}$ and $(z_n)_{n\ge1}$ such that $s_n\to-\infty$, $\|z_n\|\le c$, $z_n\to z$ in $L^2(\Omega)$ and meas $A_n\to\mu>0$, where $A_n=\{x\in\Omega\colon\ s_n\,e_1+z_n\ge0\}$. Let $\chi_n(x)=1$ if $x\in A_n$ and $\chi_n(x)=0$ if $x\in\Omega\setminus A_n$. Then if we consider a subsequence $\chi_n\to\chi$ weakly in $L^2(\Omega)$ and $\chi\ge0$ a.e. in Ω. Moreover $0\ge\int(e_1+\frac{z_n}{s_n})\,\chi_n\to\int e_1\,\chi\ge0$, then $\int e_1\,\chi=0$; hence $\chi=0$ a.e. in Ω, i.e. meas $A_n\to0$, and this is a contradiction. ∎

3.12 Definition. Let $\alpha, \beta \in \mathbb{R}$ and let $Q\colon H^1_0(\Omega) \to \mathbb{R}$ be defined by:

$$Q(u) = Q_{\alpha,\beta}(u) = \int_\Omega |Du|^2 - \alpha \int_\Omega (u^+)^2 - \beta \int_\Omega (u^-)^2 \ .$$

3.13 Lemma. *Assume (1.2) and (1.4) with $\lambda_j < \beta < \lambda_{j+1}$ for some $j \geq 1$ and $\beta \leq \alpha$. Then there exists a positive constant c_2 such that, for every $t < 0$,*

$$\sup_{v \in \mathrm{span}[e_1,\ldots,e_j]} f_t(\overline{s}\, e_1 + v) \leq f_t(\overline{s} e_1) + c_2 \ , \tag{3.14}$$

where $\overline{s} = \frac{t}{\beta - \lambda_1}$.

Proof: By definition of f_t and by hypotheses (1.2)–(1.4), in the same way of the previous lemma, we obtain there exists $c_2 > 0$ such that:

$$\begin{aligned} f_t(se_1 + v) - f_t(se_1) &= \frac{1}{2}\,\|se_1 + v\|^2 - \frac{1}{2}\,\|se_1\|^2 \\ &\quad - \int_\Omega \Big(G(x, se_1 + v) - G(x, se_1)\Big) \\ &\quad - \int_\Omega \Big(H(x, se_1 + v) - H(x, se_1)\Big) + t \int_\Omega e_1\, v \\ &\leq \frac{1}{2}\Big(Q(se_1 + v) - Q(se_1)\Big) + t \int_\Omega e_1\, v + c_2 \ . \end{aligned}$$

If we put $\Gamma(s) = \frac{1}{2}\,\alpha(s^+)^2 + \frac{1}{2}\,\beta(s^-)^2$ and $\gamma(s) = \alpha\, s^+ - \beta\, s^-$, then $\Gamma(s_1) - \Gamma(s_2) - \gamma(s_2)\,(s_1 - s_2) \leq \frac{\alpha \vee \beta}{2}\,(s_1 - s_2)^2$. Therefore, since $\beta \leq \alpha$, we get:

$$\begin{aligned} &Q(se_1 + v) - Q(se_1) = \\ &\qquad = \|se_1 + v\|^2 - \|se_1\|^2 - 2\int_\Omega \Big(\Gamma(se_1 + v) - \Gamma(se_1)\Big) \\ &\qquad \leq \|v\|^2 + 2\int_\Omega D(se_1)\, Dv - \int_\Omega (\alpha \vee \beta)\, v^2 - 2\int_\Omega \Big(\alpha(se_1)^+ - \beta(se_1)^-\Big) v \\ &\qquad = \|v\|^2 - \beta \int_\Omega v^2 + Q'(se_1)(v) \\ &\qquad \leq \frac{\lambda_j - \beta}{\lambda_j}\,\|v\|^2 + Q'(se_1)(v) \ . \end{aligned} \tag{3.16}$$

Choose $s = \overline{s} = \frac{t}{\beta - \lambda_1}$, then $Q'(se_1)(v) + t \int e_1 v = 0$. At this point by (3.15) and (3.16) the lemma easily follows. ∎

3.17 Lemma. *In the same hypotheses of lemma (3.13) there exist $R_1 > 0$ and $t_1 < 0$ such that, for every $t < t_1$,*

$$\inf_{\substack{z\in \mathrm{span}[e_{j+1}\dots] \\ \|z\|=R_1}} f_t(\overline{s}e_1 + z) > \sup_{v\in \mathrm{span}[e_1,\dots,e_j]} f_t(\overline{s}e_1 + v) \ , \tag{3.18}$$

where $\overline{s} = \frac{t}{\beta-\lambda_1}$.

Proof: By lemma (3.1) if $z \in \mathrm{span}[e_{j+1},\dots]$ and $\overline{s} = \frac{t}{\beta-\lambda_1}$ with $t < 0$ we have:

$$f_t(\overline{s}e_1 + z) - f_t(\overline{s}e_1) \le a\,\|z\|^2 - c_1 - \Big(\|z\|^2 + \|z\|^{p+1}\Big)\,\varepsilon\,(t, \|z\|) \ , \tag{3.19}$$

where $a > 0$, $c_1 > 0$, $\varepsilon\,(t, \|z\|) > 0$ and $\lim_{t\to-\infty} \varepsilon\,(t, \|z\|) = 0$ uniformely for $\|z\| \le \mathrm{const}$. We can choose (for example) $R_1 > 0$ and $\varepsilon_1 > 0$ with $\varepsilon_1 < \frac{a}{2}$ such that $2c_2 + c_1 < \frac{a}{2}\,R_1^2 - \varepsilon_1\,R^{p+1}$. Hence there exists $t_1 < 0$ such that for $t < t_1$:

$$\begin{aligned} f_t(\overline{s}e_1 + z) + 2c_2 &< f_t(\overline{s}e_1) + \frac{a}{2}\,R_1^2 - c_1 - \varepsilon_1(R_1^2 + R_1^{p+1}) \\ &\le f_t(\overline{s}e_1) + a\,R_1^2 - c_1 - (R_1^2 + R_1^{p+1})\,\varepsilon(t, R_1) \ . \end{aligned}$$

By this fact, (3.14) and (3.19) we have that there exist $R_1 > 0$ and $t_1 < 0$ such that, for $t < t_1$:

$$\inf_{\substack{z\in \mathrm{span}[e_{j+1}\dots] \\ \|z\|=R_1}} f_t(\overline{s}e_1 + z) > f_t(\overline{s}e_1) + c_2 \ge \sup_{v\in \mathrm{span}[e_1,\dots,e_j]} f_t(\overline{s}e_1 + v) \ .$$

The claim immediately follows. ∎

3.20 Definition. Let $\alpha, \beta \in \mathbb{R}$ be such that $\lambda_j < \beta \le \alpha$ for some $j \ge 1$. We set:

$$M = M_{\alpha,\beta} = \Big\{ u \in H_0^1(\Omega) \colon \ Q'(u)(v) = 0 \ \ \forall\, v \in \mathrm{span}[e_1,\dots,e_j] \Big\} \ .$$

3.21 Remark. By standard arguments, as in [2], it follows that M is the graph of a Lipschitz map $\gamma\colon \mathrm{span}[e_{j+1},\dots] \to \mathrm{span}[e_1,\dots,e_j]$.

3.22 Lemma. In the same hypotheses of lemma (3.13), if there exists $u^* \in M$ (see (3.20)) such that $Q(u^*) < 0$, then for every $s \le 0$, we have:

$$\lim_{\substack{v\in \mathrm{span}[e_1,\dots,e_j],\sigma\ge 0 \\ \|\sigma u^* + v\|\to+\infty}} f_t(\overline{s}e_1 + \sigma u^* + v) = -\infty \ .$$

Proof: By definition of f_t, by similar arguments as in lemma (3.13) and (3.16), we get that there exists $c_3 > 0$ such that:

$$\begin{aligned} f_t(se_1+\sigma u^*+v) &\le \frac{1}{2}\,Q(se_1+\sigma u^*+v) - \frac{1}{p+1}\int_{\{se_1+u^*+v\ge k\}}(se_1+\sigma u^*+v)^{p+1} \\ &\quad + c_3 + t\int_\Omega e_1(se_1+\sigma u^*+v) \\ &\le \frac{1}{2}\,Q(se_1+\sigma u^*+v) + c_3 + t\int_\Omega e_1(se_1+\sigma u^*+v) \\ &\le \|se_1+v\|^2 - \frac{\beta}{2}\int_\Omega (se_1+v)^2 + \frac{1}{2}\,\sigma^2\,Q(u^*) \\ &\quad + Q'(u^*)\,(se_1+v) + c_3 + t\int_\Omega e_1(se_1+\sigma u^*+v) \\ &\le \frac{\lambda_j-\beta}{2\lambda_j}\,\|se_1+v\|^2 + \frac{1}{2}\,\sigma^2\,Q(u^*) + c_3 + t\,s + t\int_\Omega e_1(\sigma u^*+v)\ , \end{aligned}$$

where $Q'(u^*)(se_1+v) = 0$ since $u^* \in M$ and $se_1+v \in \mathrm{span}[e_1,\ldots,e_j]$. Since $Q(u^*) < 0$, the statement easily follows. ∎

3.23 Lemma. (f_t satisfies the "linking condition").
In the same hypotheses of lemma (3.22), then, for t negative and small enough, the functional f_t satisfies "linking condition" with respect to u_0, H_1, H_2 where (see (2.1)):

$$u_0 = \frac{t}{\beta-\lambda_1}\,e_1\,, \qquad H_1 = \mathrm{span}[e_1,\ldots,e_j]\,, \qquad H_2 = \mathrm{span}[e_{j+1},\ldots]\ .$$

Proof: From lemmas (3.17) and (3.22) there exist $R_1 > 0$ and $t_1 < 0$ such that for every $t < t_1$ we have for some $\rho > R_1$:

$$\begin{aligned} \inf_{\substack{z\in\mathrm{span}[e_{j+1},\ldots]\\ \|z\|=R_1}} f_t(\overline{s}e_1+z) &> \sup_{v\in\mathrm{span}[e_1,\ldots,e_j]} f_t(\overline{s}e_1+v) \\ &\ge \sup_{\substack{v\in\mathrm{span}[e_1,\ldots,e_j],\sigma\ge0\\ \|\sigma u^*+v\|=\rho}} f_t(\overline{s}e_1+\sigma u^*+v)\ , \end{aligned}$$

where $\overline{s} = \frac{t}{\beta-\lambda_1}$.

We wish to remark that in this case the "linking condition" of (2.1) is satisfied with $B_1 = \{\sigma u^*+v\colon\ \sigma\ge0,\ v\in\mathrm{span}[e_1,\ldots,e_j],\ \|\sigma u^*+v\|\le\rho\}$ and $B_2 = B(0,R_1)\cap\mathrm{span}[e_{j+1},\ldots]$.

At this point, by lemma (3.23), the variational principle (2.2) can be applied to obtain the following result.

3.24 Theorem. (f_t has two critical values).
Assume the same hypotheses of lemma (3.22) and suppose that the functional f_t satisfies the Palais–Smale condition (see (1.7)). Then there exist two different critical values for the functional f_t for t negative and small enough. ∎

We now characterized a subset S of $\mathbb{R}^2$ such that, if $(\alpha,\beta)\in S$, there exists $u^*\in M$ for which $Q(u^*)<0$.

3.25 Lemma. *Let:*

$$S=\left\{(\alpha,\beta\in\mathbb{R}^2\colon\ \lambda_{j+1}-\frac{m}{m+1}(\lambda_{j+1}-\lambda_j)<\beta<\lambda_{j+1},\ \frac{\lambda_{j+1}}{m}-\frac{1-m}{m}\,\beta<\alpha\right\}$$

where $m=m(j)=\inf\{\int v^2+\int((e_{j+1}+v)^+)^2\colon\ v\in\operatorname{span}[e_1,\dots,e_j]\}$ *and* $m\in[0,1]$. *Then for each* $(\alpha,\beta)\in S$ *there exists* $u^*=e_{j+1}+\gamma(e_{j+1})\in M$ *(see (3.20)) such that* $Q(u^*)<0$.

Proof: Recalling the definition of γ given in (3.21), we obtain:

$$\begin{aligned}Q(e_{j+1}+\gamma(e_{j+1}))&=\|e_{j+1}+\gamma(e_{j+1})\|^2-\alpha\int_\Omega\Big((e_{j+1}+\gamma(e_{j+1}))^+\Big)^2\\&\quad-\beta\Big((e_{j+1}+\gamma(e_{j+1}))^-\Big)^2\end{aligned}$$

$$=\|e_{j+1}+\gamma(e_{j+1})\|^2-\beta\int_\Omega\Big(e_{j+1}+\gamma(e_{j+1})\Big)^2+(\beta-\alpha)\int_\Omega\Big((e_{j+1}+\gamma(e_{j+1}))^+\Big)^2$$

$$=\lambda_{j+1}-\beta+\|\gamma(e_{j+1})\|^2-\beta\int_\Omega\gamma(e_{j+1})^2+(\beta-\alpha)\int_\Omega\Big(e_{j+1}+\gamma(e_{j+1}))^+\Big)^2$$

$$\le\lambda_{j+1}-\beta+(\lambda_j-\beta)\int_\Omega\gamma(e_{j+1})^2+(\beta-\alpha)\int_\Omega\Big(e_{j+1}+\gamma(e_{j+1}))^+\Big)^2\,.$$

If $\lambda_j-\beta\le\beta-\alpha$, then $Q(e_{j+1}+\gamma(e_{j+1}))\le\lambda_{j+1}-\beta+(\beta-\alpha)m$. There exist α and β for which $\lambda_{j+1}-\beta+(\beta-\alpha)\,m<0$. In fact, since $m<1$, we can choose $\beta\in(\lambda_j,\lambda_{j+1})$ such that $\frac{\beta-\lambda_{j+1}}{m}>\lambda_j-\beta$, that is $\beta\ge\beta_0=\frac{m\lambda_j+\lambda_{j+1}}{m+1}$, and next we take α so that $\frac{\beta-\lambda_{j+1}}{m}>\beta-\alpha>\lambda_j-\beta$, that is $2\beta-\lambda_j\ge\alpha>\frac{\lambda_{j+1}}{m}-\frac{1-m}{m}\,\beta$.

If $\lambda_j-\beta\ge\beta-\alpha$, that is $\alpha>2\beta-\lambda_j$, then $Q(e_{j+1}+\gamma(e_{j+1}))\le\lambda_{j+1}-\beta+(\lambda_j-\beta)\,m$. Taking $\beta\ge\beta_0$, then $\lambda_{j+1}-\beta+(\lambda_j-\beta)\,m<0$. Hence the statement. Finally, by (3.24) and (3.25), we get immediatly the theorem (1.6). ∎

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