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A HADAMARD TYPE THEOREM FOR THE STATIC SPACE-TIME

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Abstract: Necessary and sufficient conditions on the static space-time are determined, in order that the exponential function $\exp_p$ be a diffeomorphism.

It is well known that the Hopf–Rinow theorem states that a connected and complete Riemannian manifold is geodesically connected.

We can express this concept saying that, if M_0 is such a manifold, then the exponential map

$$\exp_p\colon\, T_p\, M_0 \to M_0$$

is onto, for every $p \in M_0$.

Hadamard theorem goes beyond this result and shows that, if M_0 is simply connected and complete and if its sectional curvature is $k \le 0$, then the map $\exp_p$ is a diffeomorphism, for every $p \in M_0$; in particular:

i) There are not couples of conjugate points on M_0;

ii) M_0 is diffeomorphic to $\mathbb{R}^n$ $(n = \dim M_0)$;

iii) For any couple of points $p, q \in M$, there exists a unique geodesic

$$\gamma\colon\, \mathbb{R} \to M_0$$

such that

$$\Big(\gamma(0) = p;\quad \gamma(1) = q\Big)\ .$$

The property (iii) above implies not only the existence of γ, that is to claim $\exp_p$ is onto, but also the uniqueness of it, that is: $\exp_p$ is one to one.

No result like Hadamard theorem is known about semi-Riemannian manifolds; on the contrary, there are counter-examples to the Hopf–Rinow theorem in the case of static Lorentz manifolds (e.g.: see the anti-De Sitter space, on [9], [13]).

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However sufficient conditions which guarantee the validity of the Hopf–Rinow theorem for static space-times, have been established in [7].

The Lorentz metric $g = g(P)$ considered there applies to a manifold M of the following form:

$$M = M_0 \times \mathbb{R} \ ,$$

where M_0 is furnished with a Riemannian metric (index $= 0$) denoted by: $\langle \cdot\,, \cdot \rangle_{M_0}$ and $g(P)$ is defined by:

$$g(P)\Big[(\zeta,\tau),(\zeta,\tau)\Big] = \langle \zeta,\zeta\rangle - \beta(P_0)\,\tau^2$$

for every $P = (P_0, t_0) \in M$ and $(\zeta,\tau) \in T_pM$; here β is a positive real function on M_0.

In the following $g(P)$ will be denoted by $\langle \cdot\,, \cdot \rangle_M$, if the context is clear.

In this work we examine necessary and sufficient conditions on the metric $g(P)$ for the function $\exp_p$ to be a diffeomorphism, for any $P \in M$. Such conditions, on the contrary of what might be expected, are not a straightforward generalization of the Hadamard ones. Before showing our results, we recall a result of [7]. Some notation now.

Let $P = (x_0, t_0)$, $Q = (x_1, t_1)$ be two events in M. We shall consider loop spaces Ω^1 on M_0 and M, defined in the following way:

$$\Omega^1 = \Omega^1(M_0, x_0, x_1) = \Big\{ x\colon\ [0,1] \to M_0\colon\ \ x \textit{ absolutely continuous and such that } \int_0^1 \langle \dot x, \dot x\rangle_{M_0}\, ds < +\infty,\ \ x(0){=}x_0,\ x(1){=}x_1 \Big\}\ .$$

Ω^1 is a Riemannian manifold, modelled on Sobolev spaces, of curves in M_0.

The geodesic on M, joining P and Q, are the critical points of the action functional f, defined by:

$$f(\gamma) = \int_0^1 \langle \dot\gamma(s), \dot\gamma(s)\rangle_M \ ds \ .$$

In [7] the following variational principle has been proved:

Let $M = M_0 \times \mathbb{R}$ be a static space-time with the metric $\langle \cdot\,, \cdot \rangle_M$.

Let $P = (x_0, t_0)$, $Q = (x_1, t_1)$ two points in M.

A curve $\gamma(s) = (x(s), t(s))$ is a critical point of f (i.e.: a geodesic) if and only if:

$\cdot$) *$x = x(s)$ is a critical point of the functional J on Ω^1, defined by:*

$$J(x) = \int_0^1 \langle \dot x, \dot x\rangle_{M_0}\ ds - \frac{\Delta^2}{\int_0^1 \frac{ds}{\beta(x)}}\ ,$$

where: $\Delta = t_1 - t_0$;

$\cdot\cdot$) $t = t(s)$ *is the function such that:*

$$t(0) = 0 \quad \text{and} \quad \dot{t}(s) = \Delta\Big(\int_0^1 (1/\beta(x))\, ds\Big)^{-1} (1/\beta(x(s))) = \Phi(x) \ .$$

Moreover if $\gamma(s) = (x(s), t(s))$ *is a geodesic, then we have:*

$$f(\gamma) = J(x) \ .$$

This variational principle allows to use the Riemannian techniques for problems over the manifold M and allows to apply the Lusternik and Schnirelman critical point theory too.

We want also to recall a result obtained in [7] which is of interest for what follows.

Using the same symbols as above, *if* M_0 *is connected and complete and if* $\beta \in C^3(M_0; \mathbb{R}^+)$ *is a function limited from above and from below by positive constants, then the Lorentz manifold* (M, g) *is geodesically connected.*

In the sequel we shall write simply:

$$J(x) = \Phi_1(x) - \Delta^2 \Phi_2(x) \ ,$$

where:

$$\Phi_1(x) = \int_0^1 \langle \dot{x}, \dot{x} \rangle_{M_0}\, ds \ ,$$

$$\Phi_2(x) = \frac{1}{\displaystyle\int_0^1 \frac{ds}{\beta(x)}} \ .$$

Our main result is the following.

Theorem (1). *Let* M, M_0, $\mathbb{R}$, g, *be defined as above. Suppose that:*

i) M_0 *is simple connected and complete;*

ii) $\beta \in C^3(M_0; \mathbb{R}^+)$ *is a function limited from above and from below by positive constants;*

iii) *the sectional curvature on* M_0 *is* $k \leq 0$.

Under such assumptions, if the function $\exp_p : T_p M \to M$ *is a diffeomorphism for any point* $P \in M$, *then the Riemannian Hessian* $H_R^\beta(P_0)$ *is a negative semidefinite quadratic form, for any point* $P_0 \in M_0$.

Viceversa, if the Riemannian Hessian $H_R^\beta(P_0)$ *is a negative definite quadratic form for any point* $P_0 \in M_0$, *then the function* $\exp_p$ *is a diffeomorphism for any point* $P \in M$.

Now we get to prove Theorem (1) and some complementary results.

Proposition (2). *Let : $\beta \in C^2(M_0; \mathbb{R}^+)$; $P_0 \in M_0$; R be the curvature tensor on M. Let τ_1 be the unitary vector field along $\mathbb{R}$, tangent to $\mathbb{R}$. The following results are valid:*

i) $\langle R_{v,\tau_1} v, \tau_1\rangle_{(P_0,t_0)} = \frac{1}{2} H_R^\beta(P_0)[v,v] - \Big(\langle v, \operatorname{grad} \sqrt{\beta}\rangle_{P_0}\Big)^2, \forall\, t_0 \in \mathbb{R}, \forall\, v \in T_{P_0} M_0.$

ii) $H_R^\beta(P_0)$ *is a negative semidefinite form iff:*

$$\langle R_{v,\tau_1} v, \tau_1\rangle_{(P_0,t_0)} \le -\Big(\langle \operatorname{grad} \sqrt{\beta}, v\rangle_{P_0}\Big)^2 \ ,$$

$\forall\, t_0 \in R, \forall\, v \in T_{P_O} M_0.$

Proof: (ii) is a straightforward consequence of (i). In order to prove (i), we recall that:

$$\begin{aligned}
\langle R_{v,\tau_1}, v, \tau_1\rangle_{(P_0,t_0)} &= \frac{-1}{\sqrt{\beta(P_0)}} H_R^{\sqrt{\beta}}(P_0)[v,v] \cdot \langle \tau_1, \tau_1\rangle_M \\
&= \Big(\frac{-1}{\sqrt{\beta(P_0)}} H_R^{\sqrt{\beta}}(P_0)[v,v]\Big) \cdot (-\beta(P_0)) \\
&= \sqrt{\beta(P_0)}\, H_R^{\sqrt{\beta}}(P_0)[v,v] , \qquad \forall\, v \in T_{P_0} M_0 \ .
\end{aligned}$$

Now we observe that:

$$\begin{aligned}
(2) \quad & H_R^{\sqrt{\beta}}(P_0)[v,v] = \Big\langle D_v(\operatorname{grad} \sqrt{\beta}), v\Big\rangle_{P_0} = \\
&= \frac{1}{2}\Big\langle D_v\Big(\frac{1}{\sqrt{\beta}} \operatorname{grad} \beta\Big), v\Big\rangle_{P_0} \\
&= \frac{1}{2}\Big\langle \Big(\Big\langle v, \operatorname{grad}\ \frac{1}{\sqrt{\beta(P)}}\Big\rangle_{M_0}\Big) \operatorname{grad} \beta + \frac{1}{\sqrt{\beta(P)}} D_v(\operatorname{grad} \beta), v\Big\rangle_{P_0} \\
&= \Big(-\frac{1}{4}\Big) \frac{1}{\beta(P_0)\sqrt{\beta(P_0)}} \Big(\langle v, \operatorname{grad} \beta\rangle_{P_0}\Big)^2 + \frac{1}{2\sqrt{\beta(P_0)}}\ \langle D_v(\operatorname{grad} \beta), v\rangle_{P_0} \\
&= \Big(-\frac{1}{4}\Big) \frac{1}{\beta(P_0)\sqrt{\beta(P_0)}} \Big(\langle v, \operatorname{grad} \beta\rangle_{P_0}\Big)^2 + \frac{1}{2\sqrt{\beta(P_0)}} H_R^\beta(P_0)[v,v], \ ,
\end{aligned}$$

$$\forall\, v \in T_{P_0} M_0 \ .$$

(i) follows from (1) and (2). ∎

Lemma (3). *Assume:*

i) *M_0 is simply connected and complete;*

ii) *$\beta \in C^3(M_0; \mathbb{R}^+)$ is a function limited from above and from below by positive constants;*

iii) *the sectional curvature on M is $K \leq 0$.*

If the Riemannian Hessian $H_R^\beta(P_0)$ is a negative definite quadratic form for any point $P_0 \in M_0$, then the quadratic form $J''(x)$ is positive definite for any curve $x = x(s)$ $(s \in I)$ which is a critical point of the functional J and for any couple of extreme points $x(0), x(1) \in M_0$.

Proof: Let x be a critical point of J. We shall show:

1) $J''(x)$ is a positive definite form if $\Phi_2''(x)$ is negative definite;

2) $\Phi_2''(x)$ is negative definite if the integral functional:

$$I(x)[v,v] = \int_0^1 H_R^\beta(x)[v,v]\, ds \ ,$$

whose domain is the set of vector fields with compact support, tangent to M_0 along x, is a negative definite form.

If $v = v(s)$ is a vector field having compact support and tangent to M_0 along x, we have:

$$\cdot) \qquad \Phi_1(x) = \int_0^1 \langle \dot{x}, \dot{x} \rangle_{x(s)}\, ds \ ;$$

$$\cdot\cdot) \qquad \Phi_1''(x)[v,v] = 2\int_0^1 \langle D_s v, D_s v \rangle_{x(s)}\, ds - 2\int_0^1 \langle R_{\dot{x},v}\, \dot{x}, v \rangle_{x(s)}\, ds \ .$$

Besides that, we have:

$$\cdot) \qquad \Phi_2(x) = \frac{1}{\int_0^1 \frac{ds}{\beta(x(s))}} \ ,$$

$$\cdot\cdot) \qquad \Phi_2'(x)[v] = \frac{1}{\left(\int_0^1 \frac{ds}{\beta(x(s))}\right)^2} \int_0^1 \frac{\langle \operatorname{grad}\beta, v \rangle_{x(s)}}{\beta^2(x(s))}\, ds \ ,$$

$$\cdots) \qquad \Phi_2\text{''}(x)[v,v] = \frac{2}{\left(\int_0^1 \frac{ds}{\beta(x(s))}\right)^3} \left(\int_0^1 \frac{\langle \operatorname{grad}\beta, v \rangle_{x(s)}}{\beta^2(x(s))}\, ds\right)^2$$

$$+ \frac{1}{\left(\int_0^1 \frac{ds}{\beta(x(s))}\right)^2} \int_0^1 (-2)\frac{\left(\langle \operatorname{grad}\beta, v \rangle_{x(s)}\right)^2}{\beta^3(x(s))}\, ds + \frac{1}{\left(\int_0^1 \frac{ds}{\beta(x(s))}\right)^2} \int_0^1 \frac{H_R^\beta(x)[v,v]}{\beta^2(x(s))}\, ds \ .$$

Now (1) comes straight, because:

$$J''(x) = \Phi_1''(x) - \Delta\Phi_2''(x) \ .$$

Also (2) is soon found, as a consequence of the following

$$\frac{2}{\left(\int_0^1 \frac{ds}{\beta(x(s))}\right)^3}\left(\int_0^1 \frac{\langle \operatorname{grad}\beta, v\rangle_{x(s)}}{\beta^2(x(s))}\,ds\right)^2 - \frac{2}{\left(\int_0^1 \frac{ds}{\beta(x(s))}\right)^2}\int_0^1 \frac{\left(\langle \operatorname{grad}\beta, v\rangle_{x(s)}\right)^2}{\beta^3(x(s))}\,ds =$$

$$= \frac{2}{\left(\int_0^1 \frac{ds}{\beta(x(s))}\right)^2}\left\{\frac{1}{\int_0^1 \frac{ds}{\beta(x(s))}}\left(\int_0^1 \frac{\langle \operatorname{grad}\beta, v\rangle_{x(s)}}{\beta^{1/2}(x(s))\,\beta^{3/2}(x(s))}\right)^2 + \right.$$

$$\left. - \int_0^1 \frac{\left(\langle \operatorname{grad}\,\beta, v\rangle_{x(s)}\right)^2}{\beta^3(x(s))}\,ds\right\} \le$$

$$\le \frac{2}{\left(\int_0^1 \frac{ds}{\beta(x(s))}\right)^2}\left\{\left(\int_0^1 \frac{ds}{\beta(x(s))}\int_0^1 \frac{\left(\langle \operatorname{grad}\beta, v\rangle_{x(s)}\right)^2}{\beta^3(x(s))}\,ds\right)\cdot \frac{1}{\int_0^1 \frac{ds}{\beta(x(s))}} + \right.$$

$$\left. - \int_0^1 \frac{\left(\langle \operatorname{grad}\beta, v\rangle_{x(s)}\right)^2}{\beta^3(x(s))}\,ds\right\} = 0 \ . \ \blacksquare$$

Lemma (4). *Assume:*

i) *M_0 is simply connected and complete;*

ii) *$\beta \in C^3(M_0; \mathbb{R}^+)$ is a function limited from above and from below by positive constants;*

iii) *the sectional curvature on M_0 is $k \le 0$.*

If the quadratic form $J''(x)$ is positive definite, for any curve $x = x(s)$ $(s \in I)$ which is a critical point of the functional J, and for any couple of extreme points $x(0), x(1) \in M_0$, then the Riemannian Hessian $H_R^\beta(P_0)$ is a negative semidefinite quadratic form, for any point $P_0 \in M_0$.

Proof: Let $P_0 \in M_0$. By [7], there exists a geodesic $\gamma(s) = (x(s), t(s))$, $s \in I$, contained in the manifold M and starting from P_0. Such geodesic depends on Cauchy data for a system of second order differential equations, then it depends on the point P_0, on the initial instant $t(0) = t_0$ and on the initial velocity $\dot\gamma(0) = (\dot x(0), \dot t(0))$. Now, as we are dealing with a static metric, it results (see introduction):

$$t(s) = t(0) + \frac{\int_0^s \frac{d\rho}{\beta(x(\rho))}}{\frac{d\rho}{\beta(x(\rho))}}\,\Delta\,, \qquad \begin{cases} s \in I \\ \Delta = t(1) - t(0)\ , \end{cases}$$

$$\dot{t}(s) = \frac{1}{\int_0^1 \frac{d\rho}{\beta(x)}} \cdot \frac{\Delta}{\beta(x(s))}\ ,$$

then $\dot{t}(s)$ depends on $x(s)$ and on Δ; therefore $\gamma(s)$ depends on the vector $\mu_1 = \dot{x}(0)$, on Δ and on the point $(P_0, t_0) \in M$.

Fixing $t(0) = t_0$ and Δ, we fix the temporal extrema of the geodesic arc $\gamma(s)$, $s \in I$, whereas fixing P and μ_1, we fix its spatial extrema; the vector $\mu(0) = \mu_0$, such that:

$$\mu_1 = \dot{x}(0) = \|\dot{x}(0)\|_{T_{P_0}}\, M_0\, \mu_0$$

determines the direction of γ.

Stated that, let $\mu_1 = \dot{x}(0)$ and Δ be fixed and $\gamma = \gamma_{\mu_1;\Delta}$ be the corresponding geodesic, with extreme points:

$$P = \gamma(0) = (x(0), t(0))\,, \qquad x(0) = P_0\ ,$$

$$Q = \gamma(1) = ((x(1), t(1))\ .$$

The curve $x = x(s)$, $s \in I$ is a critical point of J. Let $\nu_0 \in T_{P_0} M_0$ be a unitary vector and let:

$$\Big\{\nu = \nu(s) \mid \forall\, s \in I\colon\ \nu(s) \in T_{x(s)} M_0;\ \|\nu(s)\|_{T_{x(s)} M_0} = 1\Big\}$$

be the unitary vector field, obtained by parallel translation of ν_0 along x. We consider now a sequence of functions:

(1) $$s\colon\ I \to \mathbb{R} \qquad\qquad k \in \mathbb{N}$$

$$s_k(\sigma) = \frac{1}{k(k+1)}\sigma + \frac{1}{k+1}$$

and set:

$$x_k(\sigma) = x(s_k(\sigma))\ ,$$

$$t_k(\sigma) = t(s_k(\sigma))\ ;$$

from (1) it follows:

$$\dot{x}_k(\sigma) = \frac{1}{k(k+1)}\,\dot{x}(s)\,|_{s=s_k(\sigma)}\ ,$$

$$\dot{t}_k(\sigma) = \frac{1}{k(k+1)}\,\dot{t}(s)\,|_{s=s_k(\sigma)}\ ;$$

besides that, we have:

$$\begin{aligned}\Delta_k &= \int_0^1 \dot t_k(\sigma)\,d\sigma = \int_{1/k+1}^{1/k} \Big(\frac{\dot t(s)}{k(k+1)}\Big)\, k(k+1)\,ds \\ &= \int_{1/k+1}^{1/k} \dot t(s)\,ds = \int_{1/k+1}^{1/k} \frac{c}{\beta(x)}\,ds \\ &= \frac{\int_{1/k+1}^{1/k}\frac{1}{\beta(x)}\,ds}{\int_0^1 \frac{1}{\beta(x)}\,ds}\,\Delta\ ;\end{aligned}$$

here, we have used: $\beta(x)\,\dot t = c$ (see the introduction).

The curve $\gamma_k = (x_k, t_k)$ are monotone linear reparametrizations of pieces of $\gamma = (x,t)$, therefore they are all geodesics and the sequence $\{x_k\}$ is made of critical points of J.

Now we use an arbitrarily fixed function $\varphi \in C_0^\infty(I, \mathbb{R}^+)$ and build up the following vector fields with compact support:

$$\begin{cases} v_k(s) = v_k(x(s))\,|_{s=s_k(\sigma)} = \varphi(\sigma_k(s))\,\nu(s), & \frac{1}{k+1} \le s \le \frac{1}{k}, \\ v_k(s) = 0, & s \in I\backslash[\frac{1}{k+1}, \frac{1}{k}]\ ; \end{cases}$$

here, σ_k is the inverse function of s_k.

We can also write: $v_k(\sigma) = \varphi(\sigma)\,\nu(s_k(\sigma))$; then observe that:

$$\begin{aligned}\int_0^1 \langle D_\sigma v_k, D_\sigma v_k\rangle_{x_k(\sigma)}\,d\sigma &= \int_0^1 \langle D_{\dot x_k} v_k, D_{\dot x_k} v_k\rangle_{x_k(\sigma)}\,d\sigma = \\ &= \int_0^1 \Big\langle D_{\dot x_k}\Big(\varphi(\sigma)\,\nu(s_k(\sigma))\Big), D_{\dot x_k}\Big(\varphi(\sigma)\,\nu(s_k(\sigma))\Big)\Big\rangle_{x_k(\sigma)} d\sigma \\ &= \int_0^1 \Big\langle \dot\varphi(\sigma)\,\nu(s_k(\sigma)) + \varphi(\sigma)\,D_{\dot x_k}\nu(s_k(\sigma)),\ \dot\varphi(\sigma)\,\nu(s_k(\sigma)) \\ &\quad + \varphi(\sigma)\,D_{\dot x_k}\nu(s_k(\sigma))\Big\rangle_{x_k(\sigma)} d\sigma \\ &= \int_0^1 \Big(\frac{d\varphi}{d\sigma}\Big)^2 d\sigma = \frac{1}{k(k+1)}\int_{1/k+1}^{1/k}\Big(\frac{d\varphi}{ds}\Big)^2 ds\ ,\end{aligned}$$

indeed:

$$\begin{aligned}\langle \dot x_k, \varphi\rangle_{x_k(\sigma)} &= \Big\langle x_k^*\Big(\frac{d}{d\sigma}\Big), \varphi\Big\rangle_{x_k(\sigma)} \\ &= \Big\langle \frac{d}{d\sigma}, \varphi\circ x_k\Big\rangle_{x_k(\sigma)} = \frac{d}{d\sigma}\,\varphi(x_k(\sigma)) = \frac{d\varphi(\sigma)}{d\sigma}\ .\end{aligned}$$

Therefore, we have (see Lemma (3)):

$$(3)\qquad J''(x_k)[v_k,v_k] = \Phi_1''(x_k)[v_k,v_k] - \Delta_k^2\Phi_2''(x_k)[v_k,v_k] =$$
$$= 2\int_0^1 \langle D_\sigma v_k, D_\sigma v_k\rangle_{x_k(\sigma)}\, d\sigma - 2\int_0^1 \langle R_{\dot{x}_k,v_k}\dot{x}_k, v_k\rangle_{x_k(\sigma)}\, d\sigma +$$
$$-\Delta_k^2\Bigg\{ \frac{2}{\Big(\int_0^1 \frac{d\sigma}{\beta(x_k)}\Big)^3}\Big(\int_0^1 \frac{\langle \operatorname{grad}\beta, v_k\rangle_{x_k(\sigma)}}{\beta^2(x_k)}\, d\sigma\Big)^2 +$$
$$-\frac{2}{\Big(\int_0^1 \frac{d\sigma}{\beta(x_k)}\Big)^2}\int_0^1 \frac{\Big(\langle \operatorname{grad}\beta, v_k\rangle_{x_k(\sigma)}\Big)^2}{\beta^3(x_k)}\, d\sigma +$$
$$+\frac{1}{\Big(\int_0^1 \frac{d\sigma}{\beta(x_k)}\Big)^2}\int_0^1 \frac{1}{\beta^2(x_k)}\, H_R^\beta(x_k)[v_k,v_k]\, d\sigma \Bigg\} > 0\ ,$$

whence:

$$(4)\qquad 2\int_0^1 \Big(\frac{d\varphi}{d\sigma}\Big)^2 d\sigma - 2\int_0^1 \langle R_{\dot{x}_k,v_k}, \dot{x}_k, v_k\rangle_{x_k(\sigma)}\, d\sigma +$$
$$-\Bigg(\frac{\Delta_k}{\int_0^1 \frac{d\sigma}{\beta(x_k)}}\Bigg)^2 \int_0^1 \frac{1}{\beta^2(x_k)}\, H_R^\beta(x_k)[v_k,v_k]\, d\sigma >$$
$$> 2\Bigg(\frac{\Delta_k}{\int_0^1 \frac{d\sigma}{\beta(x_k)}}\Bigg)^2 \Bigg\{ \frac{1}{\int_0^1 \frac{d\sigma}{\beta(x_k)}}\Big(\int_0^1 \frac{\langle \operatorname{grad}\beta, v_k\rangle_{x_k(\sigma)}}{\beta^2(x_k)}\, d\sigma\Big)^2 +$$
$$-\int_0^1 \frac{\Big(\langle \operatorname{grad}\beta, v_k\rangle_{x_k(\sigma)}\Big)^2}{\beta^3(x_k)}\, d\sigma \Bigg\}$$

and using (2):

$$(5)\qquad 2\int_0^1 \Big(\frac{d\varphi}{d\sigma}\Big)^2 d\sigma - 2\int_0^1 \langle R_{\dot{x}_k,v_k}, \dot{x}_k, v_k\rangle_{x_k(\sigma)}\, d\sigma +$$
$$-\Big(\frac{1}{k(k+1)}\Big)^2 \Bigg(\frac{\Delta}{\int_0^1 \frac{ds}{\beta(x)}}\Bigg)^2 \inf_{\sigma\in I}\Big(\frac{1}{\beta^2(x_k)}\, H_R^\beta(x_k)[\nu,\nu]\Big) \int_0^1 \varphi^2(\sigma)\, d\sigma >$$
$$> 2\Big(\frac{1}{k(k+1)}\Big)^2 \Bigg(\frac{\Delta}{\int_0^1 \frac{ds}{\beta(x)}}\Bigg)^2 \Bigg\{ \frac{k(k+1)}{\int_{1/k+1}^{1/k} \frac{ds}{\beta(x)}}\Big(\int_{1/k+1}^{1/k} \frac{\langle \operatorname{grad}\beta, v_k\rangle_{x(s)}}{\beta^2(x)}\, ds\Big)^2 +$$
$$-k(k+1)\int_{1/k+1}^{1/k} \frac{\langle \operatorname{grad}\beta, v_k\rangle_{x(s)}}{\beta^3(x)}\, ds \Bigg\}.$$

Afterwards, we are going to apply the mean theorem for continuous functions to the second part of inequality (5); so we obtain:

$$(6)\qquad 2\int_0^1\left(\frac{d\phi}{d\sigma}\right)^2 d\sigma - 2\int_0^1 \langle R_{\dot{x}_k,v_k}\dot{x}_k, v_k\rangle_{x_k(\sigma)}\, d\sigma + \\ -\left(\frac{\Delta}{k(k+1)\int_0^1 \frac{ds}{\beta(x)}}\right)^2 \inf_{\frac{1}{k+1}\le s\le\frac{1}{k}}\left(\frac{1}{\beta^2(x)}\, H_R^\beta(x)[\nu,\nu]\right)\int_0^1 \varphi^2(\sigma)\, d\sigma >$$

$$> 2\left(\frac{\Delta}{k(k+1)\int_0^1 \frac{ds}{\beta(x)}}\right)^2\left\{\beta(x(s_1^{(k)}))\,\frac{\left(\langle \operatorname{grad}\beta, v_k\rangle_{x(s_2^{(k)})}\right)^2}{\beta^4(x(s_2^{(k)}))} - \frac{\left(\langle \operatorname{grad}\beta, v_k\rangle_{x(s_3^{(k)})}\right)^2}{\beta^3(x(s_3^{(k)}))}\right\};$$

here, the $s_i \in]\frac{1}{k+1},\frac{1}{k}[$ are suitable points.

At last observe that, by the completeness of M, the geodesic $\gamma=\gamma(s)$ can be extended from $s\in I$ up to $s\in\mathbb{R}$; after making that, let's divide the inequality (6) by Δ^2 and go to the limit for $(\Delta\to+\infty)$, then let's multiplicate the resulting inequality by $(k(k+1))^2$ and go again to the limit for $(k\to+\infty)$. We shall obtain the thesis of the lemma. ∎

Proposition (5). *We suppose that*

1) *$\gamma=(x,t)$ is a geodesic contained in M;*

2) *$f(\gamma)=\int_0^1\langle\dot\gamma,\dot\gamma\rangle_M\, ds$;*

3) *$\Phi(x)=t$ is the function obtained in [7] (see the introduction);*

let us denote

4) *v a vector field having compact support and tangent to M_0 along x;*

5) *(w,τ) a vector field having compact support and tangent to M along γ.*

Then:

$$J''(x)[v,w] = f''(\gamma)\left[(v,\Phi'(x)[v]),(w,\tau)\right]\,.$$

Proof: On smooth curves $y=y(s)$ contained in M_0, it is identically (see [7]):

$$(1)\qquad f'_t(y,t)\,|_{t=\Phi(y)} = 0\;;$$

therefore we have identically:

$$\frac{d}{dy} f'_t(y,\Phi(y)) = 0\;,$$

that is, for any vector field $w = w(s)$ having compact support and tangent to M_0 along y, for any vertical vector field τ along $\Phi(y)$, having compact support, we have:

$$f''_{tx}(y, \Phi(y))[w, \tau] + f''_{tt}(y, \Phi(y)) \left[\Phi'(y)[w], \tau\right] = 0 \; . \tag{2}$$

Now recall that:

$$\begin{aligned}
&f''(\gamma)\left[(v, \tau_1), (w, \tau_2)\right] = \\
&\quad = \int_0^1 \left\{2\left\langle D_s v, D_s w\right\rangle_{M_0} - 2\left\langle R_{\dot{x},v}\dot{x}, w\right\rangle_{M_0} - H_R^{\beta}(x)[v, w]\, \dot{t}^2\right\} ds \\
&\qquad - \int_0^1 \left\{2\left\langle \operatorname{grad}\beta, v\right\rangle_{M_0}\, \dot{t}\,\dot{\tau}_2 + 2\left\langle \operatorname{grad}\beta, w\right\rangle_{M_0}\, \dot{t}\,\dot{\tau}_1\right\} ds - \int_0^1 2\,\beta(x)\,\dot{\tau}_1\,\dot{\tau}_2\, ds \\
&\quad = f''_{xx}(\gamma)[v, w] + f''_{xt}(\gamma)[v, \tau_2] + f''_{xt}(\gamma)[w, \tau_1] + f''_{tt}(\gamma)[\tau_1, \tau_2] \\
&\quad = \left(f''_{xx}(\gamma)[v, w] + f''_{xt}(\gamma)[w, \tau_1]\right) + \left(f''_{xt}(\gamma)[v, \tau_2] + f''_{tt}(\gamma)[\tau_1, \tau_2]\right) \; ,
\end{aligned}$$

so that, (2) implies:

$$\begin{aligned}
&f''(x, \Phi(x))\left[(v, \Phi'(x)[v]), (w, \tau_2)\right] = \\
&\qquad\qquad = f''_{xx}(x, \Phi(x))[v, w] + f''_{x,t}(x, \Phi(x))\left[w, \Phi'(x)[v]\right] .
\end{aligned} \tag{3}$$

Recall also that:

$\cdot)$ $\quad J(x) = f(x, \Phi(x))$,

$\cdot\cdot)$ $\quad J'(x)[v] = f'_x(x, \Phi(x))[v] + f'_t(x, \Phi(x))\left[\Phi'(x)[v]\right]$,

$\cdots)$ $$\begin{aligned}J''(x)[v, w] = {}& f''_{xx}(x, \Phi(x))[v, w] + f''_{xt}(x, \Phi(x))\left[v, \Phi'(x)[w]\right] + \\ &+ f''_{xt}(x, \Phi(x))\left[w, \Phi'(x)[v]\right] + f''_{tt}(x, \Phi(x))\left[\Phi'(x)[v], \Phi'(x)[w]\right] \; ;\end{aligned}$$

here, $(\cdots)$ can be obtained easily, using a variation of x corresponding to the directions of v and w and observing that $\beta(x)\,\dot{t} = \text{constant}$ (see [7]). At this point, the thesis springs out from (2), (3), $(\cdots)$. ∎

Corollary (6). *The following a) and b) are equivalent.*

a) $J''(x)[v, w] = 0$, *for any vector field* w*;*

b) $f''(\gamma)\left[(v, \Phi'(x))[v], (w, \tau)\right] = 0$, *for any vector field* (w, τ)*.*

So that, the null spaces of $J''(x)$ *and of* $f''(\gamma)$ *have the same dimension.*

Proof: The equivalence a)↔b) is straightforward.

We shall show the second statement. Let (v,τ_1) belong to the null space of $f''(\gamma)$:

$$f''(\gamma)\,[(v,\tau_1),(w,\tau_2)] = \Big(f''_{xx}(\gamma)[v,w] + f''_{xt}(\gamma)[w,\tau_1]\Big) + \\ + \Big(f''_{xt}(\gamma)[v,\tau_2] + f''_{tt}(\gamma)[\tau_1,\tau_2]\Big) = 0, \quad \forall\,(w,\tau_2)\ .$$

So, it is identically:

$$\text{(1)} \qquad \begin{cases} f''_{xx}(\gamma)[v,w] + f''_{xt}(\gamma)[w,\tau_1] = 0, & \forall\, w, \\ f''_{xt}(\gamma)[v,\tau_2] + f''_{tt}(\gamma)[\tau_1,\tau_2] = 0, & \forall\,\tau_2\ . \end{cases}$$

Comparing the second equation of the above system (1) with the equation (2) of the proposition (5), we obtain:

$$f''_{xt}(\gamma)[v,\tau_2] + f''_{tt}(\gamma)[\tau_1,\tau_2] = f''_{xt}(\gamma)[v,\tau_2] + f''_{tt}(\gamma)\,\big[\Phi'(x)[v],\tau_2\big]\,, \quad \forall\,\tau_2$$

$$\Rightarrow \qquad f''_{tt}(\gamma)[\tau_1,\tau_2] = f''_{tt}(\gamma)\,\big[\Phi'(x)[v],\tau_2\big]\,, \quad \forall\,\tau_2$$

$$\Rightarrow \qquad f''_{tt}(\gamma)\,\big[\tau_1 - \Phi'(x)[v],\tau_2\big] = 0, \quad \forall\,\tau_2$$

$$\Rightarrow \qquad \int_0^1 \beta(x)\,\frac{d}{ds}\Big(\tau_1 - \Phi'(x)[v]\Big)\,\dot\tau_2\,ds = 0, \quad \forall\,\tau_2\ .$$

If $(\tau_2 = \tau_1 - \Phi'(x)[v])$, then:

$$\int_0^1 \left\| \frac{d}{ds}\left(\tau_1 - \Phi'(x)[v]\right)\right\|^2_{T_{\gamma(s)}M} ds = 0\ ,$$

so,

$$\tau_1 = \Phi'(x)[v] + \text{constant}\ ;$$

but, as $\tau_1(0) = 0 = \Phi'(x)[v]\,|_{s=0}$, then:

$$\tau_1 = \Phi'(x)[v]\ . \blacksquare$$

Remark (7). A vector field (v,τ) belongs to the null space of $f''(\gamma)$ iff it is:

$$(v,\tau) = (v,\Phi'(x)[v])$$

and v belongs to the null space of $J''(x)$.

Proof of Theorem (1): We already know, by [7], that the function $\exp_p$ is onto, for any point $P \in M$. In order to prove it is one to one, it will suffice to show that all the critical points of the functional J are minima; afterwards, the unicity of the geodesic joining two given points on M will follow from well known results of the critical points theory (see e.g.: [8], Theorem (6, 5, 3), page 354).

Viceversa, if $\exp_p$ is a diffeomorphism for any point $P \in M$, then no couple of conjugate points with respect to the action functional exists, on any geodesic γ contained in M; then corollary (6) and remark (7) imply the lack of couples of conjugate points, with respect to the functional J, on the critical curves of J.

This fact shows that $\exp_p$ is a diffeomorphism for any $P \in M$, iff the form J'' is positive definite on any critical curve of J, independently of the extreme points of that, so Lemmas (3) and (4) complete the proof.

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