PORTUGALIAE MATHEMATICA
Vol. 52 Fasc. 4 – 1995

WHEN IS A 0-1 KNAPSACK A MATROID ?

J. Orestes Cerdeira[1] and Paulo Barcia[2]

Abstract: We give a polynomial time algorithm for deciding whether the set of solutions of a 0-1 knapsack is a matroid.

1 – Introduction

Wolsey [3] gave a necessary and sufficient condition for the set of the feasible solutions of an arbitrary 0-1 knapsack to be a matroid. However, from that condition a polynomial time algorithm does not directly follow.

Recently Amado and Barcia [1] showed how matroids can be used, within a lagrangean relaxation approach, to obtain strong bounds for 0-1 knapsacks.

They described a polynomial time algorithm to decide whether a knapsack is a member of a special family of matroids. Yet, as pointed out in [1], knapsacks exist which are matroids and do not belong to that family.

Here we turn the result of Wolsey into a polynomial time algorithm to decide whether an arbitrary 0-1 knapsack is a matroid.

We also show that, unless $P = NP$, there is no polynomial time algorithm for deciding whether the greedy algorithm produces a maximum weight solution for a 0-1 knapsack problem.

Received: February 10, 1995.
Keywords: 0-1 knapsack, matroids, subset sum problem.
[1] Centro de Matemática e Aplicações Fundamentais (Projecto 6F91).
[2] Centro de Matemática e Aplicações Fundamentais (Projecto 6F91). The work of this author was partially supported by Projecto 8768MIC of JNICT.

2 – Preliminaries

Let $a_1, a_2, ..., a_n$ be integer coefficients of the linear inequality

$$\sum_{j=1}^{n} a_j\, x_j \le b \;, \tag{1}$$

and assume $b \ge a_1 \ge a_2 \ge ... \ge a_n > 0$. If $N = \{1, 2, ..., n\}$, then $\mathcal{F} = \{J \subseteq N\colon a(J) = \sum_{j\in J} a_j \le b\}$ is the set of the 0-1 solutions of the knapsack defined by inequality (1). Clearly, the pair $M = (N, \mathcal{F})$ is an independence system.

Definition 1. A maximal independent set $C \subseteq N$ is a *ceiling* of M if whenever $j \in C$ and $j-1 \notin C$, implies $(C-\{j\}) \cup \{j-1\} \notin \mathcal{F}$.

Definition 2. A minimal dependent set $S = \{j_1, ..., j_r\} \subseteq N$ $(j_1 < ... < j_r)$ is a *strong cover* of M if $(S-\{j_1\}) \cup \{k\} \in \mathcal{F}$, where k is the smallest integer greater than j_1 and $k \notin S$.

Wolsey [3] proved the following:

Theorem 3. *M is a matroid iff M has a unique ceiling,*

Theorem 4. *If the number of strong covers is less than or equal to 2, then M is a matroid.*

Here we show that deciding whether M is a matroid amounts to check the independence of at most two sets which we specify. In case M is a matroid, we show that these sets are strong covers, and no other strong cover exists, i.e., that the converse of Theorem 4 also holds.

3 – The main result

Let G be the greedy set of $(N, \mathcal{F})$ with respect to the weights $a_1, ..., a_n$, i.e., the solution obtained by the greedy algorithm for the problem of maximizing $\{a(J)\colon J \in \mathcal{F}\}$.

Recall that the greedy algorithm for $(N, \mathcal{F})$ starts with $G = \{1\}$ and, for $j = 2, ..., n$, adds j to G whenever $a(G) + a_j \le b$.

G consists of $t \ge 1$ pairwise disjoint blocks $G(1), ..., G(t)$ of consecutive elements of N, where if $j \in G(i)$ and $j' \in G(i{+}1)$, then $a_j > a_{j'}$. We use $\bar{G}(i)$ to denote the set of all elements of N which lie between $G(i)$ and $G(i{+}1)$, $i = 1, ..., t{-}1$,

$\bar{G}(t) = \{j \in N\colon\ j > l$, for all $l \in G(t)\}$. Note that $\bar{G}(i) \neq \emptyset$, for $i = 1, ..., t-1$ and $\bar{G}(t) = \emptyset$ iff $n \in G$. For $i = 1, ..., t$ define $N(i) = \bigcup_{j \leq i}(G(j) \cup \bar{G}(j))$, and assume $N(0) = \emptyset$.

Clearly G is a ceiling. Moreover, as G consists of the $|G \cap (N(i) - N(i-1))|$ smallest integers of $N(i) - N(i-1)$, $i = 1, ..., t$, any set A satisfying all the inequalities $|A \cap N(i)| \leq |G \cap N(i)|$ is in $\mathcal{F}$.

Lemma 5. *If $C \neq G$ is a ceiling of M, then $|C \cap N(i)| > |G \cap N(i)|$, for some $1 \leq i \leq t-1$ or $i = t$ if $n \notin G$.*

Proof: Supose C satisfies $|C \cap N(i)| \leq |G \cap N(i)|$, for $i = 1, ..., t$. Since G and C are different ceilings, $C \nsubseteq G$ and $G \nsubseteq C$. Take the smallest integers $g \in G - C$ and $c \in C - G$. If $c < g$, then $G \cap \{1, ..., c-1\} = C \cap \{1, ..., c-1\}$. If we let i be such that $c \in N(i) - N(i-1)$, we would have $|G \cap N(i)| < |C \cap N(i)|$, a contradiction.

We therefore have $c > g$ and, consequently, $G \cap \{1, ..., c-1\} \supset C \cap \{1, ..., c-1\}$. If $C' = (C - \{c\}) \cup \{g\}$, then $|C' \cap N(i)| \leq |G \cap N(i)|$, for $i = 1, ..., t$, and C cannot be a ceiling, since $C' \in \mathcal{F}$. ∎

Define $S(i)$ as the set of the $\sum_{j \leq i} |G(j)| + 1$ greatest integers in $N(i)$, $i = 1, ..., t$.

Theorem 6. *If M is a matroid, then $S(i)$, $i = 1, ..., t-1$ and $S(t)$ if $n \notin G$ are strong covers of M. No other strong cover exists.*

Proof: Take any $S(i)$ on the conditions of the theorem. Since $\bigcup_{j \leq i} G(j)$ is a maximal independent set in $N(i)$ with cardinality $|S(i)| - 1$, it follows, from the matroidal nature of M, that $S(i) \notin \mathcal{F}$.

To see that $S(i) = \{s, ..., g\}$ $(s < ... < g)$ is a minimal dependent set, remove from $S(i)$ its greatest element g. Note that $g \notin G$. As $S(i) - \{g\}$ consists of the $\sum_{j \leq i} |G(j)|$ greatest integers in $N(i) - \{g\}$, while G has the same number of elements in $N(i) - \{g\}$, we can conclude that removing any element from $S(i)$ produces an independent set.

We have just proved that $S(t)$ is a strong cover, whenever $n \notin G$.

Consider now $i < t$. The set $(S(i) - \{s\}) \cup \{g+1\}$ consists of the $\sum_{j \leq i} |G(j)|$ greatest integers in $N(i)$ together with $g+1$. The greedy set G has the same number of elements in $N(i)$ and it also includes the element $g+1$. Therefore, $(S(i) - \{s\}) \cup \{g+1\} \in \mathcal{F}$ which completes the proof that all the $S(i)$ in the above conditions are strong covers.

We now show that no other strong cover exists.

Recall that any set A satisfing $|A \cap N(i)| \leq |G \cap N(i)|$, $i = 1, ..., t$, is independent. If S is dependent $|S \cap N(i')| \geq |S(i')| = |G \cap N(i')| + 1$, for some $i' \in \{1, ..., t\}$. Suppose S is a strong cover different from all the sets $S(i)$ of the theorem. Let s' be the smallest integer in S and k be the smallest integer greater than s' which is not in S. Note that $k \in N(i')$, since otherwise $S \supset S(i')$ would not be minimal. Thus, $(S - \{s'\}) \cup \{k\}$ includes at least $|S(i')|$ elements in $N(i')$. As $S(i') \notin \mathcal{F}$ consists of the $|S(i')|$ greatest integers in $N(i')$, $(S - \{s'\}) \cup \{k\}$ cannot be in $\mathcal{F}$. ∎

The following result concerning the structure of G, whenever M is a matroid, appears in [3] in terms of ceilings.

Theorem 7. *If M is a matroid, $t \leq 3$. Moreover if $t = 3$, then $n \in G$.* ∎

Theorems 6 and 7 show that the converse of the implication in Theorem 4 also holds. Thus,

Theorem 8. *M is a matroid iff the number of strong covers is less than or equal to 2.* ∎

The same two theorems give the following possible configurations for the greedy set G and the strong covers, whenever M is a matroid.

- **i**) G_0: $t = 1$ and $n \in G_0$ (i.e. $G_0 = N$). There are no strong covers.
- **ii**) G_1: $t = 1$ and $n \notin G_1$; or $t = 2$ and $n \in G_1$. The unique strong cover is $S(1)$.
- **iii**) G_2: $t = 2$ and $n \notin G_2$; or $t = 3$ and $n \in G_2$. The strong covers are $S(1)$ and $S(2)$.

We now state and prove our main result.

Theorem 9. *M is a matroid iff $G = G_0$, or $G = G_1$ and $S(1) \notin \mathcal{F}$, or $G = G_2$ and $S(1), S(2) \notin \mathcal{F}$.*

Proof: If $G = G_0$, clearly M is the free matroid.

It remains to be shown that $S(1) \notin \mathcal{F}$ when $G = G_1$, and $S(1), S(2) \notin \mathcal{F}$ when $G = G_2$, implies M to be a matroid.

Supose $G = G_2$ and M is not a matroid. Then there is some ceiling $C \neq G_2$ which, according to Lemma 5, is such that $|C \cap N(i)| > |G_2 \cap N(i)|$, for some $i = 1, 2$. Since $C \cap N(i) \in \mathcal{F}$ and $S(i)$ consists of the $\sum_{j \leq i} |G_2(j)| + 1$ $(\leq |C \cap N(i)|)$ greatest integers in $N(i)$, we would have $S(i)$ also in $\mathcal{F}$.

The proof for $G = G_1$ is similar. ∎

4 – Final remark

Theorem 9 states that deciding whether the set of the 0-1 solutions of inequality (1) is a matroid can be carried out in polynomial time. It seems natural to ask if one can decide in polynomial time whether the greedy set G maximizes $\{a(J)\colon\ J \in \mathcal{F}\}$.

We use the completeness of the subset sum problem (SSP) (the problem of deciding whether there is a subset J of N for which $a(J) = b$) to show that

Theorem 10. *If there is a polynomial time algorithm for deciding whether G maximizes $\{a(J)\colon\ J \in \mathcal{F}\}$, then $P = NP$.*

Proof: We show how to solve the SSP for inequality (1) using an algorithm which decides whether $a(G)$ is maximum.

If $a(G) = b$ the correct answer to the SSP is obviously *yes*. If $G = N$, then the answer is *no* iff $a(G) < b$.

If $G \neq N$ and $a(G) < b$, consider first the case $n \notin G$. Define $a_0 = b - 1$, and the inequality

$$(2) \qquad \sum_{j=0}^{n} a_j\, x_j \le b\ .$$

The greedy solution for inequality (2) is $G' = \{0\}$. If $a(G')$ is maximum for (2), clearly the correct answer to the SSP is *no*. If $a(G')$ is not maximum for (2), then there is some set $J \not\ni 0$ in $\mathcal{F}$ such that $a(J) > a_0 = b - 1$, and *yes* would be the correct answer.

In case $n \in G$, let $N := N - G(t)$ and $b := b - a(G(t))$, and use the above argument.

The result follows from the completeness of SSP (Garey and Jonhson [2]). ∎

REFERENCES

[1] Amado, L. and Barcia, P. – Matroidal relaxations for 0-1 knapsack problems, *Operations Research Letters,* 14 (1993), 147–152.

[2] Garey, M.R. and Jonhson, D.S. – *Computers and intractability: a guide to the theory of NP completeness*, W.H. Freeman & company, San Franscico, 1979.

[3] Wolsey, L.A. – Faces for a linear inequality in 0-1 variables, *Mathematical Programming,* 8 (1975), 165–178.

J. Orestes Cerdeira,
Departamento de Matemática, Instituto Superior de Agronomia,
Tapada da Ajuda, 1399 Lisboa Codex – PORTUGAL
E-mail: orestes@isa.utl.pt

and

Paulo Barcia,
Universidade Nova de Lisboa, Faculdade de Economia,
Travessa Estevão Pinto, Campolide, 1000 Lisboa – PORTUGAL
E-mail: barcia@fe.unl.pt