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THE C^1 INTERIOR OF ZERO ENTROPY DIFFEOMORPHISMS

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Abstract: Morse–Smale diffeomorphisms on a two-sphere are C^1 dense in the interior of zero entropy diffeomorphisms.

1 – Introduction

Let M be a compact connected riemannian manifold of dimension two and $\mathrm{Diff}^r(M)$ denote its diffeomorphisms endowed with the uniform C^r topology, $1 \le r \le \infty$.

It is known that Axiom A diffeomorphisms are C^0 dense in $\mathrm{Diff}^r(M)$ but it remains an open question the similar conclusion on the other more interesting topologies. Meanwhile they are indeed not C^r dense on Diff^r(2-torus $\times$ 2-sphere), $1 \le r \le \infty$, nor C^2 dense on Diff^r(2-sphere), in spite of the C^1 generic density of the periodic points on the non-wandering set — see for instance [1] and [7].

We attempt here to enlighten the still mysterious picture in dimension two. We prove that Morse–Smale diffeomorphisms are dense in the interior (in the C^1 topology) of the set

$$\mathcal{E} = \Big\{ f \in \mathrm{Diff}^2(\mathbf{S}^2)\colon\ h_{\mathrm{top}}(f) = 0 \Big\} ,$$

where $h_{\mathrm{top}}(f)$ denotes the topological entropy of f and $\mathbf{S}^2$ is the two-sphere. Notice that in dimension two a positive topological entropy is a non-empty open property, so the restriction implicit in the definition of $\mathcal{E}$ is somehow expected.

Morse–Smale systems are special islands in the C^1 interior of $\mathcal{E}$ (abbreviated into $\mathring{\mathcal{E}}_1$) but there is so far no proof that they are the only ones (just as hap-

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pens among $\mathrm{Diff}^r(\mathbf{S}^1)$) or even if $\mathcal{E}$ is the C^1 closure of $\overset{\circ}{\mathcal{E}}_1$, due to a still weak understanding of the bifurcation processes possible within $\mathcal{E}$.

2 – Preliminaries

For $f \in \mathrm{Diff}^r(M)$, a point $x \in M$ is non-wandering if for each neighbourhood $\mathcal{U}$ of x there is a positive integer N such that $f^N(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. In the sequel the set of non-wandering points of f will be denoted by $\Omega(f)$.

A diffeomorphism f satisfies the Axiom A if

a) $\Omega(f)$ has a hyperbolic structure;

b) the periodic points, $\mathrm{Per}(f)$, are dense in $\Omega(f)$.

In compact manifolds without boundary of dimension two, this definition is repetitive since

Lemma 1. *In dimension two,* $[\Omega(f)$ *hypebolic* $\Rightarrow \Omega(f) =$ *closure* $(\mathrm{Per}(f))]$.

Proof: A brief sketch o the argument given in [8] follows as this: let f be a diffeomorphism whose non-wandering set is hyperbolic; then its limit set is the closure of the periodic points and has a spectral decomposition in basic sets. This decomposition turns out to be the one of $\Omega(f)$. ∎

Morse–Smale diffeomorphisms are the ones with finite non-wandering set which has only periodic hyperbolic orbits, whose stable and unstable manifolds intersect transversally.

2 – The C^0 topology

Let $\mathcal{E}$ be the set $\{f \in \mathrm{Diff}^2(M) : h_{\mathrm{top}}(f) = 0\}$. The interior of $\mathcal{E}$ in this topology turns to be empty, since, for example, we may construct a family of diffeomorphisms on the two-sphere with topologically very thin horseshoes C^0 approaching a saddle.

Lemma 2. *Axiom A diffeomorphisms are* C^0 *dense in* $\mathrm{Diff}^r(M)$.

Proof: Given f in $\mathrm{Diff}^r(M)$, the construction of g satisfying Axiom A and C^0 close to f depends on a clever triangulation of the manifold whose simplexes define a filtration for some isotopy transform I_f of f. By surgery, I_f appears with hyperbolic limit set and without cycles. See [9] and [10] for details. ∎

Choose f in $\mathcal{E}$. By Lemma 2, we may find g satisfying the Axiom A which is C^0 close to f.

Lemma 3. *If g belongs to $\mathcal{E} \cap \{\text{Axiom A}\}$, then its non-wandering set is finite.*

Proof: As g is Axiom A and has zero topological entropy, the spectral decomposition of $\Omega(g)$ in basic sets $\Omega(g) = \Lambda_1 \cup \Lambda_2 \cup ... \cup \Lambda_s$ does not allow a Λ_i with infinitely many points. In fact we would have in such a Λ_i, according to the equivalence relation that determines the spectral partition, subsets contributing to positive entropy: homoclinic intersections of invariant manifolds producing horseshoes. ∎

Lemma 4. *In the hypothesis of Lemma 3, g may be C^1 approximated by a Ω-stable diffeomorphism G such that $\Omega(G) = \Omega(g)$.*

Proof: Since the non-wandering set of G is hyperbolic and finite, Theorem B from [8] assures the existence of G. Notice that, as $\Omega(g)$ is finite, $h_{\text{top}}(G) = 0$; moreover G is in the C^1 interior of $\mathcal{E}$ due to the invariance of the entropy by topological conjugacy. ∎

Lemma 5. *G is a Morse–Smale diffeomorphism.*

Proof: The missing property, that is, the transversality of the intersections between stable and unstable manifolds of elements of $\Omega(G)$, is ensured by Theorem 2 of [5]. ∎

It is not known whether we may find a diffeomorphism with zero entropy but not C^0 close to one satisfying Axiom A and whose topological entropy is zero. However if f has zero entropy and satisfies Axiom A, we may conclude that f is C^0 approximated by a Morse–Smale diffeomorphism, in account of the fact that C^1 closeness yields C^0 closeness. Hence

Theorem A.

i) *Morse–Smale diffeomorphisms are C^1 dense in $\mathcal{E} \cap \{\text{Axiom A diffeomorphisms}\}$.*

ii) *Morse–Smale diffeomorphisms are C^0 dense in the C^0 closure of the set $\mathcal{E} \cap \{\text{Axiom A}\}$.* ∎

4 – The C^1 topology and the two-sphere

Recall that a diffeomorphism possesses a homoclinic tangency if one of its periodic hyperbolic points has stable and unstable manifolds intersecting non-transversally.

The peculiarities of the two-sphere already intervened in [7] — where geometric properties of invariant subsets, inducing persistence of non-hyperbolic non-wandering points, imposed topological restrictions in $\mathrm{Diff}(\mathbf{S}^2)$ — and in [6] through the possibility of promoting almost homoclinic orbits to homoclinic points, which makes extensive use of the Jordan curve theorem.

Lemma 6. *Let f be a C^2 diffeomorphism of the two-sphere with all periodic points hyperbolic and infinitely many. Then f can be approximated in the C^1 topology by a diffeomorphism with homoclinic points.* ∎

This Lemma is the main result in [6] and depends on recent progress due to Araújo and Mañé on the understanding of the asymptotic behaviour of diffeomorphisms on manifolds of dimension two, [3].

A stronger property on periodic hyperbolic orbits of a diffeomorphism is the key for the following Lemma. We denote by $\mathcal{F}^1(M)$ the C^1 interior in $\mathrm{Diff}^1(M)$ of the family of diffeomorphisms with all periodic points hyperbolic.

Lemma 7. $f \in \mathcal{F}^1(M) \Rightarrow f$ *satisfies Axiom A.* ∎

This has been established in [2], emerging from the ideas of Mañé which led to the proof of the stability conjecture.

Let $\overset{\circ}{\mathcal{E}}_1$ be the set $\{f \in \mathrm{Diff}^2(M)\colon \exists\, \mathcal{V}$ neighbourhood of f in the C^1 topology such that $\mathcal{V} \subseteq \mathcal{E}\}$. Notice that $\overset{\circ}{\mathcal{E}}_1$ is non-empty, contrary to what happens with the C^0 interior of $\mathcal{E}$, since it contains all Morse–Smale systems.

Theorem B. *Morse–Smale diffeomorphisms on the two-sphere are dense in* $\overset{\circ}{\mathcal{E}}_1$.

Proof: Let f be an element of $\overset{\circ}{\mathcal{E}}_1$ and $\mathcal{V}$ a C^1 neighbourhood of f contained in $\mathcal{E}$. We may approximate f by F in $\mathcal{V}$ such that all periodic points of F are hyperbolic, since Kupka–Smale systems are C^r generic. As F belongs to $\mathcal{V}$, it has zero topological entropy.

If F satisfies Axiom A then, by Lemma 3, the non-wandering set of F is finite and, by Lemmas 4 and 5, F may be C^1 approximated by a Morse–Smale diffeomorphism, yielding the theorem for the given f.

If F does not satisfy Axiom A then, by Lemma 1, this occurs because $\Omega(F)$ is not hyperbolic. But the set of periodic points of F, say $\mathrm{Per}(F)$, is hyperbolic and must be finite by Lemma 6: if not, we could find a diffeomorphism H in $\mathcal{V}$ exhibiting a generic homoclinic tangency whose unfolding would produce an element H_1 in $\mathcal{V}$ with positive entropy. But if $\mathrm{Per}(F)$ is hyperbolic and finite, then F belongs to the interior of the subset of $\mathrm{Diff}^1(M)$ with all periodic points hyperbolic. By Lemma 7 this implies that F satisfies Axiom A, which is a contradiction. ∎

In particular we proved that

Corollary. *Generically in* $\overset{\circ}{\mathcal{E}}_1$ *all diffeomorphisms satisfy Axiom A.*

5 – Other manifolds of dimension two

The conclusions in this context are less mature then the ones in the previous section, apart from the major work on the dynamics in dimension two achieved in [3] and which reads as follows:

Lemma 8. *Let f be a C^2 diffeomorphism on a manifold M with dimension two with all periodic points hyperbolic. Then one of the following assertions is true:*

- **i)** *f possesses a finite number of hyperbolic attractors and contracting irrational rotations, say $A_1, ..., A_n$, and a finite number of hyperbolic repellers and expanding irrational rotations, say $R_1, ..., R_m$, such that for Lebesgue almost every point in M its α-limit lies in R_j for some j and the ω-limit lies in A_k for some k;*
- **ii)** *f is C^1 close to a diffeomorphism which exhibits a homoclinic tangency.*

Recall that a hyperbolic attractor of $f \in \mathrm{Diff}^r(M)$ is a transitive hyperbolic invariant set having a neighbourhood $\mathcal{U}$ such that the closure of $f(\mathcal{U})$ is contained in $\mathcal{U}$ and the attractor is the intersection of the positive iterates of $\mathcal{U}$ by f. Simple examples are the attracting periodic orbits, usually called sinks.

Following [3], we say that Λ is a contracting irrational rotation of f if

- Λ is homeomorphic to a circle and is invariant by an iterate N of f (hence $f^N|_\Lambda$ has zero topological entropy);
- f^N is conjugate to an irrational rotation;
- the tangent bundle of M restricted to Λ has a continuous Df^N invariant splitting into a direct sum $E \oplus T\Lambda$ such that $\|Df^N|_{E(x)}\| < 1$ for all x in Λ.

Analogous definitions for hyperbolic repellors, sources and expanding irrational rotations.

Definition. We say that $f \in \mathrm{Diff}^r(M)$ is an *almost Morse–Smale diffeomorphism* if Lebesgue almost every point in M has its α-limit in a source and its ω-limit in a sink and these are finitely many.

A straight corollary of Lemma 8 reduces the possible features in $\overset{\circ}{\mathcal{E}}_1$ to the following description:

Theorem C. *The almost Morse–Smale systems are dense in* $\overset{\circ}{\mathcal{E}}_1$.

Notice that for each C^2 Axiom A diffeomorphisms there is at least one attractor and moreover Lebesgue almost all point of the manifold has its ω-limit in the (finite) family of the attractors — see for instance [4]. The lack of differentiability obliges us to weaken this assertion among C^1 diffeomorphisms, reducing it to the topological prevalence of almost Morse–Smale systems in $\overset{\circ}{\mathcal{E}}_1$.

Proof: As in the proof of Theorem B, we may start with an element of $\overset{\circ}{\mathcal{E}}_1$ whose periodic points are all hyperbolic. We have already remarked that the assertion ii) of Lemma 8 is impossible within $\overset{\circ}{\mathcal{E}}_1$. If f is in $\overset{\circ}{\mathcal{E}}_1$, it cannot have hyperbolic non-trivial attractors or repellors, since these are necessarily of Plyckin type and so contribute to positive topological entropy. Therefore f can only have a finite number of sinks and sources, besides the finite number of contracting or expanding irrational rotations. Moreover, these kind of rotations are not C^r generic, so we may C^1 approximate f by another element of $\overset{\circ}{\mathcal{E}}_1$ whose physically observable dynamics (that is, the asymptotic behaviour of Lebesgue almost every orbit) are sources and sinks as described in Lemma 8 i). Therefore f is C^1 close to an almost Morse–Smale system as claimed. ∎

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