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DEGENERATE ELLIPTIC EQUATION INVOLVING A SUBCRITICAL SOBOLEV EXPONENT

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Abstract: We prove the existence of a solution of degenerate elliptic equation (1) involving a subcritical Sobolev exponent. To solve (1) we establish the existence of a solution of the constrained minimization problem (3). A relative compactness of a minimizing sequence is obtained by examining a possible loss of a mass at infinity of a minimizing sequence.

1 – Introduction

The purpose of this article is to investigate the existence of a nontrivial solution of the degenerate equation

$$-D_i\Big(a(x)\,D_i u\Big) + \lambda\, u = K(x)\,|u|^{p-2}\,u \quad \text{in} \;\; \mathbb{R}^N \tag{1}$$

in a weighted Sobolev space which will be defined in Section 2, where $\lambda > 0$ is a parameter, $2 < p < \frac{2N}{N-2}$ and $N \geq 3$. We assume that $a(x)$ and $K(x)$ are continuous and bounded in $\mathbb{R}^N$ and moreover $a(x) \geq 0$ and $a(x) \not\equiv 0$ on $\mathbb{R}^N$ and $\alpha \leq K(x) \leq \beta$ on $\mathbb{R}^N$, for some constants $\alpha > 0$ and $\beta > 0$. We establish the existence of a nontrivial solution under assumptions on a and K, which control the location of zeros of $a(x)$ and the behaviour of $a(x)$ and $K(x)$ at infinity. The latter assumption can be replaced by the periodicity assumption on $K(x)$. However, we only need a periodicity assumption either on K or a. The case of a periodic function a is only treated for a uniformly elliptic equation.

Unlike the case of unbounded domains, degenerate equations in bounded domains, in particular the Dirichlet problem, have a quite extensive literature [MS], [SA], where further bibliographical references can be found.

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A variational problem (3) (Section 2) associated with (1) is characterized by a lack of compactness. In Section 3 we give a description of a possible loss of mass at infinity of a mimmizing sequence in quantitative terms. This will be used to show that a minimizing sequence is relatively compact.

2 – Preliminaries

The appropriate Sobolev space for equation (1) is $H^1_a(\mathbb{R}^N)$, defined as a completion of C_0^∞ with respect to the norm

$$\|u\|_a^2 = \int_{\mathbb{R}^N} \Big(a(x)\,|Du|^2 + \lambda\, u^2\Big)\,dx\ .$$

The dual space is denoted by $H_a^{-1}(\mathbb{R}^N)$, that is $H^1_a(\mathbb{R}^N)^* = H_a^{-1}(\mathbb{R}^N)$. Since a is a bounded function, the Sobolev space $H^1(\mathbb{R}^N)$ is continuously embedded in $H^1_a(\mathbb{R}^N)$.

In this paper we always denote in a given Banach space X a weak convergence by "$\rightharpoonup$" and a strong convergence by "$\to$".

A function $u \in H^1_a(\mathbb{R}^N)$ is a solution of (1) if

$$\int_{\mathbb{R}^N} \Big(a(x)\,Du\,D\phi + \lambda\,u\,\phi - K(x)\,|u|^{p-2}\,u\,\phi\Big)\,dx = 0 \tag{2}$$

for each $\phi \in C_0^\infty(\mathbb{R}^N)$.

To find a solution to equation (1), we consider the constrained minimization problem

$$M_{a,K} = \inf\bigg\{\int_{\mathbb{R}^N} a(x)\,|Du|^2\,dx;\ u \in H^1_a(\mathbb{R}^N),\ \int_{\mathbb{R}^N} K(x)\,|u|^p\,dx = 1\bigg\}\ . \tag{3}$$

To ensure that $M_{a,K} > 0$ we impose the following condition on a

(**A**) There exists $R_0 > 0$ such that

$$\{x;\, a(x) = 0\} \subset B(0,R_0) \quad \text{and} \quad \frac{1}{a} \in L^q(B(0,R_0))$$

for some $q > \frac{Np}{2N+2p-Np}$.

Then we have the following result:

Proposition 1. *Suppose that (A) holds and that* $\inf_{\mathbb{R}^N - B(0,R_0)} a(x) > 0$. *Then there exists a constant* $C > 0$ *such that*

$$\Big(\int_{\mathbb{R}^N} |u|^p\,dx\Big)^{\frac{1}{p}} \le C\Big(\int_{\mathbb{R}^N} \Big(a(x)\,|Du|^2 + \lambda\,u^2\Big)\,dx\Big)^{\frac{1}{2}} \tag{4}$$

for all $u \in H^1_a(\mathbb{R}^N)$.

Proof: We follow the argument from paper [PA] (Proposition 2.1). We may assume, by taking R_0 larger if necessary, that $\{x;\, a(x) = 0\} \subset B(0, R_0 - 2)$ and $\inf_{\mathbb{R}^N - B(0,R_0-2)} a(x) > 0$. Let $r = \frac{2q}{1+q}$. Then $q > \frac{Np}{2N+2p-Np}$ implies $p < \frac{Nr}{N-r}$ $(1 < r < 2 < N)$. Consequently by the Sobolev embedding theorem $H_0^{1,r}(B(0,R_0))$ is continuously (compactly) embedded in $L^p(B(0,R_0))$. This fact will be used to establish (4). Toward this end we define for every $R > 0$ a function $\phi_R \in C^1(\mathbb{R}^N)$ such that $\phi_R(x) = 1$ on $B(0,R)$, $\phi_R(x) = 0$ on $\mathbb{R}^N - B(0,R+1)$ and $0 \le \phi_R(x) \le 1$ on $\mathbb{R}^N$. Applying the Hölder inequality we get

$$
\begin{aligned}
\int_{B(0,R_0)} |Du|^r\,dx &\le \int_{B(0,R_0+1)} |D(u\,\phi_{R_0})|^r\,dx \\
&= \int_{B(0,R_0+1)} a^{\frac{q}{1+q}}\,|D(u\,\phi_{R_0})|^{\frac{2q}{q+1}}\,\frac{1}{a^{\frac{q}{q+1}}}\,dx \\
&\le C\Big(\int_{B(0,R_0+1)} \frac{1}{a^q}\,dx\Big)^{\frac{1}{q+1}} \Big(\int_{B(0,R_0+1)} \Big(a\,|Du|^2 + \lambda\,u^2\Big)\,dx\Big)^{\frac{q}{q+1}}
\end{aligned} \tag{5}
$$

for some constant $C > 0$. Inequality (5) combined with the Sobolev inequality implies

$$
\begin{aligned}
\Big(\int_{B(0,R_0-1)} |u|^p\,dx\Big)^{\frac{1}{p}} &\le C\Big(\int_{B(0,R_0)} |u\,\phi_{R_0-1}|^p\,dx\Big)^{\frac{1}{p}} \\
&\le C\Big(\int_{B(0,R_0)} |D(u\,\phi_{R_0-1})|^r\,dx\Big)^{\frac{1}{r}} \\
&\le C\Big(\int_{B(0,R_0)} \Big(|Du|^r + \lambda\,|u|^r\Big)\,dx\Big)^{\frac{1}{r}} \\
&\le C\Big(\int_{B(0,R_0+1)} \Big(a\,|Du|^2 + \lambda\,u^2\Big)\,dx\Big)^{\frac{q}{r(q+1)}} \\
&= C\Big(\int_{B(0,R_0+1)} \Big(a\,|Du|^2 + \lambda\,u^2\Big)\,dx\Big)^{\frac{1}{2}}\,.
\end{aligned} \tag{6}
$$

Letting $\psi_R = 1 - \phi_R$, we see that $\psi_R(x) = 1$ on $\mathbb{R}^N - B(0,R+1)$. Then the Sobolev inequality implies

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^N - B(0,R_0-1)} |u|^p\,dx\Big)^{\frac{1}{p}} &\le \Big(\int_{\mathbb{R}^N - B(0,R_0-2)} |u\,\psi_{R_0-2}|^p\,dx\Big)^{\frac{1}{p}} \\
&\le C\Big(\int_{\mathbb{R}^N} \Big(a\,|Du|^2 + \lambda\,u^2\Big)\,dx\Big)^{\frac{1}{2}}\,.
\end{aligned} \tag{7}
$$

Here we have used the assumption $\inf_{\mathbb{R}^N - B(0,R_0-2)} a(x) > 0$. Estimates (6) and (7) then imply the assertion of Proposition 1. ∎

Since we always assume that $\alpha \le K(x) \le \beta$ on $\mathbb{R}^N$, for some constants $0 < \alpha < \beta$ (see Introduction), estimate (4) can be rewritten as

$$\Big(\int_{\mathbb{R}^N} K(x)\,|u|^p\,dx\Big)^{\frac{1}{p}} \le C\Big(\int_{\mathbb{R}^N}\Big(a(x)\,|Du|^2 + \lambda\,u^2\Big)\,dx\Big)^{\frac{1}{2}} \tag{8}$$

for some constant $C > 0$. As an immediate consequence of Proposition 1, we have $M_{a,K} > 0$.

It may happen that $M_{a,K} = 0$ if condition (A) is not satisfied. For instance this occurs if

$$a(x) \le C\,|x|^b \quad \text{for} \quad |x| \le \delta \tag{9}$$

for some constants $\delta > 0$ and $b > \frac{2N+2p-Np}{p}$ and $a(x) > 0$ for $x \ne 0$.

Indeed, let $w \in C_0^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} K(x)\,|w|^p\,dx = 1$ and set

$$\phi(x) = \frac{w(x\,\sigma)\,\sigma^{\frac{N}{p}}}{\Big(\int_{\mathbb{R}^N} K\big(\frac{x}{\sigma}\big)\,|w(x)|^p\,dx\Big)^{\frac{1}{p}}} \quad \text{for} \quad \sigma > 0\ .$$

Then

$$\begin{aligned} M_{a,K}\Big(\int_{\mathbb{R}^N} K\Big(\frac{x}{\sigma}\Big)\,|w(x)|^p\,dx\Big)^{\frac{2}{p}} &\le \\ &\le \int_{\mathbb{R}^N}\Big(a\Big(\frac{x}{\sigma}\Big)\,|Dw(x)|^2\,\sigma^{\frac{2N+2p-Np}{p}} + \lambda\,w(x)^2\,\sigma^{\frac{2N}{p}-N}\Big)dx \\ &\le C\int_{\mathbb{R}^N} |x|^b\,|Dw(x)|^2\,\sigma^{\frac{2N+2p-Np}{p}-b}\,dx + \lambda\int_{\mathbb{R}^N} w(x)^2\,\sigma^{\frac{2N}{p}-N}\,dx \;\to\; 0 \end{aligned}$$

as $\sigma \to \infty$, where C is a positive constant independent of σ.

It is clear that if a satisfies (9) then $\int_{B(0,R_0)} \frac{1}{a^q}\,dx = \infty$.

To proceed further we introduce a functional $F\colon H_a^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$F(u) = \frac{1}{2}\int_{\mathbb{R}^N}\Big(a(x)\,|Du|^2 + \lambda\,u^2\Big)\,dx - \frac{1}{p}\int_{\mathbb{R}^N} K(x)\,|u|^p\,dx\ ,$$

which is of class C^1. It is routine calculation to show that (see Theorem 2.1 [LTW]):

Proposition 2. *Suppose that (A) holds and that* $\inf_{\mathbb{R}^N - B(0,R_0)} a(x) > 0$ *and let* $\{u_m\} \subset H_a^1(\mathbb{R}^N)$ *be a minimizing sequence for problem (3). Then*

$v_m = M_{a,K}^{\frac{1}{p-2}}\, u_m$ *satisfies*

$$\textbf{(i)} \quad F(v_m) \to \Big(\frac{1}{2} - \frac{1}{p}\Big)\, M_{a,K}^{\frac{p}{p-2}} \quad \textit{as} \;\; m \to \infty \;,$$

$$\textbf{(ii)} \quad F'(v_m) \to 0 \quad \textit{in} \;\; H_a^{-1}(\mathbb{R}^N) \;\; \textit{as}\; m \to \infty \;.$$

This result implies that if a minimizing sequence $\{u_m\} \subset H_a^1(\mathbb{R}^N)$ has a limit point u, then $M_{a,K}^{\frac{1}{p-2}} u$ satisfies equation (1).

3 – Existence result

In order to show that a minimizing sequence of (3) is relatively compact in $H_a^1(\mathbb{R}^N)$ we introduce quantities which control a possible loss of mass of this sequence at infinity.

Let $a(x)$ satisfy (A) and suppose that $\inf_{\mathbb{R}^N - B(0,R_0)} a(x) > 0$. If $\{u_m\} \subset H_a^1(\mathbb{R}^N)$ is a minimizing sequence for (3), then $\{u_m\}$ is bounded in $H_a^1(\mathbb{R}^N)$ and restricted to $\mathbb{R}^N - B(0,R_0)$ is bounded in $H^1(\mathbb{R}^N - B(0,R_0))$. It also follows from Proposition 1 that $\{u_m\}$ restricted to $B(0,R_0)$, is bounded in $H^{1,r}(B(0,R_0))$, $p < \frac{Nr}{N-r}$. Therefore we may assume that $u_m \rightharpoonup u$ in $H_0^1(\mathbb{R}^N)$ and $u_m \to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$. It is clear that the following quantities are well defined:

$$\alpha_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\mathbb{R}^N - B(0,R)} |u_m|^p \, dx \;,$$

$$\beta_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\mathbb{R}^N - B(0,R)} \Big(|Du_m|^2 + \lambda\, u_m^2\Big)\, dx \;,$$

$$\alpha_{K,\infty} = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\mathbb{R}^N - B(0,R)} K(x)\, |u_m|^p \, dx$$

and

$$\beta_{a,\infty} = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\mathbb{R}^N - B(0,R)} \Big(a(x)\, |Du_m|^2 + \lambda\, u_m^2\Big)\, dx \;.$$

It is easy to check that if $\lim_{|x|\to\infty} K(x) = K(\infty)$ and $\lim_{|x|\to\infty} a(x) = a(\infty)$, then $\alpha_{K,\infty} = K(\infty)\, \alpha_\infty$ and $\beta_{a,\infty} = a(\infty)\, \beta_\infty$.

Writing for each $R > 0$

$$1 = \int_{\mathbb{R}^N} K(x)\, |u_m|^p \, dx = \int_{B(0,R)} K(x)\, |u_m|^p \, dx + \int_{\mathbb{R}^N - B(0,R)} K(x)\, |u_m|^p \, dx$$

and letting $m \to \infty$ and then $R \to \infty$ we get

$$1 = \int_{\mathbb{R}^N} K(x)\,|u|^p\,dx + \alpha_{K,\infty}\ . \tag{10}$$

Therefore to show that u is a solution of the minimization problem (3), it is enough to show that $\alpha_{K,\infty} = 0$.

Since the norm is weakly lower semicontinuous with respect to weak convergence we derive in a similar manner the inequality

$$M_{a,K} = \lim_{m\to\infty} \int_{\mathbb{R}^N} \Big(a\,|Du_m|^2 + \lambda\,u_m^2\Big)\,dx \geq \int_{\mathbb{R}^N} \Big(a\,|Du|^2 + \lambda\,u^2\Big)\,dx + \beta_{a,\infty}\ . \tag{11}$$

Finally, by writing for each $R > 0$,

$$M_{a,K}\Big(\int_{\mathbb{R}^N} K(x)\,|u_m\,\psi_R|^p\,dx\Big)^{\frac{2}{p}} \leq \int_{\mathbb{R}^N} \Big(a\,|D(u_m\,\psi_R)|^2 + \lambda(u_m\,\psi_R)^2\Big)\,dx\ ,$$

where ψ_R is a function introduced in the proof of Proposition 1, we easily derive

$$M_{a,K}(\alpha_{K,\infty})^{\frac{2}{p}} \leq \beta_{a,\infty}\ . \tag{12}$$

We commence with the following technical lemma.

Lemma 1. *Suppose that (A) holds and that* $\inf_{\mathbb{R}^N - B(0,R_0)} a(x) > 0$. *Let* $\{u_m\} \subset H^1_a(\mathbb{R}^N)$ *be a minimizing sequence for problem (3). If* $u_m \rightharpoonup u \not\equiv 0$ *in* $H^1_a(\mathbb{R}^N)$, *then* u *is a solution of problem (3).*

Proof: According to the above discussion we need to show that $\alpha_{K,\infty} = 0$. Arguing indirectly, let us assume that $\alpha_{K,\infty} > 0$. Then by (10) we have

$$0 < \int_{\mathbb{R}^N} K(x)\,|u|^p\,dx < 1\ . \tag{13}$$

Since $\lim_{m\to\infty}\langle F'(u_m\,M_{a,K}^{\frac{1}{p-2}}),\,u_m\,M_{a,K}^{\frac{1}{p-2}}\,\psi_R\rangle = 0$ uniformly in $R \geq 1$, we see that

$$\beta_{a,\infty} = \alpha_{K,\infty}\,M_{a,K}\ . \tag{14}$$

Combining (14), (10) and (11), we have

$$\int_{\mathbb{R}^N} \Big(a\,|Du|^2 + \lambda\,u^2\Big)\,dx \leq M_{a,K} \int_{\mathbb{R}^N} K\,|u|^p\,dx\ .$$

Since we always have

$$M_{a,K}\Big(\int_{\mathbb{R}^N} K|u|^p\,dx\Big)^{\frac{2}{p}} \leq \int_{\mathbb{R}^N} \Big(a|Du|^2 + \lambda\,u^2\Big)\,dx\ ,$$

we see that the last two inequalities imply

$$\int_{\mathbb{R}^N} K|u|^p\,dx \geq 1\ ,$$

which contradicts (13). ∎

We are now in a position to establish the following existence result.

Theorem 1. *Suppose that (A) holds. If $a(x) \leq a(\infty)$ on $\mathbb{R}^N$, where $a(\infty) = \lim_{|x|\to\infty} a(x)$ with the strict inequality on a set of positive measure in $\mathbb{R}^N$ and $K(x) \geq K(\infty)$ on $\mathbb{R}^N$, where $K(\infty) = \lim_{|x|\to\infty} K(x)$. Then problem (3) has a solution $u \in H^1_a(\mathbb{R}^N)$. Moreover, $u \in H^1(\mathbb{R}^N - B(0,R_0))$ and $u \in H^{1,r}(B(0,R_0))$ with $r = \frac{2q}{1+q}$.*

Proof: Let $\{u_m\} \subset H^1_a(\mathbb{R}^N)$ be a minimizing sequence for problem (3). According to the comments made at the beginning of this section we may assume that $u_m \rightharpoonup u$ in $H^1_a(\mathbb{R}^N)$ and $u \to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$. By virtue of Lemma 1, it is sufficient to show that $u \not\equiv 0$ on $\mathbb{R}^N$. Assuming that $u \equiv 0$ on $\mathbb{R}^N$, we see that $\alpha_{K,\infty} = 1$. As in the proof of Lemma 1, we check that $M_{a,K} = \beta_{a,\infty}$. We now compare $M_{a,K}$ with M_∞ defined by

$$M_\infty = \inf\left\{\int_{\mathbb{R}^N}\Big(a(\infty)\,|Du|^2 + \lambda\,u^2\Big)\,dx;\ u\in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} K(\infty)\,|u|^p\,dx = 1\right\}.$$

It is well known that this problem has a positive radially symmetric solution $\overline{u}$ with an exponential decay at infinity which is unique up to a translation (see [KW]). It follows from the definition of M_∞ that

$$M_\infty\Big(K(\infty)\int_{\mathbb{R}^N} |u_m\,\psi_R|^p\,dx\Big)^{\frac{2}{p}} \leq \int_{\mathbb{R}^N}\Big(a(\infty)\,|D(u_m\,\psi_R)|^2 + \lambda(u_m\,\psi_R)^2\Big)\,dx\ .$$

Letting $m \to \infty$ and then $R \to \infty$ gives

$$M_\infty = M_\infty(K(\infty)\,\alpha_\infty)^{\frac{2}{p}} \leq a(\infty)\,\beta_\infty = \beta_{a,\infty}\ ,$$

which is equivalent to

$$M_\infty \leq M_{a,K}\ .$$

On the other hand we have

$$M_{a,K} = M_{a,K}\Big(K(\infty)\int_{\mathbb{R}^N} |\overline{u}|^p\,dx\Big)^{\frac{2}{p}} \leq M_{a,K}\Big(\int_{\mathbb{R}^N} K(x)\,|\overline{u}|^p\,dx\Big)^{\frac{2}{p}} \leq$$

$$\leq \int_{\mathbb{R}^N}\Big(a(x)\,|D\overline{u}|^2 + \lambda\,\overline{u}^2\Big)\,dx < \int_{\mathbb{R}^N}\Big(a(\infty)\,|D\overline{u}|^2 + \lambda\,\overline{u}^2\Big)\,dx = M_\infty$$

and we arrived at a contradiction. We also show that $u_m \to u$ in $H^1_a(\mathbb{R}^N)$. For u is a solution of problem (3) we have $\alpha_{K,\infty} = 0$. Then $u_m \to u$ in $L^p(\mathbb{R}^N)$ due to the uniform convexity of the space $L^p(\mathbb{R}^N)$. The convergence $u_m \to u$ in $H^1_a(\mathbb{R}^N)$ follows then from the following identity

$$\int_{\mathbb{R}^N} \Big(a(x)\,|Dw_m - w_n|^2 + \lambda(w_m - w_n)^2\Big)\,dx =$$
$$= \Big\langle F'(w_m) - F'(w_n),\, w_m - w_n \Big\rangle + \int_{\mathbb{R}^N} K\Big(|w_m|^{p-2}\,w_m - |w_n|^{p-2}\,w_n\Big)\,(w_m - w_n)\,dx\;,$$

where $w_m = M_{a,K}^{\frac{1}{p-2}}\,u_m$. ∎

Since $F(u) = F(|u|)$, by the maximum principle the solution u can be chosen to be positive on $\mathbb{R}^N$.

Obviously the assumption "$a(x) \le a(\infty)$ on $\mathbb{R}^N$ with strict inequality on a set of positive measure in $\mathbb{R}^N$" can be replaced by "$K(x) \ge K(\infty)$ on $\mathbb{R}^N$ with strict inequality on a set of positive measure".

4 – Case of K periodic

It is known ([EL], Corollary II.2 or [L1], Corollary II.3) that for a uniformly elliptic equation on $\mathbb{R}^N$, with a and K periodic on $\mathbb{R}^N$ with the same period, problem (3) has a solution. At the end of this section we give a simple proof of this result based on an analysis of quantities $\alpha_{K,\infty}$ and $\beta_{a,\infty}$. In Theorem 2 below we show, using the above mentioned results, that if a satisfies the assumptions of Theorem 1 and K is periodic on $\mathbb{R}^N$, then problem (3) has a solution. This means that if $\{x\colon a(x) = 0\} = \emptyset$, that is equation (1) is uniformly elliptic, we only need the periodicity of K. However, we must retain the assumption on behaviour of a at infinity. In the final result of this paper, Theorem 3, we prove the existence of solution of problem (3) in the case of a uniformly elliptic equation (1), assuming that a is periodic on $\mathbb{R}^N$ and K satisfies assumptions of Theorem 1.

Theorem 2. *Suppose that a satisfies assumptions of Theorem 1. If K is a periodic function on $\mathbb{R}^N$, then problem (3) has a solution in $H^1_a(\mathbb{R}^N)$.*

Proof: Let $\{u_m\} \subset H^1_a(\mathbb{R}^N)$ be a minimizing sequence for problem (3). As in the proof of Theorem 1 we may assume that $u_m \rightharpoonup u$ in $H^1_a(\mathbb{R}^N)$, $u_m \to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$. It is enough to show that $u \not\equiv 0$ on $\mathbb{R}^N$. Assuming that $u \equiv 0$ on $\mathbb{R}^N$, we have $\alpha_{K,\infty} = 1$. It follows from the proof of Theorem 1 that $M_{a,K} = \beta_{a,\infty}$.

To proceed further we consider the minimization problem

$$M_P = \inf\Big\{\int_{\mathbb{R}^N}\Big(a(\infty)\,|Du|^2 + \lambda\,u^2\Big)\,dx;\ u\in H^1_a(\mathbb{R}^N),\ \int_{\mathbb{R}^N} K(x)\,|u|^p\,dx = 1\Big\}.$$

Because the coefficients $a(\infty)$ and $K(x)$ are periodic, this problem has a positive solution $\overline{u} \in H^1(\mathbb{R}^N)$. Considering the inequality

$$M_P\Big(\int_{\mathbb{R}^N} K(x)\,|(u_m\,\psi_R)|^p\,dx\Big)^{\frac{2}{p}} \le \int_{\mathbb{R}^N}\Big(a(\infty)\,|D(u_m\,\psi_R)|^2 + \lambda(u_m\,\psi_R)^2\Big)\,dx$$

we show that

$$M_P = M_P(\alpha_{K,\infty})^{\frac{2}{p}} \le \beta_\infty\,a(\infty) = \beta_{a,\infty} = M_{a,K}\ . \tag{15}$$

On the other hand we have

$$\begin{aligned} M_{a,K} = M_{a,K}\Big(\int_{\mathbb{R}^N} K(x)\,|\overline{u}|^p\,dx\Big)^{\frac{2}{p}} &\le \int_{\mathbb{R}^N}\Big(a(x)\,|D\overline{u}|^2\,dx + \lambda\,\overline{u}^2\Big)\,dx \\ &< \int_{\mathbb{R}^N}\Big(a(\infty)\,|D\overline{u}|^2 + \lambda\,\overline{u}^2\Big)\,dx = M_P \end{aligned}$$

and this contradicts inequality (15). ∎

In case of a uniformly elliptic equation, we can interchange assumptions on a and K in the sense that a is periodic on $\mathbb{R}^N$ and K satisfies condition $K(x) \ge K(\infty)$ on $\mathbb{R}^N$.

We need the following result which is well known and can be obtained as a by-product of the proof of Lemma 3.1 in [BC] or Theorem 4.1 in [LTW].

Lemma 2. *Suppose that $0 < a_1 \le a(x) \le a_2$ on $\mathbb{R}^N$ for some constants a_1 and a_2. Then for each minimizing sequence $\{u_m\} \subset H^1(\mathbb{R}^N)$ of problem (3), there exist a subsequence of $\{u_m\}$, denoted again by $\{u_m\}$, and a sequence $\{y_m\} \subset \mathbf{Z}^N$ such that $u_m(\cdot + y_m) \rightharpoonup u \not\equiv 0$ in $H^1(\mathbb{R}^N)$.*

Combining Lemmas 1 and 2 we obtain

Corollary 1. *Suppose that a and K are both periodic with the same period $y \in \mathbf{Z}^N$ and that $0 < a_1 \le a(x) \le a_2$ on $\mathbb{R}^N$ for some constants a_1 and a_2. Then problem (3) has a solution.*

Theorem 3. *Let $0 < a_1 \le a(x) \le a_2$ on $\mathbb{R}^N$ for some constants a_1 and a_2 and suppose that a is periodic on $\mathbb{R}^N$, that is $a(x + y) = a(x)$ for all $x \in \mathbb{R}^N$*

and $y \in \mathbf{Z}^N$. If $K(x) \ge K(\infty) = \lim_{|x|\to\infty} K(x)$ on $\mathbb{R}^N$, then problem (3) has a solution.

Proof: We have to consider the case where $K(x) \ge K(\infty)$ with strict inequality on a set of positive measure on $\mathbb{R}^N$, since otherwise the result follows by Corollary 1. Since $\{u_m\}$ is bounded in $H^1(\mathbb{R}^N)$ we may assume that $u_m \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_m \to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$. The assertion will follow from Lemma 1 if we can show that $u \not\equiv 0$. Arguing indirectly, assume that $u \equiv 0$ on $\mathbb{R}^N$. This implies that $\alpha_{K,\infty} = 1$. As in the proof of Lemma 1 we check that $\beta_{a,\infty} = M_{a,K}$. We now consider the minimization problem

$$M_{P,\infty} = \inf\left\{\int_{\mathbb{R}^N}\Big(a(x)\,|Du|^2 + \lambda\, u^2\Big)\,dx;\; u\in H^1(\mathbb{R}^N),\; \int_{\mathbb{R}^N} K(\infty)\,|u|^p\,dx = 1\right\}.$$

Since a and $K(\infty)$ are periodic on $\mathbb{R}^N$ there exists a positive function $\overline{u} \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} K(\infty)\,|\overline{u}|^p\,dx = 1 \quad \text{and} \quad M_{P,\infty} = \int_{\mathbb{R}^N}\Big(a(x)\,|D\overline{u}|^2 + \lambda\,\overline{u}^2\Big)\,dx\;.$$

Let ψ_R be a function from Proposition 1, then

$$M_{P,\infty}\Big(\int_{\mathbb{R}^N} K(\infty)\,|u_m\,\psi_R|^p\,dx\Big)^{\frac{2}{p}} \le \int_{\mathbb{R}^N}\Big(a(x)\,|D(u_m\,\psi_R)|^2 + \lambda(u_m\,\psi)^2\Big)\,dx\;.$$

Letting $m \to \infty$ and then $R \to \infty$ we obtain

$$M_{P,\infty} = M_{P,\infty}(\alpha_{K,\infty})^{\frac{2}{p}} \le \beta_{a,\infty} = M_{a,K}\;. \tag{16}$$

Since $M_{P,\infty}$ is achieved by $\overline{u} \in H^1(\mathbb{R}^N)$, we have

$$M_{a,K}\Big(\int_{\mathbb{R}^N} K(x)\,|\overline{u}|^p\,dx\Big)^{\frac{2}{p}} \le \int_{\mathbb{R}^N}\Big(a(x)\,|D\overline{u}|^2 + \lambda\,\overline{u}^2\Big)\,dx = M_{P,\infty}\;.$$

Since $\overline{u} > 0$ on $\mathbb{R}^N$, and $K(x) > K(\infty)$ on a set of positive measure in $\mathbb{R}^N$, we get that $M_{a,K} < M_{P,\infty}$ which contradicts (16). ∎

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