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AN $L^2[0,1]$ INVARIANCE PRINCIPLE FOR LPQD RANDOM VARIABLES

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Abstract: Using an explicit isometry between Hilbert spaces and an embedding of the space of signed measures we prove an invariance principle with weak convergence in $L^2[0,1]$ for random variables which are linearly positive quadrant dependent under a Lindeberg type condition and some regularity on the covariance structure.

1 – Introduction

Let $(X_n)_{n\geq 1}$ be random variables with $EX_n = 0$ and finite variances. Write $S_n = \sum_{i=1}^n X_i$ and define the partial sums processes W_n and W_n^* by

$$W_n(t) = \frac{1}{\sqrt{n}}\, S_{[nt]} \quad \text{and} \quad W_n^*(t) = \frac{1}{\sigma_n}\, S_{[nt]}\,, \qquad t \in [0,1] \tag{1}$$

where $\sigma_n^2 = ES_n^2$ and $[x]$ denotes the integral part of x. When the X_n are independent and identically distributed the Donsker–Prokhorov invariance principle says that W_n^* (resp. W_n) converges weakly to the Brownian motion W (resp. σW where $\sigma^2 = EX_i^2$) in the Skorokhod space $D[0,1]$ [1]. This invariance principle has been intensively extended under various assumptions of dependence. The $D[0,1]$ weak convergence of a sequence of processes $(\xi_n)_{n\geq 1}$ to a limiting process ξ implies the convergence in distribution of random variables $T(\xi_n)$ to $T(\xi)$ for any functional $T:\ D[0,1] \to \mathbb{R}$ continuous in the Skorokhod topology (a typical example being the supremum, $T(f) = \|f\|_\infty$). Nevertheless, for a large class of functionals defined by integral of paths of the type $\int_0^1 f(t)\,\xi_n(t)\,dt$ or $\int_0^1 G(\xi_n(t))\,dt$ the weaker $L^2[0,1]$ continuity of T is sufficient. The first $L^2[0,1]$ invariance principle goes back to Prokhorov [15] who studied the independent

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case. More recently, the $L^2[0,1]$ framework was used by Khmaladze [6] for the convergence of the empirical process and by Mason [7] for the quantile process, both in the case of independent underlying variables $(X_n)_{n\geq 1}$. For dependent sequences, Oliveira and Suquet [12] obtained the $L^2[0,1]$ weak convergence of the empirical process under milder assumptions than in $D[0,1]$. The authors proved also an $L^2[0,1]$ invariance principle for non stationary φ-mixing sequences [11]. The present paper proposes to study invariance principles in $L^2[0,1]$ under positive dependence conditions. Before describing some of the results obtained so far we will recall the notions of positive dependence that we will be interested in. For a more complete study of positive dependence we refer to Newman [8].

Definition 1. A sequence $(X_n)_{n\geq 1}$ is

- pairwise positive quadrant dependent (pairwise PQD) if, given any reals s, t, for $i \neq j$
$$P(X_i > s,\ X_j > t)\ \geq\ P(X_i > s)\, P(X_j > t)\ .$$
- linearly positive quadrant dependent (LPQD) if for any disjoint $A, B \subset \mathbb{N}$ and positive $(r_i)_{i\geq 1}$ the variables $\sum_{i\in A} r_i\, X_i$ and $\sum_{i\in B} r_i\, X_i$ are PQD;
- associated if for any finite choice of indexes $i_1, ..., i_n$ and coordinatewise nondecreasing functions f, g defined on $\mathbb{R}^n$, we have
$$\mathrm{Cov}\Big(f(X_{i_1}, ..., X_{i_n}),\ g(X_{i_1}, ..., X_{i_n})\Big) \geq 0\ .$$

As the functions $\mathbb{1}_{(r,+\infty)}(x) = 1$ if $x > r$ and 0 otherwise are nondecreasing it is easy to derive that association implies LPQD which in turn implies pairwise PQD. Newman and Wright [9] obtained an invariance principle for strictly stationary associated random variables.

Theorem 2 (Newmann, Wright [9]). *Let $(X_n)_{n\geq 1}$ be strictly stationary associated random variables with $EX_n = 0$ and finite second moments verifying*
$$0 < \sigma^2 = \mathrm{Var}(X_1) + 2\sum_{n=2}^{+\infty} \mathrm{Cov}(X_1, X_n) < \infty$$
then $W_n^*(t)$ converges weakly to a standard Brownian motion in $D[0,1]$.

Birkel [2] extended this result dropping the stationarity. For this Birkel introduced
$$u(n) = \sup_{k\geq 1} \sum_{j:\, |j-k|\geq n} \mathrm{Cov}(X_j, X_k)\ .$$

Theorem 3 (Birkel [2]). *Let $(X_n)_{n\geq 1}$ be associated random variables such that $EX_n = 0$ and $E(X_n^2) < \infty$. If*

$$u(n) \to 0\,, \quad u(1) < \infty\,, \tag{2}$$

$$\forall\,\varepsilon > 0\,, \quad \frac{1}{\sigma_n^2} \sum_{j=1}^{n} \int_{\{|X_j| \geq \varepsilon\,\sigma_n\}} X_j^2\, dP \;\to\; 0\,, \tag{3}$$

$$\inf_{n\geq 1} \frac{1}{n}\,\sigma_n^2 > 0\,, \tag{4}$$

$$\frac{1}{\sigma_n^2}\,\sigma_{nk}^2 \to k\,, \quad k \geq 1\,, \tag{5}$$

then $W_n^*(t)$ converges weakly to a standard Brownian motion in $D[0,1]$.

Recently Birkel [3] strengthening conditions (2) and (3) proved an invariance principle for LPQD random variables.

Theorem 4 (Birkel [3]). *Let $(X_n)_{n\geq 1}$ be LPQD random variables with $EX_n = 0$. Suppose that*

$$\exists\,\rho_1 > 0\colon\; u(n) = O(n^{-\rho_1})\,, \tag{6}$$

$$\exists\,\rho_2 > 0\colon\; \sup_{n\geq 1} E\Big(|X_n|^{2+\rho_2}\Big) < \infty\,, \tag{7}$$

(4) and (5) are verified. Then the conclusion of the previous theorem holds.

From this theorem it is derived the following corollary.

Corollary 5 (Birkel [3]). *Let $(X_n)_{n\geq 1}$ be a wide sense stationary LPQD sequence with $E(X_n) = 0$ and $E(X_n^2) < \infty$. If (6) and (7) are verified then W_n^* converges weakly to a standard Brownian motion in $D[0,1]$.*

2 – Results

For the proof of the $L^2[0,1]$ weak convergence we will be interested in dealing with integrals of the form $\int_0^1 W_n(t)\, f(t)\, dt$, where $f \in L^2[0,1]$ and

$$W_n(t) = \frac{1}{\sqrt{n}}\, S_{[nt]} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\, 1\!\!1_{[i/n,1]}(t)\,.$$

Then, we find

$$\int_0^1 W_n(t)\, f(t)\, dt = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \int_{i/n}^1 f(t)\, dt \;.$$

That is, we are naturally driven to consider the space of functions

$$H = \Big\{ h(s) = \int_s^1 f(t)\, dt,\;\; f \in L^2[0,1] \Big\} \;.$$

This space of functions may be equipped with an inner product such that it becomes isometric to $L^2[0,1]$. Of course, the Lipschitz functions are in H and it is easy to verify that they are dense in H.

The following theorem gives a sufficient condition for the relative compactness of the sequence $(W_n)_{n\geq 1}$. This result may be found in [10] or [11]. For the reader's convenience we include here a proof better adapted to the present framework.

Theorem 6. *If there exists a constant $C > 0$ such that*

$$\frac{1}{n} \sum_{j,m=1} |E\, X_j\, X_m| \leq C \tag{8}$$

then the sequence $(W_n)_{n\geq 1}$ is weakly relatively compact in $L^2[0,1]$.

Proof: Let $(e_i)_{i\geq 0}$ be an orthonormal basis of $L^2[0,1]$ and define $f_i(s) = \int_s^1 e_i(t)\, dt$, the corresponding basis of H. According to Prokhorov's moment condition ([15], Th. 1.13) it is enough to prove that

$$\lim_{N\to+\infty} \sup_{n\in\mathbb{N}} \int \sum_{i=N}^{\infty} \Big(\int_0^1 W_n(t)\, e_i(t)\, dt \Big)^2 dP = 0$$

and

$$\sup_{n\in\mathbb{N}} \int \sum_{i=0}^{\infty} \Big(\int_0^1 W_n(t)\, e_i(t)\, dt \Big)^2 dP < \infty \;.$$

We have

$$\begin{aligned} \sup_{n\in\mathbb{N}} \int \sum_{i=N}^{\infty} \Big(\int_0^1 W_n(t)\, e_i(t)\, dt \Big)^2 dP &= \sup_{n\in\mathbb{N}} \sum_{i=N}^{\infty} E \Big(\frac{1}{n} \sum_{j=1}^n f_i\Big(\frac{j}{n}\Big)\, X_j \Big)^2 \\ &= \sup_{n\in\mathbb{N}} \sum_{i=N}^{\infty} \frac{1}{n} \sum_{j,m=1}^n f_i\Big(\frac{j}{n}\Big)\, f_i\Big(\frac{m}{n}\Big)\, E\, X_j\, X_m \\ &\leq \sup_{n\in\mathbb{N}} \Big(\sup_{x\in[0,1]} \sum_{i=N}^{\infty} f_i^2(x) \Big) \frac{1}{n} \sum_{j,m=1}^n |E\, X_j\, X_m| \end{aligned}$$

which converges to zero according to (8) and Dini's theorem. The second condition is trivially verified by choosing $N = 0$ in the previous calculation. ∎

Corollary 7. *Suppose the sequence $(X_n)_{n\geq 1}$ is LPQD. Then the sequence $(W_n^*)_{n\geq 1}$ is weakly relatively compact in $L^2[0,1]$.*

Proof: Using the calculation as in the previous theorem we would find the upper bound

$$\sup_{n\in\mathbb{N}} \Big(\sup_{x\in[0,1]} \sum_{i=N}^{\infty} f_i^2(x) \Big) \frac{1}{\sigma_n^2} \sum_{j,m=1}^{n} |E\, X_j\, X_m| \ .$$

Now, as the random variables are LPQD, their covariances are non negative, so the last factor is equal to 1. Finally, Dini's theorem gives the convergence to zero we sought. ∎

Next follows a technical lemma needed in the proof of Theorem 9.

Lemma 8. *Let $(u_n)_{n\geq 1}$ be a sequence of real numbers such that*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} u_k = \tau$$

then, for each $h \in H$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} h^2\Big(\frac{k}{n}\Big)\, u_k = \tau\, \|h\|_2^2 \ .$$

Proof: We verify the convergence for h a Lipschitz function. Denote $v_n = \sum_{k=1}^{n} u_k$. We may write $n^{-1}\, v_n = \tau + \varepsilon_n$ where $\varepsilon_n \to 0$. Then it follows

$$u_k = k(\tau + \varepsilon_k) - (k-1)\,(\tau + \varepsilon_{k-1}) = \tau + k\,\varepsilon_k - (k-1)\,\varepsilon_{k-1} \ ,$$

so to prove the lemma it suffices to prove

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} h^2\Big(\frac{k}{n}\Big) \Big(k\,\varepsilon_k - (k-1)\,\varepsilon_{k-1}\Big) = 0 \ .$$

As h is Lipschitz there exists a constant $\alpha > 0$ such that $|h(x) - h(y)| \leq \alpha\,|x-y|$, so

$$\left| \frac{1}{n} \sum_{k=1}^{n} h^2\Big(\frac{k}{n}\Big) \Big(k\,\varepsilon_k - (k-1)\,\varepsilon_{k-1}\Big) \right| =$$

$$= \left| \frac{1}{n} \sum_{k=1}^{n-1} k\, \varepsilon_k \Big(h^2\Big(\frac{k}{n}\Big) - h^2\Big(\frac{k+1}{n}\Big) \Big) + \varepsilon_n\, h^2(1) \right|$$

$$\leq \frac{2\,\alpha\, \|h\|_\infty}{n} \sum_{k=1}^{n-1} |\varepsilon_k| + |\varepsilon_n|\, h^2(1) \ \to\ 0 \ ,$$

according to Cesaro's theorem. For the general case remark that $\|h\|_2 \leq \|h\|_\infty \leq C\,\|h'\|_2$ (where h' denotes the almost everywhere derivative of the absolutely continuous function h) and use standard density arguments. ∎

We now state an invariance principle in $L^2[0,1]$ for the sequence $(W_n)_{n\geq 1}$. As for the result proved by Birkel, the main problem is to have some control on the covariances placed outside of the principal diagonal of the covariance matrix. The essence is to impose conditions that imply that the sum of those covariances became negligible. We achieve this in a somewhat different way than that used by Birkel [3], which saves us from imposing some speed convergence to zero of the above mentioned sums, as Birkel was forced to do with $u(n)$ (cf. condition (6) on Theorem 4). Besides, we will need only the existence of moments of order 2, instead of (7), which supposes the existence of moments of order greater than 2.

Theorem 9. *Let $(X_n)_{n\geq 1}$ be LPQD random variables with $EX_n = 0$. For each $p \in \mathbb{N}$ put $k = [\frac{n}{p}]$ and $\xi_{j,p} = \sum_{i=(j-1)\,p+1}^{jp} X_i$, $j = 1,...,k-1$, and $\xi_{k,p} = \sum_{kp+1}^{n} X_i$. Suppose the following conditions are verified*

$$\lim_{n\to+\infty} \frac{1}{n}\, E(S_n^2) = \sigma^2 > 0 \ , \tag{9}$$

$$\lim_{k\to+\infty} \frac{1}{k} \sum_{j=1}^{k} E\,\xi_{j,p}^2 = a_p \quad \text{and} \quad \lim_{p\to+\infty} \frac{a_p}{p} = \sigma^2 \ , \tag{10}$$

$$\forall\,\delta > 0\,, \quad \frac{1}{n} \sum_{i=1}^{n} \int_{\{|X_i|>\delta\sqrt{n}\}} X_i^2\, dP \ \to\ 0 \ . \tag{11}$$

Then $(W_n)_{n\geq 1}$ converges weakly in $L^2[0,1]$ to σW, where W is a standard Brownian motion.

Proof: As the random variables are LPQD, the condition (8) may be written as

$$\sup_{n\in\mathbb{N}} \frac{1}{n}\, E(S_n^2) \leq C < \infty \ ,$$

for some constant $C > 0$. But this is an immediate consequence of (9), so we have the relative compactness of the sequence $(W_n)_{n\geq 1}$. To prove the invariance principle we must verify the convergence in distribution

$$\int_0^1 W_n(t)\, f(t)\, dt = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \int_{i/n}^1 f(t)\, dt \;\to\; \int_0^1 \sigma\, W(t)\, f(t)\, dt\,, \qquad f \in L^2[0,1]\ .$$

It is well known that for $f \in L^2[0,1]$ the random variable $\int_0^1 \sigma W(t)\, f(t)\, dt$ is Gaussian with mean 0 and variance $\|h\|_2$ where $h(x) = \int_x^1 f(t)\, dt$. Using the space H, this means we are interested in the random variables

$$S_n(h) = \frac{1}{\sqrt{n}} \sum_{i=1}^n h\Big(\frac{i}{n}\Big)\, X_i\,, \qquad h \in H\ ,$$

and we may consider functions h which are Lipschitz. Notice that we are interested in proving a central limit theorem for a triangular array. To accomplish this we will use a method similar to the proof of Theorem 3 by Newman and Wright [9], which consists in approximating $\varphi_{S_n(h)}$, the characteristic function of $S_n(h)$, by the product of the characteristic functions of the blocks $\xi_{j,p}$, $j = 1, ..., k$. As a first step

$$\Big|\varphi_{S_n(h)}(t) - \varphi_{S_{kp}(h)}(t)\Big| \leq |t| \operatorname{Var}^{1/2}\Big(S_n(h) - S_{kp}(h)\Big) \tag{12}$$

which converges to zero as $k \to +\infty$, using (9) and the fact that h is Lipschitz. Next we approach h by a simple function keeping an approximation of the corresponding characteristic functions

$$\left|\varphi_{S_{kp}(h)}(t) - E\exp\Big(\frac{it}{\sqrt{kp}} \sum_{j=1}^k h\Big(\frac{j}{k}\Big) \xi_{j,p}\Big)\right| \leq$$
$$\leq |t| \operatorname{Var}^{1/2}\Big(S_{kp}(h) - \frac{1}{\sqrt{kp}} \sum_{j=1}^k h\Big(\frac{j}{k}\Big)\, \xi_{j,p}\Big)\ .$$

Expanding the variance and using the Lipschitz property of h, that is $|h(x) - h(y)| \leq \alpha\, |x - y|$, we easily find

$$\operatorname{Var}\Big(S_{kp}(h) - \frac{1}{\sqrt{kp}} \sum_{j=1}^k h\Big(\frac{j}{k}\Big)\, \xi_{j,p}\Big) \leq \frac{\alpha^2}{k^3 l} \sum_{j,m=1}^{kp} E\, X_j\, X_m\ ,$$

hence, from (9), there exists a constant $C_1 > 0$, independent from p, such that

$$\left|E\exp(it\, S_{kp}(h)) - E\exp\Big(it\, \frac{1}{\sqrt{kp}} \sum_{j=1}^k h\Big(\frac{j}{k}\Big)\, \xi_{j,p}\Big)\right| \leq \frac{C_1\, \alpha\, |t|}{k} \tag{13}$$

which converges to zero as $k \to +\infty$. The next step is to approximate the characteristic function of $(kp)^{-1/2} \sum_{j=1}^{k} h(j/k)\, \xi_{j,p}$ by what we would find if the blocks where independent. Using Theorem 1 of Newman and Wright [9], it follows, using (9) and (10), that for k large enough and some constant $C_2 > 0$,

$$\begin{aligned} (14)\qquad &\left| E \exp\Bigl(it\, \frac{1}{\sqrt{kp}} \sum_{j=1}^{k} h\Bigl(\frac{j}{k}\Bigr)\, \xi_{j,p}\Bigr) - \prod_{j=1}^{k} E \exp\Bigl(\frac{it}{\sqrt{kp}}\, h\Bigl(\frac{j}{k}\Bigr)\, \xi_{j,p}\Bigr) \right| \le \\ &\le \frac{1}{2} \sum_{\substack{j,m=1 \\ j \neq m}}^{k} \frac{t^2}{kp} \left| h\Bigl(\frac{j}{k}\Bigr)\, h\Bigl(\frac{m}{k}\Bigr) \right| E(\xi_{j,p}\, \xi_{m,p}) \\ &\le \frac{t^2\, \|h\|_\infty^2}{2\, kp} \sum_{\substack{j,m=1 \\ j \neq m}}^{k} E(\xi_{j,p}\, \xi_{m,p}) \le \frac{t^2\, \|h\|_\infty^2}{2}\, C_2 \Bigl(\sigma^2 - \frac{a_l}{l}\Bigr) . \end{aligned}$$

So it remains to prove that the product $\prod_{j=1}^{k} E \exp(it(kp)^{-1/2}\, h(j/k)\, \xi_{j,p})$ converges to the characteristic function of a Gaussian distribution where the $\xi_{j,p}$, $j = 1, ..., k$, may be supposed independent. Using Lemma 8 it follows from (10) that

$$s_n^2(h) = \frac{1}{kp} \sum_{j=1}^{k} h^2\Bigl(\frac{j}{k}\Bigr)\, E\, \xi_{j,p}^2 \;\to\; \frac{a_p}{p}\, \|h\|_2^2 .$$

So to prove the Lindeberg condition for the triangular array $(kp)^{-1/2}\, h(j/k)\, \xi_{j,p}$, $j = 1, ..., k$, $k \in \mathbb{N}$, it is enough to prove that, for every $\varepsilon > 0$,

$$(15)\qquad \sum_{j=1}^{k} \int_{\{|h(\frac{j}{k})|\, |\xi_{j,p}| > \varepsilon\, s_n(h)\, \sqrt{kp}\}} \frac{1}{kp}\, h^2\Bigl(\frac{j}{k}\Bigr)\, \xi_{j,p}^2\, dP \;\to\; 0 .$$

An upper bound for this integral is, for k large enough and using Lemma 4 from Utev [17],

$$\begin{aligned} (16)\qquad &\frac{\|h\|_\infty^2}{k} \sum_{j=1}^{k} \sum_{i=(j-1)\,p+1}^{jp} \int_{\{|X_i| > \frac{\varepsilon}{2} \sqrt{\frac{a_p}{p}}\, \frac{\|h\|_2}{\|h\|_\infty}\, \sqrt{\frac{k}{p}}\}} X_i^2\, dP \le \\ &\le \frac{\|h\|_\infty^2}{k} \sum_{j=1}^{kp} \int_{\{|X_i| > \frac{\varepsilon}{2p}\, \sqrt{\frac{a_p}{p}}\, \frac{\|h\|_2}{\|h\|_\infty}\, \sqrt{kp}\}} X_i^2\, dP , \end{aligned}$$

which converges to zero, according to (11). Now summing up the inequalities (12), (13), (14) and (16), we get, for p fixed

$$\limsup_{n \to +\infty} \left| E \exp(it\, S_n(h)) - \exp\Bigl(-\frac{\sigma^2}{2}\, t^2\, \|h\|_2^2\Bigr) \right| \le C\, t^2\, \|h\|_\infty^2 \Bigl(\sigma^2 - \frac{a_p}{p}\Bigr)$$

and now letting $p \to +\infty$ we have the central limit theorem that ends the proof. ∎

It is easily verified that, supposing the wide sense stationary, condition (10) is superfluous, thus we have the following result.

Corollary 10. *Let $(X_n)_{n\geq 1}$ be a wide sense stationary LPQD sequence of random variables with $EX_n = 0$. If (9) and (11) are verified then $W_n(t)$ converges weakly in $L^2[0,1]$ to σW, where W is a standard Brownian motion.*

It is evident that (6) implies (9) and (7) implies (11).

The method of proof used by Birkel [2] for associated random variables may be adapted to the LPQD case providing a result with the same conditions of our previous theorem in what concerns the existence of moments.

Theorem 11. *Let $(X_n)_{n\geq 1}$ be LPQD random variables with $EX_n = 0$. Assume*

$$u(n) = \sup_{k\geq 1} \sum_{j:\, |j-k|\geq n} \operatorname{Cov}(X_j, X_k) \to 0\,, \qquad u(1) < \infty\ , \tag{17}$$

$$\inf_{n\in\mathbb{N}} \frac{1}{n}\, \sigma_n^2 > 0 \quad \text{ and } \quad \frac{1}{\sigma_n^2}\, \sigma_{nk}^2 \to k, \quad k \geq 1\ , \tag{18}$$

$$\forall\, \varepsilon > 0\,, \quad \frac{1}{\sigma_n^2} \sum_{j=1}^{n} \int_{\{|X_i| > \varepsilon\, \sigma_n\}} X_i^2\, dP \ \to\ 0\ . \tag{19}$$

Then the sequence $W_n^*(t)$ converges weakly in $L^2[0,1]$ to a standard Brownian motion.

Proof: According to Corollary 7 the sequence $(W_n^*)_{n\geq 1}$ is relatively compact, so we need only to consider the convergence of $\int_0^1 W_n^*(t)\, f(t)\, dt$, $f \in L^2[0,1]$. The proof follows from an adaptation of Lemma 3, Lemma 4 and Theorem 3 in Birkel [2], to which we refer the reader. We mention only the steps which are not already contained in Birkel's proof.

First, the identification of the limit, that is, we must prove that

$$s_n^2(h) = \frac{1}{\sigma_n^2} \sum_{j=1}^{k} h^2\!\left(\frac{j}{k}\right) E\, \xi_{j,p}^2 \ \to\ \|h\|_2^2\ .$$

For this put $\tau_{k,p} = \sum_{j=1}^{k} E\, \xi_{j,p}^2$, where the random variables are defined as in Theorem 9, and define the probability measures $\mu_{k,p} = \tau_{k,p}^{-1} \sum_{j=1}^{k} E\, \xi_{j,p}^2\, \delta_{j/k}$, where δ_x

denotes the Dirac mass at point x. Then

$$s_n^2(h) = \frac{\tau_{k,p}^2}{\sigma_{kp}^2} \int h^2 \, d\mu_{k,p} \ .$$

According to the proofs of Lemma 4 and Theorem 3 in Birkel [2] it is possible to construct a sequence p_n (hence, a sequence k_n) such that $p_n \to +\infty$ and $\tau_{k,p}^2 \sigma_{kp}^{-2} \to 1$. Now we check that $\mu_n = \mu_{k_n,p_n}$ converges weakly to the uniform distribution on $[0,1]$. Here it is enough to prove $\mu_n[0,t] \to t$, $t \in [0,1]$. In fact

$$\mu_n[0,t] = \frac{1}{\tau_{k,p}^2} \sum_{j=1}^{[nt]} E\, \xi_{j,p}^2 = \frac{\sigma_{kp}^2}{\tau_{k,p}^2} \frac{1}{\sigma_{kp}^2} \left(\sigma_{[kpt]}^2 - \gamma_{k,p,t}\right) ,$$

and, from Lemma 1 in [2], it follows $\sigma_{kp}^{-2} \sigma_{[kpt]}^2 \to t$. On the other hand

$$0 \le \frac{\gamma_{k,p,t}}{\sigma_{kp}^2} \le \frac{\sigma_{kp}^2 - \tau_{k,p}^2}{\sigma_{kp}^2} \ \to\ 0 \ .$$

That is, $\mu_n[0,t] \to t$. The function h being bounded and continuous, it follows $s_n^2(h) \to \|h\|_2^2$.

Second problem, the use of the Lindeberg condition. Instead of the expression in Lemma 4 of [2], we find in our setting

$$\frac{p}{s_n^2(h)\, \sigma_n^2} \sum_{j=1}^{n} \int_{\{|X_j| > \varepsilon\, s_n^2(h)\, \sigma_n\, p^{-1}\}} X_i^2 \, dP \ ,$$

that is, we have the factor $s_n(h)$ that did not appear in Birkel's setting. We have just proved that $s_n^2(h) \to \|h\|_2^2$, so this integral reduces to the one considered in Lemma 4 of [2] up to a constant, which naturally does not affect the convergence to zero. ∎

3 – An example

We give now an example showing that our Theorem 9 is not contained in the Birkel's Theorem 4, nor even in Theorem 3 when the X_n are associated. It illustrates the fact that Birkel's conditions are more sensible to perturbations than the assumptions of Theorem 9.

Let $(X_n)_{n\ge1}$ be a stationary and LPQD sequence with $EX_n = 0$ and $EX_n^2 = 1$. We write $\mathrm{Cov}(X_i, X_j) = \gamma(|j-i|)$ and assume that $\sum_{n\ge1} \gamma(n) < \infty$ together with

$\gamma(n) > 0$ for each $n \in \mathbb{N}$. It is easily verified that $(X_n)_{n\geq 1}$ satisfies the conditions (9), (10) and (11) of Theorem 9. Let us define the perturbed sequence $(X'_n)_{n\geq 1}$ by

$$X'_n = c_n\, X_n \quad \text{ where } \; c_n = \begin{cases} q^{1/2} & \text{if } n = 2^q,\ q \in \mathbb{N}, \\ 1 & \text{else }. \end{cases}$$

The perturbed sequence remains LPQD. Its Birkel coefficient $u'(n)$ is now

$$u'(n) = \sup_{k\geq 1} u'_k(n) \quad \text{ with } \; u'_k(n) = \sum_{j:\, |j-k|\geq n} c_j\, c_k\, \mathrm{Cov}(X_j, X_k)\ .$$

For each n we can find k large enough such that $k+n = 2^q$ and $q^{1/2}\,\gamma(n) \geq 1$ so $\sup_{k\geq 1} u'_k(n) \geq 1$ and $u'(n)$ does not converge to zero.

Next we check conditions (9) to (11) for $(X'_n)_{n\geq 1}$. Write $S'_n = \sum_{i=1}^n X'_i$, and $\xi'_{j,p}$ for the blocs relative to the X'_i.

To (9): By stationarity of $(X_n)_{n\geq 1}$, we have

$$\frac{1}{n}\, E\, S_n^2 \to \sigma^2 = \mathrm{Var}(X_1) + 2 \sum_{k\geq 1} \gamma(k) < \infty\ .$$

In the decomposition

$$E\, S_n'^2 = E\, S_n^2 + \sum_{\substack{1\leq i,j\leq n \\ c_i c_j > 1}} (c_i\, c_j - 1)\, E\, X_i\, X_j\ ,$$

the second term is bounded above by

$$\sum_{\substack{1\leq i,j\leq n \\ c_i c_j > 1}} c_i\, c_j\, E\, X_i\, X_j = \sum_{\substack{1\leq i,j\leq n \\ c_i > 1,\ c_j > 1}} c_i\, c_j\, E\, X_i\, X_j + 2 \sum_{\substack{1\leq i\leq n \\ c_i = 1}} \sum_{\substack{1\leq j\leq n \\ c_j > 1}} c_j\, E\, X_i\, X_j = T_1 + T_2\ ,$$

where

$$T_1 \leq \sum_{\substack{1\leq i,j\leq n \\ c_i > 1,\ c_j > 1}} c_i\, c_j = \Big(\sum_{i=1,\ c_i > 1}^{n} c_i \Big)^2 = O(\ln^3 n)$$

and

$$T_2 \leq 4\,\sigma^2 \sum_{j=1,\ c_j > 1}^{n} c_j = O(\ln^{3/2} n)\ .$$

Hence $n^{-1}(T_1 + T_2) \to 0$ and $n^{-1} E S_n'^2$ converges to σ^2.

To (10): Observing that the number of blocs $\xi'_{j,p}$ having at least one perturbed term is dominated by $\log_2(kp)$ and that the variance of such a perturbed bloc is bounded by $p^2 \log_2(kp)$, we have the estimate

$$0 \le \sum_{j=1}^{k} E\,\xi'^2_{j,p} - \sum_{j=1}^{k} E\,\xi^2_{j,p} \le p^2 (\log_2 n)^2 \ ,$$

so $\lim_{k\to\infty} k^{-1} \sum_{i=1}^{k} E\,\xi'^2_{j,p} = a_p = E\,S_p^2$.

To (11): Use the crude estimate

$$\sum_{i=1,\ c_i>1}^{n} \int_{\{c_i\,|X_i|>\delta\,\sqrt{n}\}} c_i^2\,X_i^2\,dP \le \sum_{i=1,\ c_i>1}^{n} c_i^2\,E\,X_i^2 = \sum_{i=1,\ c_i>1}^{n} c_i^2 = O(\ln^2 n) \ .$$

Remark. If we choose the sequence $(X_n)_{n\ge1}$ associated, the same conclusion hold: $(X'_n)_{n\ge1}$ verify the $L^2[0,1]$ invariance principle by Theorem 9 but $(X_n)_{n\ge1}$ does not satisfy the conditions of Theorem 3.

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