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GLOBAL SOLVABILITY OF A MIXED PROBLEM FOR A NONLINEAR HYPERBOLIC-PARABOLIC EQUATION IN NONCYLINDRICAL DOMAINS

J. Ferreira and N.A. Lar'kin *

Presented by Hugo Beirão da Veiga

Abstract: In this paper we study the global existence and uniqueness of regular solutions to the mixed problem for the nonlinear hyperbolic-parabolic equation

$$K_1(x,t)\,u_{tt} + K_2(x,t)\,u_t - \Delta u + f_1(t)\,|u|^\rho\,u = f(x,t) \quad \text{in } \widehat{Q}\ ,$$

$$u = 0 \quad \text{at } \widehat{\Sigma}_t\ ,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)\,, \quad x \in \Omega_0\ ,$$

where $\widehat{Q}$ is a noncylindrical domain of $\mathbb{R}^{n+1}$ with the lateral boundary $\widehat{\Sigma}_t$ and K_1, K_2, f_1 are functions which satisfy some appropriate conditions.

1 – Introduction

Hyperbolic-parabolic equations belong to a class of equations of a variable type, see Lar'kin, Novikov and Yanenko [6]. These equations are interesting not only from the point of view of the general theory of PDE but also due to various applications in Mathematical Physics and Mechanics.

The most famous representative of this class is the transonic Karman equation

$$u_t\,u_{tt} - u_{xx} = 0\ ,$$

which models flow of a compressible gas in the transonic region, where the velocity of a gas changes from subsonic values to supersonic ones. Respectively, a type

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of the Karman equation changes from elliptic to hyperbolic, depending on the sign of u_t. In the supersonic region, including the sonic curve, where $u_t = 0$, the Karman equation is hyperbolic-parabolic, and the variable t can be considered as the time variable.

As a rulle, domains in which this equation is considered, are noncylindrical.

For example, flow of a gas in supersonic part of a Laval Nozzle which expands with x, can be simulated by hyperbolic-parabolic equations in noncylindrical domains.

A great number of papers dealt with hyperbolic-parabolic equations in cylindrical domains, but very few of them are devoted to regular solutions in noncylindrical domains. It seemed for us worthwhile to study this problem in the present paper.

Let Ω be a bounded domain of $\mathbb{R}^n$ with a sufficiently smooth boundary Γ, $Q = \Omega \times (0,\infty)$, $\Sigma = \Gamma \times (0,\infty)$ and $K \in C^4(0,\infty)$.

Let us consider the subsets Ω_t of $\mathbb{R}^n$ given by

$$\Omega_t = \Big\{x \in \mathbb{R}^n;\ x = K(t)\,y,\ y \in \Omega\Big\}, \qquad 0 \le t \le T \le \infty\ ,$$

whose boundaries are denoted by Γ_t, and the noncylindrical domain $\widehat{Q} \in \mathbb{R}^{n+1}$:

$$\widehat{Q} = \Big\{(x,t) \in \mathbb{R}^n \times (0,\infty);\ x \in \Omega_t\Big\} = \bigcup_{0\le t<\infty} \Omega_t \times \{t\} \tag{1}$$

with the lateral boundary

$$\widehat{\Sigma}_t = \bigcup_{0\le t<\infty} \Gamma_t \times \{t\}$$

such that $\nu_t \le 0$, $K_1\nu_t^2 - \sum_{i=1}^n \nu_{x_i}^2 \le 0$. Here ν_t, ν_{x_i} are projections of an outer normal vector to $\widehat{\Sigma}_t$ on the corresponding axis. The noncylindrical domain $\widehat{Q}$ defined by (1) is time like.

In $\widehat{Q}$ we consider for the hyperbolic-parabolic equation the following mixed problem:

$$\begin{aligned} &K_1(x,t)\,u_{tt} + K_2(x,t)\,u_t - \Delta u + f_1(t)\,|u|^\rho\,u = f(x,t) \quad \text{in } \widehat{Q}\ ,\\ &u = 0 \quad \text{on } \widehat{\Sigma}_t\ ,\\ &u(x,0) = u_0(x),\quad u_t(x,0) = u_1(x)\,, \qquad x \in \Omega_0\ , \end{aligned} \tag{2}$$

where $f_1\colon [0,\infty) \to \mathbb{R}$, $K_1(x,t)$ and $K_2(x,t)$ are two real functions defined in $\widehat{Q}$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Linear and nonlinear wave equations in noncylindrical domains have been treated by many authors. Lions [9] introduced the penalty method to solve the existence problem. Using this method, Medeiros [10] proved the existence of weak solutions to the problem

$$u_{tt} - \Delta u + \beta(u) = f \tag{3}$$

for a wide class of $\beta(u)$ such that $\beta(u)\,u \geq 0$. Cooper and Bardos [1] proved the existence and uniqueness of weak solutions of (3), for the case $\beta(u) = |u|^\alpha\, u$ ($\alpha \geq 0$) and when $\widehat{\Sigma}_t$ is globally "time like", without the increasing condition on $\widehat{Q}$. Cooper and Medeiros [2] included the above results in a general model

$$u_{tt} - \Delta u + f(u) = 0\ ,$$

where f is continuous, $sf(s) \geq 0$ and $\widehat{\Sigma}_t$ is globally "time like". Inoue [4] succeeded in proving the existence of classical solutions to (3) for the case $n = 3$ and $\beta(u) = u^3$ when the body is "time like" at each point.

Ferreira [3] studied the existence of weak solutions to the mixed problem for the equation

$$K_1(x)\,u_{tt} + K_2(x)\,u_t + A(t)\,u + H(u) = f\,, \qquad K_1 \geq 0\ .$$

Da Prato and Grisvard [11] established existence, uniqueness and regularity results in our type of noncylindrical domains $\hat{Q}$ for the following problem

$$\begin{aligned} &u_{tt} - \Delta u - \rho\,\Delta u_t = 0 \quad \text{in } \widehat{Q}\ ,\\ &u + \rho\,u_t = 0 \quad \text{at } \Gamma_t,\ \ 0 < t < T\ ,\\ &u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x)\,, \qquad x \in \Omega_0\ . \end{aligned} \tag{4}$$

Some paper dealt also with regular solutions in nondegenerate case [4, 11]. Degenerating of nonlinear hyperbolic equations brings essential difficulties in a case of noncylindrical domains, because a geometry of a domain influences correctness of problem (2). See Lar'kin [5], when a domain is characteristic.

The goal of this paper is to prove existence and uniqueness of regular solutions to problem (2) for all $t \in [0,\infty)$ in noncylindrical domains (1).

Our approach consists of changing of variables, $v(y,t) = u(K(t)y,t)$. Under this transformation problem (2) in $\widehat{Q}$ is formulated in the cylindrical domain $Q = \Omega \times [0,\infty)$ as follows:

$$K_3(y,t)\,v_{tt} + K_4(y,t)\,v_t - \sum_{i,j=1}^n \frac{\partial}{\partial y_i}\Big(a_{ij}(y,t)\,\frac{\partial v}{\partial y_j}\Big) +$$

$$+\sum_{i=1}^{n} b_i(y,t)\,\frac{\partial v_t}{\partial y_i} + \sum_{i=1}^{n} c_i(y,t)\,\frac{\partial v}{\partial y_i} + f_1(t)\,|v(y,t)|^\rho\, v(y,t) = g(y,t) \quad \text{in } \ Q \ ,$$

$$(5)\qquad \begin{aligned} & v = 0 \quad \text{on } \ \Sigma = \Gamma \times [0,\infty) \ , \\ & v(0) = v_0(y) = u_0(K(0)y)\,, \quad y \in \Omega \ , \\ & v_t(0) = u_1(K(0)y) + \frac{K'(0)}{K(0)} \sum_{i=1}^{n} y_i\,\frac{\partial v_0}{\partial y_i} = v_1(y)\,, \quad y \in \Omega \ , \end{aligned}$$

where

$$(6)\qquad \begin{aligned} K_1(x,t) &= K_1(K(t)y,t) \equiv K_3(y,t) \ , \\ K_2(x,t) &= K_2(K(t)y,t) \equiv K_4(y,t) \ , \\ f(x,t) &= f(K(t)y,t) \equiv g(y,t) \ , \end{aligned}$$

and

$$\begin{aligned} a_{ij}(y,t) &= (\delta_{ij} - K'^2 K_3\, y_i\, y_j)\, K^{-2} \ , \\ b_i(y,t) &= -2K_3\, K'\, K^{-1}\, y_i \ , \\ c_i(y,t) &= \Big[(1-n)\, K'^2\, K_3 - K''^2\, K_3\, K - K'\, K\, K_4\Big]\, K^{-2}\, y_i \\ &\quad - K'^2\, K^{-2} \sum_{j=1}^{n} y_i\, y_j\, \frac{\partial K_3}{\partial y_j} \ . \end{aligned}$$

The paper is organized as follows:

2 – Notations and assumptions.

3 – Existence of regular solutions.

4 – Uniqueness.

5 – Proof of Theorem 3.1.

2 – Notations and assumptions

By $\mathcal{D}(\Omega)$ we denote the space of infinitely differentiable functions with a compact support contained in Ω. The inner products and norms in $L^2(\Omega)$ and $H_0^1(\Omega)$ will be represented by $(\cdot,\cdot)(t)$, $|\cdot|(t)$, $((\cdot,\cdot))(t)$, $\|\cdot\|(t)$ respectively. By $H^{-1}(\Omega)$ we denote the dual space of $H_0^1(\Omega)$. If X is a Banach space, then we denote by $L^p(0,\infty;X)$, $1 \le p \le \infty$ the Banach space of vector valued functions $u\colon [0,\infty) \to X$, which are measurable and $\|u(t)\|_X \in L^p(0,\infty)$, with the norms:

$$\|u\|_{L^p(0,\infty;X)} = \Big[\int_0^\infty \|u(t)\|_X^p \, dt\Big]^{1/p}, \qquad 1 \le p < \infty \ ,$$

$$\|u\|_{L^\infty(0,\infty;X)} = \operatorname*{ess\,sup}_{0 \le t < \infty} \|u(t)\|_X \ .$$

We define $L^q(0,\infty;L^p(\Omega_t))$, the space of functions $w \in L^q(0,\infty;L^p(\mathbb{R}^n))$, such that $w = 0$ in $\mathbb{R}^n \backslash \Omega_t$

$$\|w\|_{L^q(0,\infty;L^p(\Omega_t))} = \Big[\int_0^\infty \|w(t)\|_{L^p(\Omega_t)}^q \, dt\Big]^{1/q}$$

and

$$\|w\|_{L^\infty(0,\infty;L^p(\Omega_t))} = \operatorname*{ess\,sup}_{0 \le t < \infty} \|w(t)\|_{L^p(\Omega_t)} \ .$$

If $w \in L^p(\Omega_t) \cap H_0^1(\Omega_t)$, we continue it by 0 in $\mathbb{R}^n \backslash \Omega_t$. Then we observe that $L^q(0,\infty;L^p(\Omega_t))$ is a closed subspace of $L^q(0,\infty;L^p(\mathbb{R}^n))$ for $1 \le q \le \infty$. In the same way we define $L^q(0,\infty;H_0^1(\Omega_t))$ as the space of functions $w \in L^q(0,\infty;H^1(\mathbb{R}^n))$ such that $w = 0$ in $\mathbb{R}^n \backslash \Omega_t$ with the norm

$$\|w\|_{L^q(0,\infty;H_0^1(\Omega_t))} = \Big[\int_0^\infty \|w(t)\|_{H_0^1(\Omega_t)}^q \, dt\Big]^{1/q}$$

for $1 \le q < \infty$, and

$$\|w\|_{L^\infty(0,\infty;H_0^1(\Omega_t))} = \operatorname*{ess\,sup}_{0 \le t < \infty} \|w(t)\|_{H_0^1(\Omega_t)} \ .$$

Let us consider the following family of operatos in $\mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$

$$A(t) = -\sum_{i,j=1}^n \frac{\partial}{\partial y_i}\Big(a_{ij}(y,t)\,\frac{\partial}{\partial y}\Big), \qquad t \ge 0 \ ,$$

where

$$a_{ij} = a_{ji} \quad \text{and} \quad a_{ij} \in W^{3,\infty}(0,\infty;C^0(\overline{\Omega})) \tag{7}$$

for all $i,j = 1,...,n$.

We suppose that

$$\sum_{i,j=1}^n a_{ij}(y,t)\,\xi_i\,\xi_j \ge \alpha\,|\xi|^2 \ , \tag{8}$$

where α is a positive constant.

For $u, v \in H_0^1(\Omega)$ we denote $a(t,u,v)$:

$$a(t,u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(y,t) \, \frac{\partial u}{\partial y_i} \, \frac{\partial v}{\partial y_j} \, dy \ .$$

From the hypothesis on a_{ij}, we obtain that $a(t,u,v)$ is symmetric and

(9) $$a(t,u,u) \geq \alpha \|u\|^2 \quad \text{for all } u \in H_0^1(\Omega)\,, \qquad t \in [0,\infty) \ .$$

Suppose that functions K_1, K_2, K, f_1, ρ satisfy the following conditions:

A.1:

$$K_1(x,t) \geq 0 \quad \text{ in } \widehat{Q} \ ,$$

$$K_1(x,0) \geq \eta_0 > 0 \quad \text{ in } \Omega_0 \ ,$$

$$K_1 \in W^{3,\infty}(0,\infty; C^0(\overline{\Omega}_t)) \ ,$$

$$K_2 \in W^{1,\infty}(0,\infty; C^0(\overline{\Omega}_t)) \ ,$$

$$\mu(x,t) = K_2(x,t) - \frac{1}{2} |K_{1t}(x,t)| \geq \delta_0 > 0 \quad \text{ in } \widehat{Q} \ ,$$

$$\left|\frac{\partial K_1}{\partial x_i}\right| \leq C K_1 + \eta\,, \qquad i = 1, ..., n \ ,$$

where η is a sufficiently small positive number.

A.2:

$$K \in C^4(0,\infty) \ ,$$

$$\min_{0 \leq t < \infty} K(t) = \alpha_0 > 0\,, \qquad \max_{0 \leq t < \infty} K(t) = \alpha_1 > 0 \ ,$$

$$\sup_{0 \leq t < \infty} K'(t) = \gamma < \frac{1}{M}\,, \qquad M = \sup_{\mathbb{R}^n} \{|y|,\ y \in \Omega\} \ ,$$

$$K'(t) \geq 0,\ \ |K''(t)|,\, |K'''(t)|,\, |K^{(iv)}(t)| \leq C\,, \qquad \forall\, t \in [0,\infty) \ ,$$

$$m_1 = \int_0^\infty K'(t)\, dt < \infty\,, \qquad m_2 = \int_0^\infty |K''(t)|\, dt < \infty \ ,$$

$$m_3 = \int_0^\infty |K'''(t)|\, dt < \infty\,, \qquad m_4 = \int_0^\infty |K^{(iv)}(t)|\, dt < \infty \ ,$$

$$m_5 = \int_0^\infty (K'(t))^2\, dt < \infty\,, \qquad m_6 = \int_0^\infty (K''(t))^2\, dt < \infty \ ,$$

$$m_7 = \int_0^\infty (K'(t))^3\, dt < \infty\,, \qquad m_8 = \int_0^\infty |K''(t)|^3\, dt < \infty \ .$$

A.3:

$$\{f_1, f_1'\} \in (L^1(0,\infty) \cap L^\infty(0,\infty))^2 \ ,$$

$$f_1'(t) \le 0\,, \quad \forall\, t \in [0,\infty) \ ,$$

$$f_1(t) \ge 0\,, \quad \forall\, t \in [0,\infty) \ ,$$

$$0 < \rho \le \frac{2}{n-2} \ \text{ if } n > 2 \quad \text{and} \quad 0 < \rho < \infty \ \text{ if } n = 1 \ \text{ or } \ n = 2 \ .$$

3 – Existence of regular solutions

Theorem 3.1. *Let $u_0 \in H_0^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and $f \in H^1(0,\infty; L^2(\Omega_t))$. Assume that A.1–A.3 take a place. Then there exists a unique function $u(x,t)$ defined in $\widehat{Q}$ such that*

$$\begin{aligned} & u \in L^\infty(0,\infty; H_0^1(\Omega_t) \cap H^2(\Omega_t)) \ , \\ & u_t \in L^\infty(0,\infty; H^1(\Omega_t))\,, \qquad u_{tt} \in L^2(\widehat{Q}) \ , \\ & K_1 u_{tt} \in L^\infty(0,\infty; L^2(\Omega_t)) \ ; \end{aligned} \tag{10}$$

for a.e. $t \in (0,\infty)$ the identity holds

$$\Big(\Big\{K_1\, u_{tt} + K_2\, u_t - \Delta u + f_1(t)\, |u|^\rho u\Big\},\ w\Big)(t) = (f, w)(t) \ , \tag{11}$$

where w is an arbitrary function from $L^2(\mathbb{R}^n)$,

$$\begin{aligned} & u(0) = u_0 \ , \\ & u_t(0) = u_1 \ , \\ & u = 0 \quad \text{on} \ \ \hat{\Sigma}_t \ . \end{aligned} \tag{12}$$

Remark 3.1. Here and in the sequel we use notations of [8].

Proof of Theorem 3.1 will be given in section 5. At first we will study our problem in a cylinder Q.

Domains Q and $\widehat{Q}$ are related by the diffeomorphism $h\colon\, \widehat{Q} \to Q$ defined by

$$h(x,t) = \left(\frac{x}{K(t)},\, t\right) \quad \text{for} \ \ (x,t) \in \widehat{Q} \ ,$$

and $h^{-1}\colon Q \to \widehat{Q}$ defined by

$$h(y,t) = (K(t)y,t) \ . \tag{13}$$

For each $u \in L^2(\widehat{Q})$; $v(y,t) = u(K(t)y,t)$.

By change of variables $x = K(t)y$, we obtain $v \in L^2(Q)$.

Taking into account A.1–A.2, it is easy to verify that

B.1:

$$\begin{aligned}
&K_3(y,t) \ge 0 \quad \text{in } \ Q \ ,\\
&K_3(y,0) \ge \eta_0 > 0 \quad \text{in } \ \Omega \ ,\\
&K_3 \in W^{3,\infty}(0,\infty; C^0(\overline{\Omega})) \ ,\\
&K_4 \in W^{1,\infty}(0,\infty; C^0(\overline{\Omega})) \ ,\\
&r(y,t) = K_4 - \frac{1}{2}\left|K_3' - \frac{K'(t)}{K(t)} \sum_{i=1}^n y_i \frac{\partial K_3}{\partial y_i}\right| \ge \delta_0 > 0 \quad \text{in } \ Q \ ,\\
&\left|\frac{\partial K_3}{\partial y_i}\right| \le C K_3 + \eta \,, \qquad \eta \ \text{ is a sufficiently small positive number} \ .
\end{aligned}$$

B.2:

$$\begin{aligned}
&a_{ij} = a_{ji} \quad \text{and} \quad a_{ij} \in W^{3,\infty}(0,\infty; C^0(\overline{\Omega})) \ ,\\
&a(t,v,v) \ge \alpha \, \|v\|^2_{H^1_0(\Omega)} \ \text{ in } Q \quad (\alpha > 0) \ .
\end{aligned}$$

Let f, u_0, u_1 be as in 3.1. By (13) we obtain

$$\begin{aligned}
&v_0 \in H^2_0(\Omega) \ ,\\
&v_1 \in H^1_0(\Omega) \ .
\end{aligned} \tag{14}$$

Theorem 3.2. *Under conditions of Theorem 3.1, for any $f \in H^1(0,\infty; L^2(\Omega))$ there exists a unique function $v(y,t)$ satisfying initial data (4),*

$$\begin{aligned}
&v \in L^\infty(0,\infty; H^1_0(\Omega) \cap H^2(\Omega)) \ ,\\
&v_t \in L^\infty(0,\infty; H^1_0(\Omega)) \,, \qquad v_{tt} \in L^2(Q) \ ,\\
&K_3 v_{tt} \in L^\infty(0,\infty; L^2(\Omega)) \ ;
\end{aligned} \tag{15}$$

for a.e. $t \in (0,\infty)$ *the identity holds*

$$\left(\left\{K_3\, v_{tt} + K_4\, v_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y,t)\,\frac{\partial v}{\partial y_j}\right) + \sum_{i=1}^{n} b_i\,\frac{\partial v_t}{\partial y_i} + \right.\right.$$
$$\left.\left. + \sum_{i=1}^{n} c_i\,\frac{\partial v}{\partial y_i} + f_1(t)\,|v|^\rho\, v\right\},\, w\right)(t) = (g,w)(t)\;. \tag{16}$$

Here w *is an arbitrary function from* $L^2(\Omega)$.

Proof: For small $\varepsilon > 0$ we consider in a cylinder Q the following mixed problem

$$\begin{aligned}
&K_{3\varepsilon}\, v^\varepsilon_{tt} + K_4\, v^\varepsilon_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y,t)\,\frac{\partial v^\varepsilon}{\partial y_j}\right) + \sum_{i=1}^{n} b_i(y,t)\,\frac{\partial v^\varepsilon_t}{\partial y_i} + \\
&\qquad + \sum_{i=1}^{n} c_i(y,t)\,\frac{\partial v^\varepsilon}{\partial y_i} + f_1(t)\,|v^\varepsilon|^\rho\, v^\varepsilon = g(y,t) \quad \text{in } Q\;,\\
&v^\varepsilon = 0 \quad \text{on } \Sigma = \Gamma\times[0,\infty)\;,\\
&v^\varepsilon(y,0) = v_0(0) = u_0(K(0)y)\,, \quad y\in\Omega\;,\\
&v^\varepsilon_t(y,0) = u_1(K(0)y) + \frac{K'(0)}{K(0)}\sum_{i=1}^{n} y_i\,\frac{\partial v_0}{\partial y_i} = v_1(y)\,, \quad y\in\Omega\;,
\end{aligned} \tag{17}$$

where $K_{3\varepsilon} = K_3 + \varepsilon$.

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis in $H^2_0(\Omega)$. For each $m\in\mathbb{N}$ we define

$$u^{m,\varepsilon}(y,t) = \sum_{\ell=1}^{m} g_{\ell m\varepsilon}(t)\, w_\ell(y)\;,$$

where unknown functions $g_{\ell m\varepsilon}(t)$ are solutions to the following Cauchy problem for the system of ordinary differential equations

$$\begin{aligned}
&(K_{3\varepsilon}\, v^{m,\varepsilon}_{tt}, w_\ell) + (K_4\, v^{m,\varepsilon}_t, w_\ell) + a(t, v^{m,\varepsilon}, w_\ell) - \\
&\quad - 2\frac{K'(t)}{K(t)}\sum_{i=1}^{n}\left(K_3\, y_i\,\frac{\partial v^{m,\varepsilon}_t}{\partial y_i},\, w_\ell\right) + \sum_{i=1}^{n}\left(c_i(t)\,\frac{\partial v^{m,\varepsilon}}{\partial y_i},\, w_\ell\right) + \\
&\quad + \left(f_1(t)\,|v^{m,\varepsilon}|^\rho\, v^{m,\varepsilon},\, w_\ell\right) = (g, w_\ell)\,, \quad 1\le\ell\le m\;,\\
&g_{\ell m\varepsilon}(0) = (v_0, w_\ell)\;,\\
&g'_{\ell m\varepsilon}(0) = (v_1, w_\ell)\;.
\end{aligned} \tag{18}$$

This problem has solutions $g_{\ell m\varepsilon} \in C^2([0,T_{m\varepsilon}))$, $0 < T_{m\varepsilon} < T$. The a priori estimates, we shall obtain, will permit us to extend the approximate solutions $v^{m,\varepsilon}$ to the interval $[0,\infty)$ and also pass to the limit as $m \to \infty$, $\varepsilon \to 0$.

A PRIORI ESTIMATE 1. In our calculations we wil omit indices m,ε. Multiplying (18) by $2g_{\ell t}$, summing over ℓ, using the hypothesis B.1–B.2 and A.3, we find

$$\begin{aligned}(19)\quad & \frac{d}{dt}\Big[|\sqrt{K_{3\varepsilon}}\,v_t|^2(t) + a(t,v(t),v(t)) + (f_1(t),M(u))\Big] + \\ & + \Big(2K_4 - K_3' + \frac{K'(t)}{K(t)}\sum_{i=1}^n y_i\,\frac{\partial k_3}{\partial y_i},\ |v_t|^2\Big) - (f_1'(t),M(u)) - \\ & - a'(t,v(t),v(t)) - 4\,\frac{K'(t)}{K(t)}\sum_{i=1}^n\Big(\frac{\partial v_t}{\partial y_i},\ K_3\,y_i\,v_t\Big) + \\ & + 2\sum_{i=1}^n\Big(c_i(t)\,\frac{\partial v}{\partial y_i},\ v_t\Big) = 2\,(g(t),v_t)\ ,\end{aligned}$$

where $M(u) = \int_0^u |s|^\rho\, s\, ds \geq 0$.

Integrating (19) from 0 to t, using the hypothesis B.1–B.2, A.2–A.3, and observing that $K_{3\varepsilon}\,v_t^2 \geq K_3\,v_t^2 \geq 0$, we obtain

$$\begin{aligned}(20)\quad & |\sqrt{K_3}\,v_t|^2(t) + \alpha\|v\|^2_{H_0^1(\Omega)} \leq \\ & \leq C + \int_0^t f_2(\tau)\,\Big(|\sqrt{K_3}\,v_\tau|^2(\tau) + \|v\|^2_{H_0^1(\Omega)}(\tau)\Big)\,d\tau + \int_0^t |g|^2(\tau)\,d\tau\ ,\end{aligned}$$

where $f_2(t) \in L^1(0,\infty)$. Hence, by Gronwall's Lemma

$$(21)\qquad |\sqrt{K_3}\,v_t|^2(t) + \alpha\|v\|^2_{H_0^1(\Omega)} + \delta_0\int_0^t |v_\tau|^2(\tau)\,d\tau \leq C\ ,$$

where C is a positive constant independent of m and $t \in [0,\infty)$.

A PRIORI ESTIMATE 2. Now we differentiate equation (17) with respect to t, multiply the result by $2g_{\ell tt}$ and summ over ℓ to obtain

$$\begin{aligned}(22)\quad & \frac{d}{dt}\Big[|\sqrt{K_{3\varepsilon}}\,v_{tt}|^2(t) + a(t,v_t(t),v_t(t)) + 2\,a'(t,v(t),v_t(t))\Big] + \\ & + \bigg(2\Big(K_4 + \frac{1}{2}\Big(K_3' - \frac{K'(t)}{K(t)}\sum_{i=1}^n y_i\,\frac{\partial K_3}{\partial y_i}\Big)\Big),\ |v_{tt}|^2\bigg) + 2(K_4' v_t, v_{tt}) - \\ & - 2a''(t,v(t),v_t(t)) - 3a'(t,v_t(t),v_t(t)) + 2\sum_{i=1}^n\Big(\Big(b_i\,\frac{\partial v_t}{\partial y_i}\Big)',\ v_{tt}\Big) + \\ & + 2\sum_{i=1}^n\Big(\Big(c_i\,\frac{\partial v}{\partial y_i}\Big)',\ v_{tt}\Big) + 2((f_1(t)|v|^\rho v)', v_{tt}) = 2(g', v_{tt})\ .\end{aligned}$$

Integrating (22) from 0 to t, using the hypothesis A.2–A.3 and B.1–B.2 and observing that $K_{3\varepsilon}\, v_{tt}^2 \geq K_3\, v_{tt}^2 \geq 0$, we have

$$
\begin{aligned}
(23) \qquad & |\sqrt{K_3}\, v_{tt}|^2(t) + \alpha \|v_t\|^2 + \delta_0 \int_0^t |v_{\tau\tau}(\tau)|^2\, d\tau \leq \\
& \leq C_1 + |(K_3 v_{tt}(0), v_{tt}(0))| + \int_0^t f_2(\tau) \Big[|\sqrt{K_3}\, v_{\tau\tau}|^2(\tau) + \|v_\tau(\tau)\|^2 \Big]\, d\tau + \int_0^t |g_\tau(\tau)|^2\, d\tau\,.
\end{aligned}
$$

Remark 3.2. We need an estimate for $v_{tt}(0)$. Putting $t = 0$ in (17) and using hypothesis about the function K_3, we obtain $|v_{tt}(0)| \leq C$, where a constant C does not depend on m, $t \in [0,\infty)$.

Now, using Remark 3.2, observing that $f_2(t) \in L^1(0,\infty)$, by Gronwall's Lemma we get

$$
(24) \qquad |\sqrt{K_3}\, v_{tt}|^2(t) + \alpha \|v\|^2_{H_0^1(\Omega)} + \frac{\delta_0}{4} \int_0^t |v_{\tau\tau}(\tau)|^2\, d\tau \leq C\,,
$$

where C is a positive constant independent of m and $t \in [0,\infty)$.

Let us now study the nonlinear term.

Since $f_1(t) \in L^1(0,\infty) \cap L^\infty(0,\infty)$, we have from (21) and (24)

$$
(25) \qquad \Big\| f_1(t)\, |v^{m,\varepsilon}|^{\rho+1} \Big\|_{L^2(0,\infty;L^2(\Omega))} \leq C\,.
$$

By compactness arguments

$$
(26) \qquad f_1(t)\, |v^{m,\varepsilon}|^\rho\, v^{m,\varepsilon} \;\to\; f_1(t)\, |v^\varepsilon|^\rho\, v^\varepsilon \quad \text{a.e. in } Q\,, \qquad m \to \infty\,.
$$

From (25), (26) we conclude:

$$
(27) \qquad f_1(t)\, |v^{m,\varepsilon}|^\rho\, v^{m,\varepsilon} \;\to\; f_1(t)\, |v^\varepsilon|^\rho\, v^\varepsilon \quad \text{weakly in } L^2(Q)\,.
$$

From the a priori estimates obtained we can see that there exists a subsequence of $(v^{m,\varepsilon})$, which we still denote by $(v^{m,\varepsilon})_{m\in\mathbb{N}}$, such that

$$
\begin{aligned}
& v^{m,\varepsilon} \to v^\varepsilon \quad \text{weak}^* \text{ in } L^\infty(0,\infty; H_0^1(\Omega))\,, \\
& v_t^{m,\varepsilon} \to v_t^\varepsilon \quad \text{weak}^* \text{ in } L^\infty(0,\infty; H_0^1(\Omega))\,, \\
& v_{tt}^{m,\varepsilon} \to v_{tt}^\varepsilon \quad \text{weakly in } L^2(Q)\,, \\
& K_{3\varepsilon}\, v_{tt}^{m,\varepsilon} \to K_{3\varepsilon}\, v_{tt}^\varepsilon \quad \text{weak}^* \text{ in } L^\infty(0,\infty; L^2(\Omega))\,, \\
& f_1(t)\, |v^{m,\varepsilon}|^\rho\, v^{m,\varepsilon} \to f_1(t)\, |v^\varepsilon|^\rho\, v^\varepsilon \quad \text{weakly in } L^2(Q)\,.
\end{aligned}
$$

Letting m tend to ∞, we conclude

$$(K_{3\varepsilon}\, v^\varepsilon_{tt},\, w)(t) + (K_4\, v^\varepsilon_t, w)(t) + \Big(\sum_{i,j=1}^n \frac{\partial}{\partial y_i}\Big(a_{ij}(y,t)\,\frac{\partial v^\varepsilon}{\partial y_j}\Big),\, w\Big)(t) +$$

$$+ \Big(\sum_{i=1}^n b_i\,\frac{\partial v^\varepsilon_t}{\partial y_i},\, w\Big)(t) + \Big(\sum_{i=1}^n e_i\,\frac{\partial v^\varepsilon}{\partial y_i},\, w\Big)(t) + \Big(f_1(t)\,|v^\varepsilon|^\rho\, v^\varepsilon,\, w\Big)(t) =$$

$$= (g,w)(t) \quad \text{for a.e. } \ t\in(0,\infty)\ ,$$

where w is an arbitrary function from $H^1_0(\Omega)$.

Obviously, initial conditions (17) are satisfied. Observe that estimates obtained are also independent of ε. Therefore, by the same argument we can pass to the limit when ε goes to zero in $\{v^\varepsilon\}$. Thus we obtain a function

$$\begin{aligned}
&v\in L^\infty(0,\infty;H^1_0(\Omega))\ ,\\
&v_t\in L^\infty(0,\infty;H^1_0(\Omega))\ ,\\
&v_{tt}\in L^2(Q)\,, \qquad K_3\, v_{tt}\in L^\infty(0,\infty;L^2(\Omega))\ ,
\end{aligned}$$

satisfying the identity

$$\sum_{i,j=1}^n \Big(\frac{\partial}{\partial y_i}\Big(a_{ij}\,\frac{\partial v}{\partial y_j}\Big),\, Z\Big)(t) =$$

$$= \bigg(\Big\{g - K_3\, v_{tt} - K_4\, v_t - \sum_{i=1}^n \Big[b_i\,\frac{\partial v_t}{\partial y_i} + c_i\,\frac{\partial v}{\partial y_i}\Big] - f_1(t)\,|v|^\rho\, v\Big\},\ Z\bigg)(t) \equiv$$

$$\equiv (P(y,t),Z)(t) \quad \text{for a.e. } \ t\in(0,\infty)\ ,$$

where Z is an arbitrary function from $H^1_0(\Omega)$ and $P\in L^2(\Omega)$.

It follows from the properties of a function $v(y,t)$ that $P(y,t)\in L^\infty(0,\infty;L^2(\Omega))$. The theory of elliptic equations gives us

$$v\in L^\infty(0,\infty;H^1_0(\Omega)\cap H^2(\Omega))\ .$$

This completes the existence part of Theorem 3.2.

4 – Uniqueness

Let v_1, v_2 be two distinct solutions to (16). Putting $w = 2(v_1 - v_2)$, we obtain:

$$\frac{d}{dt}\Big[|\sqrt{K_3}\, w_t|^2(t) + a(t, w(t), w(t))\Big] +$$
$$+ (2K_4 - K_{1t},\, w_t^2) - a'(t, w(t), w(t)) + 2\Big(\sum_{i=1}^n b_i \frac{\partial w_t}{\partial y_i},\, w_t\Big) +$$
$$+ 2\Big(\sum_{i=1}^n c_i \frac{\partial w}{\partial y_i},\, w_t\Big) + 2\Big(f_1(t)\,|v_1|^\rho\, v_1 - f_1(t)\,|v_2|^\rho\, v_2,\, w_t\Big) = 0\,, \tag{28}$$
$$w = 0 \quad \text{on} \ \ \Sigma\,,$$
$$w(0) = 0\,, \qquad w_t(0) = 0\,.$$

Green's formula gives

$$2\sum_{i=1}^n \Big(b_i \frac{\partial w_t}{\partial y_i},\, w_t\Big) = -\sum_{i=1}^n \Big(\frac{\partial b_i}{\partial y_i},\, w_t^2\Big)$$

and

$$-\sum_{i=1}^n \Big(\frac{\partial b_i}{\partial y_i}, w_t^2\Big) = \sum_{i=1}^n \Big(2K_3 K' K^{-1}, w_t^2\Big) + 2\,\frac{K'(t)}{K(t)}\, \sum_{i=1}^n \Big(y_i\, \frac{\partial K_3}{\partial y_i},\, w_t^2\Big)\,. \tag{29}$$

With regard to the nonlinear term, we obtain

$$\begin{aligned} 2\,\Big|\Big(f_1(t)\,|v_1|^\rho\, v_1 - f_1(t)\,|v_2|^\rho\, v_2,\, w_t\Big)\Big| &\le \\ &\le 2\, f_1(t) \int_\Omega \Big|\Big(|v_1|^\rho v_1 - |v_2|^\rho,\, w_t\Big)\Big|\, dy \\ &\le 2\, f_1(t)\, C_\rho \int_\Omega \Big[|v_1(t)|^\rho + |v_2(t)|^\rho\Big]\, |w(t)|\, |w_t(t)|\, dy\,. \end{aligned} \tag{30}$$

Since injection $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is continuous, if $\frac{1}{n} + \frac{1}{2} + \frac{1}{q} = 1$ and $\rho\, n \le q$, then $|u|^\rho_{L^\rho}$, $|v|^\rho_{L^\rho} \in L^n(\Omega)$. From (30) we find

$$2\,\Big|\Big(f_1(t)\,|v_1|^\rho\, v_1 - f_2(t)\,|v_2|^\rho\, v_2,\, w_t\Big)\Big| \le C_\rho\, f_1(t)\, \|w\|\, |w_t|\,. \tag{31}$$

Integrating (28) from 0 to $t < \infty$, using the hypothesis A.2–A.3, B.1–B.2, (28), (29), (31) and the inequality of Schwartz, we have

$$|\sqrt{K_3}\,w_t|^2(t)+\alpha\int_\Omega|\nabla w|^2(t)\,dy+ \\ +\int_0^t\Big(2\,K_4-\frac{1}{2}\,K_3'+\frac{K'(\tau)}{K(\tau)}\sum_{i=1}^n y_i\,\frac{\partial K_3}{\partial y_i}\Big)\,|w_\tau|^2(\tau)\,d\tau+ \\ +\int_0^t\int_\Omega 2n\,K_3\,K'\,K^{-1}\,|w_\tau|^2(\tau)\,dy\,d\tau\leq \\ \leq C_\varepsilon\int_0^t f_2(\tau)\,|\sqrt{K_3}\,w_\tau|^2(\tau)\,d\tau+C_\varepsilon\int_0^t f_2(\tau)\,|\nabla w|^2(\tau)\,d\tau+\varepsilon\int_0^t|w_\tau|^2(\tau)\,d\tau\;.$$

From here

$$|\sqrt{K_3}\,w_t|^2(t)+\alpha\int_\Omega|\nabla w|^2(t)\,dy\leq C\int_0^t f_3(\tau)\,\Big(|\nabla w|^2(\tau)+|\sqrt{K_3}\,w_\tau|^2(\tau)\Big)\,d\tau\;,$$

where $f_3(t)=\max\{f_1(t),f_2(t)\}$, $\forall\,t\in[0,\infty)$.

Since $f_3(t)\in L^1(0,\infty)$, we have by Gronwall's lemma $\nabla w(t)\equiv 0$ a.e. $t\in[0,\infty)$. With $w|_\Sigma=0$ we conclude that $w(t)\equiv 0$ in Q, hence $v_1=v_2$. The proof of Theorem 3.2 is completed. ∎

5 – Proof of Theorem 3.1

Let v be the solution from Theorem 3.2 and u defined by (13). Then $u\in L^\infty(0,\infty;H_0^1(\Omega_t)\cap H^2(\Omega_t))$; $u_t\in L^\infty(0,\infty;H^1(\Omega_t))$, $u_{tt}\in L^2(\widehat{Q})$; $K_1\,u_{tt}\in L^\infty(0,\infty;L^2(\Omega_t))$, $u(0)=u_0$ and $u_t(0)=u_1$.

If $w\in L^2(0,\infty;H_0^1(\Omega_t))$, let $\phi(y,t)=w(K(t)y,t)$ for $(y,t)\in Q$. We note that (16) is valid. Changing the variable $x=K(t)\,y$, we obtain (11) from (16).

Let u_1, u_2 be two solutions to (11), and v_1, v_2 be the functions obtained through the isomorphism h. Then v_1, v_2 are the solutions to (16).

By the uniqueness result of Theorem 3.2, we have $v_1=v_2$, so $u_1=u_2$.

Thus the proof of Theorem 3.1 is completed. ∎

Remark 5.1. Results of Theorem 3.1 can be easily generalized for more general equations

$$K_1(x,t)\,u_{tt}+K_2(x,t)\,u_t+A(t)\,u+f_1(t)\,H(u)=f\;,$$

where $A(t)$ is a strictly elliptic operator and a smooth function $H(u)$ satisfies the condition $H(u)\,u\geq 0$.

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Jorge Ferreira,
Departamento de Matemática, Universidade Estadual de Maringá,
87020-900, Maringá - PR – BRASIL

and

Nickolai A. Lar'kin,
The Institute of Theoretical and Applied Mechanics,
Novosibirsk-90, 630090 – RUSSIA