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AN ALGEBRA OF ABSTRACT VECTOR VARIABLES

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Abstract: In this paper we introduce an abstract algebra of vector variables that generalizes both polynomial algebra and Clifford algebra. This abstractly defined algebra and its endomorphisms contains all the basic $SO(m)$-invariant polynomials and operators used in Clifford analysis.

I – Introduction

Consider the Clifford algebra $\mathbb{R}_m$ generated by the Euclidean space $\mathbb{R}^m = \operatorname{span}\{e_1, ..., e_m\}$ and determined by the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. Then a vector variable is a first order Clifford polynomial of the form $\underline{x} = \sum x_j e_j$. The corresponding Dirac operator or vector derivative in the variable $\underline{x}$ is the operator $\partial_{\underline{x}} = \sum e_j\, \partial_{x_j}$. Clifford analysis deals with the function theory of solutions of $\partial_{\underline{x}} f(\underline{x}) = 0$, called monogenic functions (see [1], [2]).

More in general one may consider operators belonging to the algebra $\operatorname{Alg}\{x_j, \partial_{x_j}, e_j\} = \operatorname{Alg}\{\underline{x}, \partial_{\underline{x}}, e_j\}$ of Clifford differential operators and in particular to $\operatorname{Spin}(m)$-invariant operators.

On Clifford-algebra valued functions one may consider the following representations. Let $s \in \operatorname{Spin}(m)$ and $a \in \mathbb{R}_m$; then first we put $h(s)[a] = s\, a\, \overline{s}$, where for $b \in \mathbb{R}_m$, $\overline{b}$ denotes the standard anti-automorphism determined by $\overline{a\, b} = \overline{b}\, \overline{a}$, $\overline{a} = -a$ for $a \in \mathbb{R}^m$. Then for $\mathbb{R}_m$-valued functions one can consider the representations

$$L(s)\, f(\underline{x}) = s\, f(\overline{s}\, \underline{x}\, s)\,, \qquad H(s)\, f(\underline{x}) = s\, f(\overline{s}\, \underline{x}\, s)\, \overline{s}\ ,$$

which admit straightforward generalizations defined on functions $f(\underline{x}_1, ..., \underline{x}_l)$ of several vector variables $\underline{x}_1, ..., \underline{x}_l \in \mathbb{R}^m$. A polynomial $P(\underline{x}_1, ..., \underline{x}_l)$ is called

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$SO(m)$-invariant if $P = H(s)\,P$, i.e. if

$$P(\underline{x}_1, ..., \underline{x}_l) = s\,P(\overline{s}\,\underline{x}_1\,s,\ ...,\ \overline{s}\,\underline{x}_l\,s)\,\overline{s}\ .$$

In [3] we have shown that the algebra of all such polynomials is generated by the set $\{\underline{x}_1, ..., \underline{x}_l, e_{1...m}\}$, $e_{1...m}$ the pseudoscalar. Prom this it follows that the algebra of $\mathbb{R}_m$-valued differential operators $P(\underline{x}, \partial_{\underline{x}})$ that are invariant under the representation L of Spin(m) is generated by $\{\underline{x}, \partial_{\underline{x}}, e_{1...m}\}$. In the case of H-invariant operators on $\mathbb{R}_m$-valued functions we can consider both left and right operators $P(\underline{x}, \partial_{\underline{x}})f(\underline{x})$ and $f(\underline{x})\,P(\underline{x}, \partial_{\underline{x}})$ and it is in fact more appropriate to consider operators with values in End$(\mathbb{R}_m)$. Using the isomorphism of End$(\mathbb{R}_m)$ with the Clifford algebra $\mathbb{R}_{m,m}$, we proved in [4] that the algebra of H-invariant operators is generated by the left and right operators $f \to \underline{x}f$, $f \to f\underline{x}$, $f \to \partial_{\underline{x}}f$, $f \to f\partial_{\underline{x}}$, the pseudoscalar $f \to e_{1...m}f$ and the projection operators $f \to [f]_k$ of $\mathbb{R}_m$ onto the spaces $\mathbb{R}_m^k$ of k-vectors. Both results may readily be generalized to the case of functions $f(\underline{x}_1, ..., \underline{x}_k)$ and it can be shown that all Spin(m)-representations can be expressed either by L or by H acting on functions of several vector variables (see also [5]). This clearly motivates the study of the algebra Alg$\{\underline{x}_1, ..., \underline{x}_l\}$ generated by several vector variables, as well as the algebra of endomorphisms on this algebra. Note that this algebra indeed contains all the above mentioned Spin(m)-invariant operators, except the pseudoscalar $f \to e_{1...m}\,f$.

In this paper we introduce an abstract axiomatically defined version of the algebra Alg$\{\underline{x}_1, ..., \underline{x}_l\}$, which we call the "radial algebra" $\boldsymbol{R}(S)$ generated by a set S of "abstract vector variables". Note that these "abstract vector variables" are no longer vectors in some Clifford algebra. There is indeed no a priori defined linear space V to which the variables $x \in S$ belong. The algebra $\boldsymbol{R}(S)$ is hence independent of any dimension m or quadratic form Q that could be specified for a linear space V leading to the interpretation of V as the space of vectors in a Clifford algebra. Nevertheless, radial algebras still have all the properties of both Clifford algebras and polynomial algebras. Moreover, in case S is finite, the algebra $\boldsymbol{R}(S)$ is isomorphic to an algebra Alg$\{\underline{x}_1, ..., \underline{x}_l\}$ of true vector variables in some Clifford algebra. Due to this fact, radial algebras generated by finite sets of variables behave like algebras of vector variables belonging to a vector space of unspecified but finite dimension. In case S is infinite, injective representations of $\boldsymbol{R}(S)$ are only possible using infinite dimensional Clifford algebras. In any case, a universal representation of all radial algebras in terms of Clifford algebras seems only possible in some universal Clifford algebra of unspecified and unlimited dimension. This is in fact also the idea behind "geometric algebra" in the sense of [6]. The problem there is that it is not really possible to define anything like a universal geometric algebra (in the standard mathematical sense). Radial algebras however are well-defined sets which behave as if they were embedded in some fictive universal geometric algebra G.

In section II we define radial algebras and establish their representation by Clifford polynomials in case S is finite.

In section III we study fundamental examples of endomorphisms on radial algebras. In particular we give an axiomatic definition of abstract vector derivatives. Surprisingly this leads to the existence of a scalar m which in the Clifford polynomial representation would play the role of the dimension. But now the dimension m is a purely abstract scalar and may hence be considered as a parameter.

In the last section we characterize the algebra $\mathrm{End}\,\boldsymbol{R}(S)$ of endomorphisms in terms of endomorphisms on Clifford polynomials. We also give a direct characterization based on the use of abstract vector variables and vector derivatives.

II – Radial algebras and their Clifford polynomial representation

The starting object in the definition of radial algebra is a set S of "abstract vector variables". The radial algebra $\boldsymbol{R}(S)$ is the universal algebra generated by S and subjected to the constraint

$$\text{(A1)} \qquad \text{for any} \;\; x,y,z \in S\,, \quad [\{x,y\},z] = 0\;,$$

whereby $\{a,b\} = ab + ba$ and $[a,b] = ab - ba$.

This axiom means that the anti-commutator of two abstract vector variables is a scalar, i.e. a quantity which commutes with every other element in the algebra. It is clearly inspired by the similar property for Clifford vector variables. Consider indeed the real Clifford algebra $\mathbb{R}_m$ determined by the relations

$$e_j\,e_k + e_k\,e_j = \pm 2\delta_{jk}\;,$$

then for any two vector variables $\underline{x} = \sum x_j e_j$ and $\underline{y} = \sum y_j e_j$, we have that $\underline{x}\,\underline{y} + \underline{y}\,\underline{x} = \pm 2\sum x_j y_j = 2\,\underline{x}\cdot\underline{y}$. The main difference between Clifford algebra and radial algebra lies in the fact that the abstract vector variables $x \in S$ have a merely symbolic nature; they are not vectors belonging to an a priori defined vector space V of some dimension m with some quadratic form on it.

Nevertheless, by only using (A1) one can already deduce many properties valid for the algebra of Clifford vector variables. We first define the wedge product of vectors by

$$x_1 \wedge ... \wedge x_k = \frac{1}{k!} \sum_{\pi} \mathrm{sgn}(\pi)\, x_{\pi(1)} ... x_{\pi(k)}$$

leading e.g. to the basic relation for Clifford vectors

$$x\,y = x \cdot y + x \wedge y\,, \qquad x \cdot y = \frac{1}{2}(x\,y + y\,x)\ .$$

Lemma 2.1. *Every element $F(x_1, ..., x_l) \in \boldsymbol{R}(S)$ depending on the abstract vector variables $x_1, ..., x_l$ may be written into the form*

$$F(x_1, ..., x_l) = \sum_{i_1 < ... < i_k} F_{i_1 ... i_k}(x_1, ..., x_l) x_{i_1} \wedge ... \wedge x_{i_k}\ ,$$

whereby $F_{i_1 ... i_k}$ are linear combinations of products of inner products $x_i \cdot x_j$.

Proof: For $l = 1$, every $F(x) \in \boldsymbol{R}(S)$, $x \in S$, is clearly of the form $\sum c_j\, x^j = A(x^2) + xB(x^2)$. More in general $F(x_1, ..., x_l)$ may always be written as a linear combination of products of vector variables $x_{j_1} ... x_{j_s}$ so that we only need to consider these and prove the decomposition by induction on the total degree s of the product.

In case two indices in the above product are the same, we can write the product into the form

$$x_{j_1} ... x_{j_s} = \sum \text{scalar l.o.p.}\ ,$$

whereby l.o.p. refers to a lower order product. As the coefficients are scalar, the decomposition follows by induction. Hence we may restrict to the case of products of l different vector variables like $x_1 ... x_l$. But it is in fact clear from (A1) that for any permutation π,

$$x_{\pi(1)} ... x_{\pi(l)} = x_1 ... x_l + \sum \text{scalar l.o.p.}$$

whence also

$$x_1 ... x_l = x_1 \wedge ... \wedge x_l + \sum \text{scalar l.o.p.}\ .$$

Hence the decomposition again follows by induction on the degree. ∎

Every element of the radial $\boldsymbol{R}\{x_1, ..., x_l\}$ generated by finitely many vector variables may hence be decomposed as a sum of k-vectors. Moreover, as every element of $\boldsymbol{R}(S)$ belongs to some subalgebra $\boldsymbol{R}\{x_1, ..., x_l\}$ we have a way of writing elements of $\boldsymbol{R}(S)$. The above lemma suggests the following nomenclature. Let

$\mathcal{C}(\boldsymbol{R}(S))$ be the center of $\boldsymbol{R}(S)$; then we call $\mathcal{C}(\boldsymbol{R}(S))$ the scalar subalgebra of $\boldsymbol{R}(S)$, denoted by $\boldsymbol{R}_0(S)$. In view of (A1), the scalars $x \cdot y = \frac{1}{2}\{x, y\}$ do belong to $\boldsymbol{R}_0(S)$.

An element $a \in S$ is called a k-vector if a may be written as a sum of elements of the form $bx_1 \wedge ... \wedge x_k$, $b \in \boldsymbol{R}_0(S)$, $x_j \in S$. The space of all k-vectors is denoted by $\boldsymbol{R}_k(S)$. However, there are some problems which have to be solved in order to make this nomenclature meaningful.

1) It is not immediately clear whether the decomposition in Lemma 2.1 of an element $f \in \boldsymbol{R}(S)$ is unique although all methods used to perform it lead to the same decomposition.

2) Lemma 2.1 suggests that the scalar algebra $\boldsymbol{R}_0(S)$ be generated by the abstract inner products $x \cdot y$, $x, y \in S$. But also this is not immediately clear, since it can only follow from the uniqueness of the decomposition.

3) There seems to be no extra condition following from (A1) which might link the scalar variables $x \cdot y$. They behave like fully independent real variables. To be more precise, consider for any pair $u, v \in S$ a scalar variable X_{uv} such that $X_{uv} = X_{vu}$. Then the algebra generated by all these scalar variables X_{uv} is a polynomial algebra in the variables $X_{uv} = X_{vu}$, $(u, v) \in S \times S$, which we'll denote by $\mathcal{P}\{X_{uv}\}$. The independence of the scalars $u \cdot v$ is expressed by the fact that after substitution $X_{uv} = u \cdot v$, $\boldsymbol{R}_0(S)$ turns out to be isomorphic to $\mathcal{P}\{X_{uv}\}$. But again, this is not so obvious as it may seem. Note hereby that this property doesn't hold for finite-dimensional Clifford vector variables. Indeed, for $l > m$ we have that for any collection of vector variables $\underline{x}_j = \sum_k x_{jk}\, e_k$ in $\mathbb{R}^m$

$$\det \underline{x}_i\, \underline{x}_j = \pm(\underline{x}_1 \wedge ... \wedge \underline{x}_l)^2 = 0$$

while of course for a general symmetric matrix X_{ij}, $\det X_{ij}$ needn't vanish as a polynomial in the variables X_{ij}.

All these problems involve only finitely many abstract vector variables. They will hence be clarified as soon as we have established the representation of the radial algebra $\boldsymbol{R}\{x_1, ..., x_l\}$ by Clifford polynomials. As expected, this representation is determined by the application $x_j \to \underline{x}_j = \sum x_{jk}\, e_k$. This indeed defines an algebra representation

$$\underline{\cdot}: \boldsymbol{R}\{x_1, ..., x_l\} \to \mathrm{Alg}\{\underline{x}_1, ..., \underline{x}_l\}\ .$$

Note hereby that the algebra $\boldsymbol{R}\{x_1, ..., x_l\}$ is in fact defined as the quotient algebra A/I, where A is the free algebra generated by the set $\{x_1, ..., x_l\}$ and I is its two-sided ideal generated by all the products of the form $[\{x_i, x_j\}, x_k]$. By replacing the variables x_j by $\underline{x}_j$, an epimorphism is obtained from A to $\mathrm{Alg}\{\underline{x}_1, ..., \underline{x}_l\}$ which maps I to zero. The above map is hence well defined as an algebra epimorphism from $\boldsymbol{R}\{x_1, ..., x_l\} = A/I$ to $\mathrm{Alg}\{\underline{x}_1, ..., \underline{x}_l\}$. We now come to the complete characterization of $\boldsymbol{R}\{x_1, ..., x_l\}$ by

Theorem 2.1. *The map* $\underline{\cdot}:\ \boldsymbol{R}\{x_1, ..., x_l\} \to \mathrm{Alg}\{\underline{x}_1, ..., \underline{x}_l\}$ *is an algebra isomorphism if and only if* $m \geq l$.

Proof: It is clear that in case $m < l$, $\underline{\cdot}$ can't be an isomorphism because $\underline{[x_1 \wedge ... \wedge x_l]} = \underline{x}_1 \wedge ... \wedge \underline{x}_l = 0$. For $l \leq m$ all wedge products of vectors belonging to the set $\{\underline{x}_1, ..., \underline{x}_l\}$ are linearly independent. Hence let $F \in \boldsymbol{R}\{x_1, ..., x_l\}$ be written into the form $F = \sum F_{i_1 \dots i_k} x_{i_1} \wedge ... \wedge x_{i_k}$; then $\underline{\cdot}[F] = \sum \underline{\cdot}[F_{i_1 \dots i_k}]\, \underline{x}_{i_1} \wedge ... \wedge \underline{x}_{i_k}$ and from $\underline{\cdot}[F] = 0$ it follows that $\underline{\cdot}[F_{i_1 \dots i_k}] = 0$. Hence if we can prove that for every element F in the algebra generated by $x_i \cdot x_j$, $\underline{\cdot}[F] = 0$ implies that $F = 0$, injectivity of $\underline{\cdot}$ follows.

To that end consider the algebra $\mathcal{P}(X_{ij})$ of polynomials in the scalar variables $X_{ij} = X_{ji}$. The variables X_{ij} have a purely formal character and may hence be interpreted as real or complex variables. The application

$$X_{ij} \to x_i \cdot x_j \to \underline{x}_i \cdot \underline{x}_j = \pm \sum_{k=1}^{m} x_{ik}\, x_{jk}\ ,$$

hence corresponds to the map $x_{ik} \to \pm \sum_k x_{ik}\, x_{jk}$ from the set of $l \times m$ matrices into the symmetric matrices $l \times l$ transforming polynomials in the variables X_{ij} into special polynomials in the variables x_{jk}. The advanced injectivity is now reduced to surjectivity of this map. For $m \geq l$ any complex symmetric matrix $X = (X_{ij})$ can be written into the form $X = \pm V V^t = \pm(\sum_k x_{ik}\, x_{jk})$, V being a complex $l \times m$ matrix. This also means that any polynomial $P(X_{ij})$ is determined by $P(\underline{x}_i \cdot \underline{x}_j)$ and certainly that every polynomial $P(x_i \cdot x_j)$ is determined by $\underline{\cdot}[P(x_i \cdot x_j)] = \underline{P(x_i \cdot x_j)} = P(\underline{x}_i \cdot \underline{x}_j)$. ∎

We immediately have the following

Corollary 2.1. *Every element* $a \in \boldsymbol{R}(S)$ *may be decomposed in a unique way as a finite sum of the form*

$$a = [a]_0 + [a]_1 + [a]_2 + ...$$

whereby $[a]_k \in R_k(S)$.

Note that this solves the first problem, namely the uniqueness of the decomposition established in Lemma 2.1. It readily follows from the uniqueness of the same decomposition in $\mathrm{Alg}\{\underline{x}_1, ..., \underline{x}_l\}$, $l \leq m$.

Corollary 2.2. *The center $C(\boldsymbol{R}(S))$ of $\boldsymbol{R}(S)$ is generated by the formal inner products $x \cdot y = 1/2\{x, y\}$, $x, y \in S$. Moreover it is canonically isomorphic to the polynomial algebra $\mathcal{P}\{X_{uv}\}$, $u, v \in S$, $X_{uv} = X_{vu}$.*

This settles the remaining two problems.

Corollary 2.3. *We have the following formula for the products of a vector with a k-vector*

$$x\, x_1 \wedge ... \wedge x_k = [x\, x_1 \wedge ... \wedge x_k]_{k+1} + [x\, x_1 \wedge ... \wedge x_k]_{k-1} \ ,$$

$$\begin{aligned}
[x\, x_1 \wedge ... \wedge x_k]_{k+1} &= x \wedge x_1 \wedge ... \wedge x_k \\
&= 1/2\Big(x\, x_1 \wedge ... \wedge x_k + (-1)^k\, x_1 \wedge ... \wedge x_k\, x\Big)
\end{aligned}$$

$$\begin{aligned}
[x\, x_1 \wedge ... \wedge x_k]_{k-1} &= x \cdot x_1 \wedge ... \wedge x_k \\
&= 1/2\Big(x\, x_1 \wedge ... \wedge x_k - (-1)^k\, x_1 \wedge ... \wedge x_k\, x\Big) \ .
\end{aligned}$$

For special values of k, this corollary could have been obtained by direct computation and only making use of (A1). But by making use of the Clifford algebra representation this property follows immediately from the orthogonal decomposition of $\boldsymbol{R}^m$ as a direct sum of the subspace parallell and the subspace orthogonal to the k-vector $x_1 \wedge ... \wedge x_k$. This is typical for radial algebra (and endomorphisms on radial algebra) in general; identities can often be proved in two ways: in a direct axiomatic way and by means of the Clifford-polynomial representation. This will be illustrated in the next section for identities between operators.

In the sequel we'll make use of the name "radial algebra with constraints". By this is meant a free algebra $\mathrm{Alg}(S)$ generated by some set S satisfying the axiom (A1) together with extra constraints of the form

$$F(x_1, ..., x_l) = 0$$

for some well specified symbolic expressions F belonging to $\boldsymbol{R}(S)$. Immediate examples of this are

i) orthogonality constraints

$$x\,y = -y\,x, \qquad \text{for any} \;\; x, y \in S \;;$$

ii) linear dependence of dimension l

$$x_1 \wedge ... \wedge x_l = 0, \qquad \text{for some} \;\; \{x_1, ..., x_l\} \;;$$

iii) orthogonal frame relations

$$x_i\,x_j + x_j\,x_i = \pm 2\,\delta_{ij}\,, \qquad i, j = 1, ..., m \;;$$

iv) null constraints

$$x^2 = 0, \qquad \text{for any} \;\; x \in S \;.$$

There are lots of interesting examples of constraints possible. Note that universal Clifford algebras are also radial algebras with constraints.

Another specialization has to do with the choice of the set S of variables. Examples are

i) the algebra $\boldsymbol{R}\{x_1, ..., x_l\}$;

ii) the algebra $R\{x_j\}$, $x_1, ..., x_n, ...$ a sequence of variables;

iii) the algebra $R\{x_p\}$, $p \in M$, M being a manifold;

iv) radial algebras of several families of variables,

$$R\{x_1, ..., x_l;\, u_1, ..., u_n\}\,, \qquad R\{x_j;\, y_j;\, u_j\}\,, \qquad R\{x_p, y_q\}, \;\; p \in M, \;\; q \in N \;.$$

This leads to new examples of radial algebras with constraints. For example one can consider the algebra $R\{x_j, y_j\}$ depending on orthogonal families of vector variables, meaning that $x_j\,y_k = -y_k\,x_j$. There is a richness of possibilities in defining radial algebras of special types, each having a special geometric interpretation.

III – Fundamental endomorphisms on radial algebra

In this section we give an axiomatic definition of the fundamental elements of the algebra of endomorphisms $\mathrm{End}(\boldsymbol{R}(S))$. As expected, they are abstract versions of well known operators from Clifford algebra as well as Clifford analysis (see also [1]–[7] esp. [4]).

(i) Involution, anti-involution

Like for Clifford numbers we can define the main involution $a \to \widetilde{a}$ and anti-involution $a \to \overline{a}$ on $\boldsymbol{R}(S)$ simply by the axioms

$$\overline{a\,b} = \overline{b}\,\overline{a}\,, \qquad \widetilde{a\,b} = \widetilde{a}\,\widetilde{b}\,, \qquad \overline{x} = \widetilde{x} = -x \ \text{ for } x \in S \ .$$

Using the Clifford vector representation of $\boldsymbol{R}(S)$, these maps essentially coincide with the involution and anti involution on Clifford algebras.

(ii) Vector multiplication

We may consider the basic multiplyers

$$x\colon\, a \to x\,a, \quad x|\colon\, a \to \widetilde{a}\,x\,, \qquad x \in S \ .$$

It is easy to see that the set $W = \{x, x| : x \in S\}$ generates a subalgebra of $\mathrm{End}(\boldsymbol{R}(S))$ which is a radial algebra with constraints

$$x\,y| = -y|\,x\,, \qquad x, y \in S \ ,$$
$$x\,y + y\,x = -(x|\,y| + y|\,x|)\,, \qquad x, y \in S \ .$$

It may be useful to consider the "affine variables"

$$X = \frac{1}{2}\,(x - x|), \quad X' = \frac{1}{2}\,(x + x|)\,, \qquad x \in S \ ,$$

generating a radial algebra with constraints

$$XY = -YX\,, \qquad X'Y' = -Y'X'\,, \qquad \{X, Y'\} = -\{Y, X'\} \ .$$

(iii) Directional derivatives

Let $x, y \in S$; then on elements $F \in \boldsymbol{R}(S)$ one may define the operators $D_{y,x} \in \mathrm{End}(\boldsymbol{R}(S))$ by means of the axioms

(D1) $\quad D_{y,x}[F\,G] = D_{y,x}[F]\,G + F\,D_{y,x}[G]$,

(D2) $\quad D_{y,x}[x] = y\,, \qquad D_{y,x}[z] = 0 \ \text{ for } z \in S \ \text{ with } z \neq x$.

The first axiom states that $D_{y,x}$ behaves like a scalar first order differential operator. The second axiom specifies that $D_{y,x}$ corresponds to the directional derivative with respect to x in the direction y. If $F(x)$ refers to the presence of the element x in the expression F, we can indeed prove the following

Lemma 3.1. *For any $F \in \boldsymbol{R}(S)$ we have that*

$$D_{y,x}F(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big(F(x+\epsilon\, y) - F(x)\Big) \ .$$

Proof: An arbitrary element $F \in \boldsymbol{R}(S)$ can always be written into the canonical form

$$F = f_1\, F_1 + x\, f_2\, F_2 \ ,$$

whereby $f_1, f_2 \in \boldsymbol{R}_0(S)$ are scalars and where F_1 and F_2 do not contain the variable x, i.e. they belong to $\boldsymbol{R}(S\backslash\{x\})$. We now evaluate $D_{y,x}[F]$ using both axioms:

$$\begin{aligned} D_{y,x}[f_1\, F_1 + x\, f_2\, F_2] &= D_{y,x}[f_1]\, F_1 + f_1\, D_{y,x}[F_1] + D_{y,x}[x]\, f_2\, F_2 \\ &\quad + x\, D_{y,x}[f_2]\, F_2 + x\, f_2\, D_{y,x}[F_2] \\ &= D_{y,x}[f_1]\, F_1 + y\, f_2\, F_2 + x\, D_{y,x}[f_2]\, F_2 \ . \end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned} F(x+\epsilon\, y) - F(x) &= f_1(x+\epsilon\, y)\, F_1 + (x+\epsilon\, y)\, f_2(x+\epsilon\, y)\, F_2 - f_1\, F_1 - x\, f_2\, F_2 \\ &= \Big[f_1(x+\epsilon\, y) - f_1\Big]F_1 + \Big[(x+\epsilon\, y)\, f_2(x+\epsilon\, y) - x\, f_2\Big]F_2 \end{aligned}$$

and so

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{F(x+\epsilon\, y) - F(x)}{\epsilon} &= \left[\lim_{\epsilon\to 0} \frac{f_1(x+\epsilon\, y) - f_1(x)}{\epsilon}\right] F_1 \\ &\quad + x\left[\lim_{\epsilon\to 0} \frac{f_2(x+\epsilon\, y) - f_2(x)}{\epsilon}\right] F_2 + y\, f_2\, F_2 \ . \end{aligned}$$

Hence it suffices to prove that $D_{y,x}[f] = \lim_{\epsilon\to 0} \frac{f(x+\epsilon y) - f(x)}{\epsilon}$ for $f \in \boldsymbol{R}_0(S)$. This requires only a bit of calculations to check it for the generators of $\boldsymbol{R}_0(S)$:

(i) $$\begin{aligned} D_{y,x}[x^2] &= D_{y,x}[x\, x] = x\, D_{y,x}[x] + D_{y,x}[x]\, x = x\, y + y\, x \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[(x+\epsilon\, y)^2 - x^2\Big] \ ; \end{aligned}$$

(ii) $$\begin{aligned} D_{y,x}[\{x,y\}] &= D_{y,x}[x]\, y + x\, D_{y,x}[y] + D_{y,x}[y]\, x + y\, D_{y,x}[x] \\ &= 2\, y^2 = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[(x+\epsilon\, y)y + y(x+\epsilon\, y) - \{x,y\}\Big] \ ; \end{aligned}$$

(iii) $D_{y,x}[\{x,z\}] = D_{y,x}[x]\,z + x\,D_{y,x}[z] + D_{y,x}[z]\,x + z\,D_{y,x}[x]$
$$= y\,z + z\,y = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[(x+\epsilon\,y)\,z + z(x+\epsilon\,y) - \{x,z\}\Big],\quad z\neq y\,;$$

(iv) $f \in \boldsymbol{R}_0(S\backslash\{x\})\colon\ D_{y,x}[f] = 0$. ∎

For $x = y$ the operator $D_{x,x}$ corresponds to the classical Euler operator E_x with respect to the variable x. As can be expected, the abstractly defined Euler operator $E_x = D_{x,x}$ measures the degree of homogeneity with respect to the variable x. Indeed, the variable $x \in S$ occurs k times in a product $x_1 ... x_l$ if and only if
$$E_x\,x_1 ... x_l = k\,x_1 ... x_l\ .$$
Hence for a general expression $F(x) \in \boldsymbol{R}(S)$, the relation
$$E_x\,F(x) = k\,F(x)$$
means that F is homogeneous of degree k in the variable x. One may hence also consider the projection $\langle F\rangle_{x,k}$ of $F \in \boldsymbol{R}(S)$ on the space of homogeneous elements (polynomials) of degree k in x.

(iv) Vector derivatives

The formal vector derivative ∂_x is supposed to be the abstract equivalent of the vector derivative or Dirac operator
$$\partial_{\underline{x}} = \pm\sum_{j=1}^{m} e_j\,\partial_{x_j}\ ,$$
where $\{e_1, ..., e_m\}$ satisfies the relations $e_j\,e_k + e_k\,e_j = \pm 2\delta_{jk}$. The advantage of such an operator is evident from the fact that all differential operators with respect to x_j are expressable in terms of the vector derivative $\partial_{\underline{x}}$. In particular the directional derivative $D_{\underline{w},\underline{x}}$, $\underline{w}$ a fixed vector not depending on $\underline{x}$, is given by
$$D_{\underline{w},\underline{x}} = \underline{w}\cdot\partial_{\underline{x}} = \frac{1}{2}(\underline{w}\,\partial_{\underline{x}} + \partial_{\underline{x}}\underline{w})\ .$$
In [6], this identity together with the fact that ∂_x is formally a vector is used as an abstract definition for ∂_x. In that case the operator ∂_x is indeed determined by all inner products $w\cdot\partial_x$ with fixed vectors. But in radial algebra this is a little

problematic. First of all there is no a priori defined vector space V to which the variables $x \in S$ belong. Hence it doesn't make much sense to say that

i) ∂_x is a vector;

ii) $y \cdot \partial_x = D_{y,x}$, for any fixed vector $y \in V$.

It does make sense to replace (ii) by the condition

ii)$'$ $D_{y,x} = \frac{1}{2}\{y, \partial_x\}$ for $x, y \in S$ with $x \neq y$

and one can also require that $\partial_x \in \mathrm{End}(\boldsymbol{R}(S))$ transforms scalars into vectors. But this is not the same as saying that ∂_x is formally a vector because one can consider both operators $F \to \partial_x F$ and $F \to F\partial_x$. Hence $\partial_x \in \mathrm{End}(\boldsymbol{R}(S))$ would have to correspond to either the left operator $F \to \partial_x F$ or the right operator $F \to F\partial_x$. Moreover, there is an even harder problem with the evaluation of the action $\partial_x x$ of ∂_x on the element x. Note that for a vector variable $\underline{x}$ belonging to a m-dimensional vector space V we have that $\partial_{\underline{x}}\underline{x} = m$. The problem hereby is that if we want ∂_x to correspond to $\partial_{\underline{x}}$ in the Clifford-algebra representation $x \to \underline{x}$, we have to assume that $\partial_x[x] = m$. But this also removes the freedom of choosing the dimension m in the Clifford-polynomial representation of Theorem 2.1, which in case S is infinite leads to problems. The only alternative seems to be to assume that $\partial_x[x]$ is a constant. Then of course ∂_x need not correspond to $\partial_{\underline{x}}$ in the Clifford-polynomial representation.

We found it best to introduce the left and right operators $\partial_x[F]$ and $[F]\partial_x$ by means of the axioms

(**D1**) $$\partial_x[fF] = \partial_x[f]\,F + f\,\partial_x[F]\ ,$$
$$[fF]\,\partial_x = F[f]\,\partial_x + f[F]\,\partial_x\,, \qquad f \in \boldsymbol{R}_0(S),\quad F \in \boldsymbol{R}(S)\ ,$$

(**D2**) $$\partial_x[FG] = \partial_x[F]\,G,\quad [GF]\,\partial_x = G[F]\,\partial_x \qquad \text{if}\ \ G \in \boldsymbol{R}(S\backslash\{x\})\ ,$$

(**D3**) $$[\partial_x F]\,\partial_y = \partial_x[F\,\partial_y]\,, \qquad x, y \in S\ ,$$

(**D4**) $$\partial_x x^2 = x^2\,\partial_x = 2\,x,\quad \partial_x\{x,y\} = \{x,y\}\,\partial_x = 2\,y\,, \qquad y \neq x\ .$$

As an unpronounced axiom we also assume that for any subset T of S with $x \in T$ and $F \in \boldsymbol{R}(T)$, the value of $\partial_x[F]$ or $[F]\partial_x$ does not depend on whether $F \to \partial_x[F]$ or $F \to [F]\partial_x$ is considered as an element of $\mathrm{End}(\boldsymbol{R}(T))$ or of $\mathrm{End}(\boldsymbol{R}(S))$. We are now able to prove

Theorem 3.1. *The axioms (D1) up to (D4) lead to a consistent definition of endomorphisms $F \to \partial_x[F]$ and $F \to [F]\partial_x$, mapping scalars into vectors and* *we have that for any $y \neq x$,*

$$D_{y,x}[F] = \frac{1}{2}\Big(y\,\partial_x[F] + \partial_x[yF]\Big) \ .$$

Moreover, for any definition of vector derivatives based on (D1) up to (D4) there has to be a unique constant scalar m for which

$$\partial_x x = x\,\partial_x = m\,, \qquad x \in S \ .$$

Proof: Note that the rules (D1), (D2) and (D3) would be satisfied for general first order Clifford differential operators with constant coefficients. It is only (D4) which determines the specific nature of vector derivatives. And there is no mentioning of dimension or quadratic form there. Nevertheless, (D1) up to (D4) can only lead to well-defined endomorphisms $F \to \partial_x[F]$ and $F \to [F]\partial_x$ if something like a dimensional constant $m = \partial_x x$ is being introduced. To see this, let us evaluate $\partial_x[F]$, $F \in \boldsymbol{R}(S)$, as much as we can.

Consider again the canonical decomposition $F = f_1F_1 + xf_2F_2$, where $f_i \in \boldsymbol{R}_0(S)$, $F_i \in \boldsymbol{R}(S\backslash\{x\})$. Using (D1), (D2) we immediately get

$$\partial_x[F] = \partial_x[f_1]\,F_1 + \partial_x[f_2]\,x\,F_2 + f_2\,\partial_x\,x\,F_2 \ .$$

Hence evaluation of $\partial_x[F]$ is possible as soon as evaluation of $\partial_x[f]$, $f \in \boldsymbol{R}_0(S)$ and $\partial_x x$ is determined. As $\boldsymbol{R}_0(S)$ is generated by the elements $\{y,z\}$, $y,z \in S$, due to (D1) the evaluation of $\partial_x[f]$ is determined by the knowledge of $\partial_x\{y,z\}$. Moreover, due to (D2), $\partial_x\{y,z\} = 0$ as soon as both y and z differ from x, so that the evaluation of $\partial_x[f]$ is determined by the knowledge of $\partial_x x^2$ and $\partial_x\{x,y\}$, $y \neq x$ specified in (D4). It is also clear that the object $\partial_x[f]$, $f \in \boldsymbol{R}_0(S)$, belongs to the space $\boldsymbol{R}_1(S)$ of vectors. Evaluation of $\partial_x F$ is hence determined by the knowledge of $\partial_x x$. Using (D3) and (D4) we arrive at the identities

$$2\,\partial_x x = \partial_x[x^2\,\partial_x] = [\partial_x x^2]\,\partial_x = 2[x]\,\partial_x \ ,$$

$$2\,\partial_x x = \partial_x[\{x,y\}\,\partial_y] = [\partial_x\{x,y\}]\,\partial_y = 2[y]\,\partial_y \ ,$$

so that also for any $x, y \in S$,

$$\partial_x x = \partial_y y \ .$$

Due to the unpronounced axiom, it follows that ∂_y can only be in the algebra generated by y and is hence independent from x. But if $\partial_x x$ is independent from x, it is a scalar constant m_x. Moreover, as we have that $m_x = m_y$, the rules (D1) up to (D4) can only lead to endomorphisms $F \to \partial_x[F]$, $F \to [F]\partial_x$ provided that there is a constant m for which

$$\partial_x x = x\,\partial_x = m, \qquad \text{for all } \ x \in S \ .$$

It is clear also that if one accepts a constant like m, the meaning of $\partial_x[F]$ and $[F]\partial_x$ is well-defined and it is also clear that, independent of the value of m, $2D_{y,x}[F] = y\,\partial_x[F] + \partial_x[yF]$ for $y \neq x$. Anyway, let us demonstrate this explicitly using the axioms of ∂_x. Of course we have that

$$y\,\partial_x[f_1F_1 + xf_2F_2] = y\,\partial_x[f_1]\,F_1 + y\,\partial_x[f_2]\,x\,F_2 + m\,f_2\,y\,F_2 \ ,$$

while

$$\begin{aligned}\partial_x y[f_1F_1 + xf_2F_2] &= \partial_x[f_1]\,y\,F_1 + \partial_x[f_2]\,y\,x\,F_2 + f_2\,\partial_x[y\,x\,F_2] \\ &= \partial_x[f_1]\,y\,F_1 + \partial_x[f_2]\,y\,x\,F_2 - m\,f_2\,y\,F_2 + 2\,f_2\,y\,F_2 \ .\end{aligned}$$

Hence we find that

$$\begin{aligned}\{y,\partial_x\}[F] &= \{y,\partial_x\}[f_1]\,F_1 + \{y,\partial_x\}[f_2]\,x\,F_2 + 2\,f_2\,y\,F_2 \\ &= \{y,\partial_x\}[f_1]\,F_1 + x\{y,\partial_x\}[f_2]\,F_2 + 2\,f_2\,y\,F_2 \ ,\end{aligned}$$

where the last step is based upon the axiom (A1) of radial algebra. Looking back at the proof of Lemma 3.1 it clearly suffices to prove that both operators $\{y,\partial_x\}$ and $2D_{y,x}$ coincide when acting on scalars:

(i) $\quad \{y,\partial_x\}[x^2] = y\,\partial_x[x^2] + \partial_x[x^2]\,y = 2\,y\,x + 2\,x\,y$

$$= 2\{x,y\} = 2D_{y,x}[x^2] \ ;$$

(ii) $\quad \{y,\partial_x\}[\{x,y\}] = y\,\partial_x[\{x,y\}] + \partial_x[y\{x,y\}]$

$$= 2\,y^2 + 2\,y^2 = 4\,y^2 = 2D_{y,x}[\{x,y\}] \ ;$$

(iii) $\quad \{y,\partial_x\}[\{x,z\}] = 2\,y\,z + 2\,z\,y = 2\{z,y\} = 2D_{y,x}[\{x,z\}], \quad z \neq y \ ;$

(iv) $\quad f \in \boldsymbol{R}_0(S\backslash\{x\})\colon\ \{y,\partial_x\}[f] = 0 = 2D_{y,x}[f] \ .$ ∎

As to the value of the dimensional constant m, there are several possibilities. One can in fact choose whatever fixed value for m which one likes, it need not even be a positive integer. Nevertheless the positive integer values for m play a singular role in the expression

$$\partial_{x_1} x_1 \wedge \ldots \wedge x_l = (m - l + 1)\,x_2 \wedge \ldots \wedge x_l \ ,$$

which vanishes for $m = l - 1$. In problems involving l vector variables one has to avoid the values $m \in \{0, ..., l-1\}$ in order to keep the generality of formulas.

Moreover, rather than assigning a fixed value to m, it may be more useful to consider m to he a parameter, in which case ∂_x behaves like a vector derivative in a vector space of unspecified dimension.

(v) Other endomorphisms

We may now define the endomorphisms

$$\partial_x\colon\ F \to \partial_x F\,, \qquad \partial_x|\colon\ F \to \widetilde{F}\partial_x\ ,$$

generating a subalgebra of $\mathrm{End}(\boldsymbol{R}(S))$ which is a radial algebra with constraints

$$\partial_x\, \partial_x| = -\partial_y|\, \partial_x\,, \qquad \{\partial_x, \partial_y\} = -\{\partial_x|, \partial_y|\}\ .$$

Also here one can consider the "affine vector derivatives"

$$\partial_X = \frac{1}{2}(\partial_x + \partial_x|)\,, \qquad \partial_{X'} = \frac{1}{2}(\partial_x - \partial_x|)$$

generating a radial algebra with constraints

$$\partial_X\, \partial_Y = -\partial_Y\, \partial_X\,, \qquad \partial_{X'}\, \partial_{Y'} = -\partial_{Y'}\, \partial_{X'}\,, \qquad \{\partial_X, \partial_{Y'}\} = -\{\partial_Y, \partial_{X'}\}\ .$$

It is readily seen that

$$\{\partial_x, y\} = -\{\partial_x|, y|\} = 2D_{y,x}, \qquad \text{for}\ \ x \neq y,\ \ x, y \in S\ ,$$
$$\{\partial_x, x\} = -\{\partial_x|, x|\} = 2E_x + m\ ,$$

so that we also have the identities

(1) $$[\{\partial_x, y\}, z] = [\{\partial_x, y\}, \partial_z] = 0, \qquad \text{for}\ \ z \neq x,\ \ x, y \in S\ ,$$

(2) $$[\{\partial_x, y\}, x] = 2\,y,\ \ [\partial_y, \{\partial_x, y\}] = 2\,\partial_x\,, \qquad x, y \in S\ .$$

We also have that

(3) $$\{\partial_x, y|\} = \{\partial_x|, y\} = 0, \qquad \text{for}\ \ x \neq y,\ \ x, y \in S\ ,$$

(4) $$\{\partial_x, x|\} = -\{\partial_x|, x\} = B \quad \text{is independent of}\ \ x \in S\ \ \text{and}$$

(5) $$\{B, Y\} = 2\,Y,\ \ \{B, Y'\} = -2\,Y'\,, \qquad x, y \in S\ ,$$

(6) $$\{B, \partial_{Y'}\} = 2\,\partial_{Y'},\ \ \{B, \partial_Y\} = -2\,\partial_Y\,, \qquad x, y \in S\ .$$

One can verify these relations directly, but it is a lot simpler to make use of the Clifford algebra representation. The relations (l)–(6) together with the relations among the elements of the sets $\{x, x| : x \in S\}$, $\{\partial_x, \partial_x| : x \in S\}$ determine the subalgebra of $\mathrm{End}(\boldsymbol{R}(S))$ generated by $\{x, x|, \partial_x, \partial_x| : x \in S\}$ only in case S has infinitely many elements and this algebra does not depend on the dimensional constant m. In case S is finite, counter-examples will be given in the next section.

IV – Clifford algebra representation of $\mathrm{End}(\boldsymbol{R}(S))$

The identities among the generators x, $x|$, ∂_x, $\partial_x|$ can be verified directly. But it is easier to represent x, $x|$, ∂_x and $\partial_x|$ by true vector derivatives and multiplication defined on some space of Clifford vectors. We also have to give a characterization of the algebra $\mathrm{End}(\boldsymbol{R}(S))$ generalizing the characterization of $\boldsymbol{R}(S)$ in Theorem 2.1.

Due to this theorem, the algebra $\boldsymbol{R}\{x_1, ..., x_l\}$ generated by finitely many abstract vector variables can be identified with the algebra $\boldsymbol{R}\{\underline{x}_1, ..., \underline{x}_l\}$ generated by the true vector variables $\underline{x}_j = \sum_{k=1}^m x_{jk}\, e_k$, $\{e_1, ..., e_m\}$, $m \geq l$ being a Clifford basis with for example $e_j^2 = -1$. The algebra $\mathrm{End}(\boldsymbol{R}\{x_1, ..., x_l\})$ is hence also isomorphic to the algebra $\mathrm{End}(\boldsymbol{R}\{\underline{x}_1, ..., \underline{x}_l\})$.

As the dimension $\partial_x x$ introduced in radial algebra is purely formal and in fact of no importance for the subalgebra generated by the operators x, $x|$, ∂_x, $\partial_x|$, we can, without too much loss of generality, identify $\partial_x x$ with the dimension m in the Clifford algebra representation. This means that the operator ∂_{x_j} is identified with the true vector derivative $\partial_{\underline{x}_j} = -\sum_{k=1}^m e_k\, \partial_{x_{jk}}$. Similarly one can identify the endomorphisms x_j, $x_j|$, $\partial_{x_j}|$ with the corresponding endomorphisms $\underline{x}_j$, $\underline{x}_j|$, $\partial_{\underline{x}_j}|$ on the algebra $\boldsymbol{R}\{\underline{x}_1, ..., \underline{x}_l\}$. To see how these endomorphisms look like we represent them by operators on Clifford polynomials. Let $\mathcal{P}\{x_{jk}\}$ be the polynomial algebra generated by the set of real variables $\{x_{jk} : j = 1, ..., l;\ k = 1, ..., m\}$. Then $\boldsymbol{R}\{\underline{x}_1, ..., \underline{x}_l\}$ is a subalgebra of the tensor product $\mathcal{P}\{x_{jk}\} \otimes \boldsymbol{R}_m$. Hence the endomorphisms on $\boldsymbol{R}\{\underline{x}_1, ..., \underline{x}_l\}$ are representable by endomorphisms on $\mathcal{P}\{x_{jk}\} \otimes \boldsymbol{R}_m$ i.e. by elements from $\mathrm{End}(\mathcal{P}\{x_{jk}\}) \otimes \mathrm{End}(\boldsymbol{R}_m)$.

The algebra $\mathrm{End}(\boldsymbol{R}_m)$ is isomorphic to the Clifford algebra $\boldsymbol{R}_{m,m}$ over ultra-hyperbolic space and the isomorphism is obtained by defining the generators of $\boldsymbol{R}_{m,m}$ as elements of $\mathrm{End}(\boldsymbol{R}_m)$:

$$e_j : a \to e_j\, a\,, \qquad e_j| : a \to \widetilde{a}\, e_j\ .$$

These elements indeed satisfy the defining relations for $\boldsymbol{R}_{m,m}$

$$\{e_j, e_k\} = -\{e_j|, e_k|\} = -2\,\delta_{jk}\,, \qquad e_j\, e_k| = -e_k|\, e_j$$

(see also [4], [7]).

Moreover, by considering the "affine generators"

$$f_j = \frac{1}{2}(e_j - e_j|)\,, \qquad f_j' = \frac{1}{2}(e_j + e_j|)\ ,$$

and the primitive idempotent

$$I = I_1 ... I_m\,, \qquad I_j = -f_j\, f_j' = \frac{1}{2}(1 + e_j\, e_j|)$$

we can identify $\boldsymbol{R}_m$ as a linear space with the minimal left ideal

$$\boldsymbol{R}_m\, I = \boldsymbol{R}_{m,m}\, I = \Lambda\, V_m'\, I\ ,$$

where V_m' is the affine space span$\{f_1', ..., f_m'\}$ and $\Lambda V_m'$ is the subalgebra of $\boldsymbol{R}_{m,m}$ generated by V_m' (which is also the Grassmann algebra). The action $b[a]$ of $b \in \boldsymbol{R}_{m,m} = \mathrm{End}(\boldsymbol{R}_m)$ on an element a of $\boldsymbol{R}_m$ is hereby directly given by left multiplication

$$b[a]\, I = b\, a\, I\ .$$

Using the representation $x_j \to \underline{x}_j$, we may thus make the identifications

$$x_j \to \sum x_{jk}\, e_k\,, \qquad x_j| \to \sum x_{jk}\, e_k|\ ,$$
$$\partial_{x_j} \to -\sum \partial_{x_{jk}}\, e_k\,, \qquad \partial_{x_j}| \to -\sum \partial_{x_{jk}}\, e_k|\ ,$$

and one can now easily verify the relations among these operators. In particular we have the identifications

$$B = \{\partial_{x_j}, x_j|\} \to -\sum e_k\, e_k|\ ,$$

showing that indeed B is independent of j and representable by a Clifford bivector in $\boldsymbol{R}_{m,m}$. Note that we have the relation

$$-\Big\{\sum e_k\, e_k|,\ f_l\Big\} = 2\, f_l$$

so that, in view of the identification $X_j \to \sum x_{jk}\, e_k$, we obtain $\{B, X_j\} = 2X_j$. The other relations are similar.

It is noteworthy that the primitive idempotent I is itself the projection operator $a \to [a]_0$ of $a \in \mathbb{R}_m$ onto the scalars. Moreover, the operator B satisfies the relations

$$B[a]_k = (m-2k)\,[a]_k$$

so that the projection operators $a \to [a]_k$ of $a \in \mathbb{R}_m$ on $\mathbb{R}_m^k$ are in fact the eigenprojections of the operator B. They are hence expressable as polynomials in B and in particular we have that

$$I = [\cdot]_0 = \exp(-B)\ .$$

Next consider the representation H of $\mathrm{Spin}(m)$ given by

$$H(s)\,f(\underline{x}) = s\,f(\overline{s}\,\underline{x}\,s)\,\overline{s}\,, \qquad s \in \mathrm{Spin}(m)\ ,$$

acting on Clifford algebra valued functions. Then under the identification $\mathbb{R}_{m,m} = \mathrm{End}(\mathbb{R}_m)$ this representation can be rewritten as follows: first we consider the subgroup of $\mathrm{Spin}(m,m)$

$$\widetilde{SO}(m) = \left\{ s\,\overline{s}|:\ s \in \mathrm{Spin}(m) \right\}\ ,$$

whereby the map $a \to a|$ from $\mathbb{R}_m$ to $\mathbb{R}_m|$ is determined by the identity $a\,I = a|\,I$. Next for an $\boldsymbol{R}_m$-valued function $f(\underline{x})$ we may consider the function $f(\underline{x})\,I$ and we have that

$$L(S)\,f(\underline{x})\,I = S\,f(\overline{S}\,\underline{x}\,S)\,I = H(s)[f(\underline{x})]\,I\,, \qquad S = s\,\overline{s}|\ \in\ \widetilde{SO}(m)\ .$$

The group $\widetilde{SO}(m)$ is identifyable to the rotation group $SO(m)$ and in [4] we have shown that the algebra of invariant differential operators under the representation $L(\widetilde{SO}(m)) = H(\mathrm{Spin}(m))$ is generated by the operators $\underline{x}$, $\underline{x}|$, $\partial_{\underline{x}}$, $\partial_{\underline{x}}|$, $a \to [a]_k$ and $a \to e_{1\dots m}\,a$. But as the operators $a \to [a]_k$ are expressable as polynomials in $B = \{\partial_{\underline{x}}, \underline{x}|\}$ this algebra is really generated by the set of operators $\{\underline{x},\ \underline{x}|,\ \partial_{\underline{x}},\ \partial_{\underline{x}}|,\ e_{1\dots m}\}$. A similar result holds of course in the case of several vector variables $\underline{x}_1, \dots, \underline{x}_l$. Moreover, one may consider there the algebra $\mathrm{Inv}(l,m)$ of all the invariant elements of $\mathrm{End}(\mathcal{P}\{x_{jk}\}) \otimes \mathbb{R}_{m,m}$ under the representation L of $\widetilde{SO}(m)$.

We now have the following

Theorem 4.1. *For $m \geq l$ there is a canonical map from the algebra $\mathrm{Inv}(l,m)$ onto the algebra of 2 by 2 matrices over $\mathrm{End}(\boldsymbol{R}\{x_1, \dots, x_l\})$.*

Proof: Let $A \in \mathrm{Inv}(l,m)$. Then the restriction of A to the space of polynomials of the form $F(\underline{x}_1, \dots, \underline{x}_l)I$ where F belongs to the algebra $R\{\underline{x}_1, \dots, \underline{x}_l\}$, transforms the element FI into an element of the form GI, whereby $G \in \mathcal{P}\{x_{jk}\} \otimes \mathbb{R}_m$ satisfies the invariance relation $G(\underline{x}_1, ..., \underline{x}_l) = s\, G(\overline{s}\, \underline{x}_1\, s, \, ..., \, \overline{s}\, \underline{x}_l\, s)\, \overline{s}$. But then G must be of the form $G_1 + e_{1...m}\, G_2$, where G_1, G_2 belong to the algebra $R\{\underline{x}_1, ..., \underline{x}_l\}$ generated by the vector variables $\underline{x}_1, ..., \underline{x}_l$. As this decomposition is also unique, it follows that there is a pair of operators $A_{11}, A_{21} \in \mathrm{End}(R\{\underline{x}_1, ..., \underline{x}_l\})$ such that the action of A on FI coincides with the element $(A_{11}[F] + e_{1...m}A_{21}[F])I$.

Moreover, in a similar way, the restriction of A to elements of the form $Fe_{1...m}I$, $F \in R\{\underline{x}_1, ..., \underline{x}_l\}$, coincides with an element of the form $(A_{12}[F] + e_{1...m}A_{22}[F])I$, $A_{12}, A_{22} \in \mathrm{End}(R\{\underline{x}_1, ..., \underline{x}_l\})$. The action of A on elements of the form $(F + e_{1...m}G)I$, with $F, G \in R\{\underline{x}_1, ..., \underline{x}_l\}$ is hence determined by the action of the matrix $(A_{ij}) \in \mathrm{End}(R\{\underline{x}_1, ..., \underline{x}_l\})$ on the pair (F, G) and the application $A \to (A_{ij})$ is clearly an algebra morphism.

The surjectivity of this map can be seen as follows. Given a 2 by 2 matrix (A_{ij}) over $\mathrm{End}(R\{\underline{x}_1, ..., \underline{x}_l\})$, the above application leads to an endomorphism A of at least the algebra $\mathrm{Alg}\{\underline{x}_1, ..., \underline{x}_l, e_{1...m}\}$ generated by $\underline{x}_1, ..., \underline{x}_l$ and the pseudoscalar. One can extend A to an element $B \in \mathrm{End}(\mathcal{P}\{x_{jk}\}) \otimes \mathbb{R}_{m,m}$ in many ways and for any extension B of A, $L(S)\, BL(\overline{S})$, $S = s\,\overline{s}| \in \widetilde{SO}(m)$, is also an extension of A. This implies that the endomorphism

$$A' = \frac{1}{\mathrm{vol}(\widetilde{SO}(m))} \int_{\widetilde{SO}(m)} L(S)\, BL(\overline{S})\, dS$$

is an extension of A which is invariant under the representation L of $\widetilde{SO}(m)$. This establishes the surjectivity of the above morphism. ∎

The injectivity of the above morphism cannot be established because the $\widetilde{SO}(m)$-invariant extension constructed in the proof on Theorem 4.1 is not unique. Even if we restrict ourselves to the algebra of differential operators with polynomial coefficients there can be problems. We do know that the algebra of invariant differential operators is generated by the set $\{\underline{x}_j,\, \underline{x}_j|,\, \partial_{\underline{x}_j},\, \partial_{\underline{x}_j}|,\, e_{1...m}\}$, but for l odd, the operator $F \to [\underline{x}_1 \wedge ... \wedge \underline{x}_l F]$ vanishes as an element of $\mathrm{End}(R\{\underline{x}_1, ..., \underline{x}_l\})$ while for $m > l$ it is certainly a nonzero element of $\mathrm{End}(\mathcal{P}\{x_{jk}\}) \otimes \mathbb{R}_{m,m}$. In case l is even, the operator

$$F \to \underline{x}_1 \wedge ... \wedge \underline{x}_l F - \widetilde{F}\underline{x}_1 \wedge ... \wedge \underline{x}_l$$

vanishes identically on $R\{\underline{x}_1, ..., \underline{x}_l\}$. Moreover, these operators are always in the algebra generated by the operators $\underline{x}_j,\, \underline{x}_j|,\, \partial_{\underline{x}_j},\, \partial_{\underline{x}_j}|$ the restriction of which to $R\{\underline{x}_1, ..., \underline{x}_l\}$ behaves exactly like the subalgebra of $\mathrm{End}(R\{x_1, ..., x_l\})$ generated by the operators x_j, $x_j|$, ∂_{x_j}, $\partial_{x_j}|$. Hence there can be algebraic expressions in the algebra generated by the operators x_j, $x_j|$, ∂_{x_j}, $\partial_{x_j}|$ which vanish identically on $R\{x_1, ..., x_l\}$ but not on the extension $R(S)$ of this algebra. Another important example of this is the following. The operator B does not depend on the choice of vector variables and has the projections $a \to [a]_k$ as eigenprojections. Hence $a \to [a]_{l+1}$ is a polynomial $P_{l+1}(B)$ in B. This polynomial vanishes of course identically on any algebra $R\{x_1, ..., x_l\}$. This means that the algebra generated by the operators $\{x, x|, \partial_x, \partial_x| : x \in S\}$ is the free algebra determined by the relations (1)–(6) only in case S is infinite. In this case also, the main involution $F \to \widetilde{F}$ can no longer be written as a polynomial in B. The algebra generated by the operators $\{x, x|, \partial_x, \partial_x| : x \in S\}$ is, in case S is infinite, really different from the algebra generated by $\{x, x|, \partial_x, \partial_x|, \widetilde{\cdot} : x \in S\}$.

The algebra $\mathrm{End}(R(S))$ has another more direct representation which is typical for endomorphism algebras. Hereby the linear space $R(S)$ is represented as a minimal left ideal of $\mathrm{End}(R(S))$. Let J be the canonical projection operator $F \to \langle F\rangle$, $\langle F\rangle \in R$ being the constant part of F (i.e. the homogeneous part of degree zero of F). Then J is a primitive idempotent and it is readily seen that for any $A \in \mathrm{End}(R(S))$,

$$A\,J = A[1]\,J$$

so that $\mathrm{End}(R(S))J = R(S)J$.

We also have that for any $f, g \in R(S)$,

$$A\,g\,J[f] = A[g\,J[f]] = A[g]\,J[f], \quad \text{i.e.} \quad A\,g\,J = A[g]\,J\ ,$$

so that $\mathrm{End}(R(S))$ acts on $R(S)$ by left multiplication on $R(S)J$. The left ideal $R(S)J$ can be represented in many ways, leading to a higher flexibility of the calculus. For example one has that for any $x \in S$ and $X = \frac{1}{2}(x - x|)$, $X' = \frac{1}{2}(x + x|)$,

$$x\,J = x|\,J = X'\,J\,, \qquad X\,J = 0\ ,$$

while for any $x_1, ..., x_l \in S$ we also have that

$$x_1 \wedge ... \wedge x_l\,J = X_1' ... X_l'\,J\ .$$

It is hence clear that the space $R(S)J$ is identical to the space

$$\mathrm{Alg}\{x, x| : x \in S\}\,J = \mathrm{Alg}\{X, X' : x \in S\}\,J$$

and any element of this space can be written in a canonical way as

$$\sum P_{j_1 ... j_l}\,X_{j_1}' ... X_{j_l}'\,J$$

where $P_{j_1...j_l}$ are scalars i.e. elements of $R_0(S)$. This indeed follows from the identity

$$\begin{aligned} X\, X_1' \dots X_l'\, J &= \{X, X_1'\}\, X_2' \dots X_l'\, J - X_1'\, X\, X_2' \dots X_l'\, J \\ &= \frac{1}{2} \sum (-1)^k \,\{x, x_k\}\, X_1' \dots \hat{X}_k' \dots X_l'\, J\ . \end{aligned}$$

It means that by considering $\mathrm{Alg}\{X, X' :\ x \in S\}\, J$, the splitting between the commutative and the anti-commutative parts of the algebra is made automatically. Multiplication by X', $x \in S$, indeed raises the "anti-commutative degree of homogeneity" (measured by B) with one, while multiplication with X, $x \in S$, lowers the anti-commutative degree of homogeneity with one and raises the commutative degree of homogeneity with two.

In view of the identities

$$\begin{aligned} \partial_X X' = 0\,, \qquad \partial_{X'} X' = 1/2(m - B)\ , \\ \partial_X P = \sum P_k\, X_k'\,, \qquad \partial_{X'} P = \sum P_k\, X_k\ , \end{aligned}$$

P, P_k scalars, we arrive at the basic formulae

$$\begin{aligned} \partial_X P X_1' \dots X_l' J &= \sum P_k\, X_k'\, X_1' \dots X_l'\, J\ , \\ \partial_{X'} P X_1' \dots X_l' J &= \Big(\sum P_k\, X_k + P\, \partial_{X'}\Big)\, X_1' \dots X_l'\, J\ , \end{aligned}$$

whereby $\partial_{X'} X_1' ... X_l' = 0$ if X' is not among the variables $X_1', X_2', ..., X_l'$ and where

$$\partial_{X_j'} X_1' \dots X_l'\, J = (-1)^j\, (l - 1 - m)\, X_1' \dots \hat{X}_j' \dots X_l'\, J\ .$$

We now come to the dual picture.

Let $A \in \mathrm{End}(R(S))$; then in case S is finite there is a symbolic series $E(\partial_x)$ depending on the vector derivatives ∂_x, $x \in S$, such that $JA = JE(\partial_x)$. The expression $E(\partial_x)$ is determined by letting JA act on $R(S)J$. More in general, for the radial algebra $R(\partial_S)$ generated by the vector derivatives ∂_x, $x \in S$, we may consider the action from the right: $Jf \to JfA$, $f \in R(\partial_S)$, which again transforms Jf into something of the form $JE(\partial_x)$. But $E(\partial_x)$ itself doesn't belong to $R(\partial_S)$. In case where for every $f \in R(\partial_S)$ and $JfA = JE(\partial_x)$ the expression $E(\partial_x)$ belongs to $R(\partial_S)$ the map $Jf \to JfA$ determines an element of $\mathrm{End}(R(\partial_S))$ and the elements $A \in \mathrm{End}(R(S))$ with this property determine a subalgebra $FE(S)$ of $\mathrm{End}(R(S))$ called the endomorphisms of finite type. Note that this algebra is well-defined for any set S and in case S is finite or countable, any element of $\mathrm{End}(R(S))$ may be approximated by elements of $FE(S)$.

Next let $P(x) \in R(S)$; then we may define $\overline{P}(x)$ like for Clifford polynomials and construct the element $\overline{P}(\partial_x)$ by replacing vector variables by vector derivatives.

The Fischer inner product (P,Q) on $R(S)$ may now be defined by

$$(P(x),Q(x))\,J = J\,\overline{P}(\partial_x)\,Q(x)\,J\ .$$

Moreover, the conjugate A^+ of $A \in FE(S)$ is given by

$$(AP(x),Q(x)) = (P(x),A^+Q(x))$$

and it is readily seen that e.g. $(P(x)J)^+ = J\,\overline{P}(\partial_x)$. Using the Clifford polynomial representation it is readily seen that the conjugation $A \to A^+$ is the restriction of the conjugation for operators of $\mathrm{End}(P\{x_{jk}\} \otimes \mathbb{R}_m)$. Hereby the Fischer inner product in $P\{x_{jk}\} \otimes \mathbb{R}_m$ is defined as usual by $(P,Q) = \overline{P}(\partial_x)\,Q(x)|_{x=0}$ and restricts to the above one defined for $R(S)$. In particular we have that

$$e_j^+ = -e_j\,, \qquad e_j|^+ = e_j|\,, \qquad x_{jk}^+ = \partial_{x_{jk}}$$

and therefore, due to the Clifford polynomial representation, for $\mathrm{End}(R(S))$ we have the relations

$$x^+ = \partial_x, \quad x|^+ = -\partial_x|, \quad X^+ = \partial_X, \quad X'^+ = \partial_{X'}\,, \qquad x \in S\ .$$

We hence immediately have the dual relations

$$J\partial_x = -J\partial_x| = J\partial_{X'}\,, \qquad J\partial_X = 0\ ,$$

$$J\partial_{x_1} \wedge \dots \wedge \partial_{x_l} = J\partial_{X_1'}\dots\partial_{X_l'}\ ,$$

$$J\partial_{X_l'}\dots.\partial_{X_1'}\partial_{X'} = 1/2\sum(-1)^k\,J\partial_{X_l'}\dots\partial_{X_k'}\dots\partial_{X_1'}\{\partial_x,\partial_{x_k}\}\ ,$$

$$J\partial_{X_l'}\dots\partial_{X_1'}P^+X = J\partial_{X_l'}\dots\partial_{X_1'}\sum\partial_{X_k'}P_k^+\ ,$$

$$J\partial_{X_l'}\dots\partial_{X_1'}P^+X' = J\partial_{X_l'}\dots\partial_{X_1'}\Big(\sum\partial_{X_k}P_k^+ + X'P^+\Big)\ ,$$

whereby P, P_k are scalars in $R(S)$ and whereby $J\partial_{X_l'}\dots\partial_{X_1'}X = 0$ if X is not among the variables $X_1,\dots,X_l$ while otherwise

$$J\,\partial_{X_l'}\dots\partial_{X_1'}X_j = (-1)^j\,(l-1-m)\,J\,\partial_{X_l'}\dots\partial_{X_j'}\dots\partial_{X_1'}\ .$$

Any element $A \in FE(S)$ can in fact act from two sides, namely from the left on elements from the left ideal $R(S)J$ and from the right on elements from the right ideal $JR(\partial_S)$. Moreover, in view of the identity $(AP(x)J)^+ = J\overline{P}(\partial_x)A^+$ these two actions are similar and the calculus is symmetrical. It is not at all hard to formulate the following representation for endomorphisms in $FE(S)$, the proof of which is left as an exercise to the reader.

Theorem 4.2. *Every operator $A \in FE(S)$ formally be written as an infinite linear combination of basic operators of the form*

$$x_{a_1} ... x_{a_l} \, J \, \partial_{x_{b_1}} ... \partial_{x_{b_k}} \,, \qquad x_{a_j}, x_{b_s} \in S \ .$$

Conclusion. There are two ways of dealing with radial algebra: either directly or using a Clifford polynomial representation. The direct approach is purely symbolic but usually more complicated than the one using Clifford numbers. Nevertheless radial algebra is very simple to define axiomatically. From a mathematical point of view it is quite natural and perhaps also useful for physical sciences. It may seem that radial algebra is only a restriction of Clifford polynomial algebra, which is also the case. But in radial algebra things are independent of dimension and quadratic form. Moreover, Clifford polynomial algebra can also be seen as a special case of radial algebra although it is simpler to work with the Clifford basis e_j, $e_j|$ of $\boldsymbol{R}_{m,m}$ and to make use of scalar coordinates x_{jk} instead of the more formal vector variables $x \in S$. In fact working with the left ideal $R(S)J$ makes everything still more formal than working with $R(S)$ itself because one can express things in terms of the affine variables $X, X', x \in S$, and forget the variables $x \in S$ altogether. The use of the variables $x \in S$ indeed has to do with breaking the affine symmetry (see also [4], [7]).

REFERENCES

[1] Delanghe, R., Sommen, F. and Souček, V. – *Clifford algebra and Spinor-valued functions: a function theory for the Dirac-operator*, Mathematics and Its Applications 53, Kluwer Academic Publishers, Dordrecht, 1992.

[2] Gilbert, J. and Murray, M. – *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge University Press, Cambridge, 1990.

[3] Sommen, F. and Van Acker, N. – Monogenic differential operators, *Results in Mathematics,* 22 (1992), 781–798.

[4] Sommen, F. and Van Acker, N. – $SO(m)$-invariant differential operators on Clifford algebra-valued functions, *Foundations of Physics,* 23(11) (1993), 1491–1519.

[5] Sommen, F. – *Clifford tensor calculus*, to appear in Proceedings D.G.M. Conference in Ixtapa, Mexico, 1993.

[6] Hestenes, D. and Sobczyk, G. – *Clifford algebra to geometric calculus*, D. Reidel, Dordrecht, 1985.

[7] Doran, C., Hestenes, D., Sommen, F. and Van Acker, N. – Lie groups as Spin groups, *J. Math. Physics,* 34(8) (1993), 3642–3669.

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