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A REMARK ON PARABOLIC EQUATIONS

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Abstract: If $L = L^*$ is a seif-adjoint linear operator generating a strongly continuous semi-group on a real Hilbert space H and $\alpha \in L^\infty(\mathbb{R}^+)$, any mild solution u of $u' = Lu + \alpha(t)\,u$ satisfies $(u(0), u(t)) \geq 0$ for all $t \geq 0$. On the other hand for any $\lambda > 0$ such that $(\pi/L)^2 < \lambda < 4(\pi/L)^2$, there are solutions u of the one-dimensional semilinear heat equation $u_t - u_{xx} + u^3 - \lambda u = 0$ in $\mathbb{R}^+ \times (0, L)$, $u(t, 0) = u(t, L) = 0$ on $\mathbb{R}^+$ such that $\int_\Omega u(0, x)\, u(t, x)\, dx < 0$ for some $t > 0$.

Résumé: Si $L = L^*$ est un opérateur auto-adjoint, generateur d'un semi-groupe fortement continu sur un espace de Hilbert réel H et $\alpha \in L^\infty(\mathbb{R}^+)$, toute solution u de $u' = Lu + \alpha(t)\,u$ satisfait $(u(0), u(t)) \geq 0$ pour tout $t \geq 0$. D'autre part pour tout $\lambda > 0$ tel que $(\pi/L)^2 < \lambda < 4(\pi/L)^2$, il existe des solutions u de l'équation de la chaleur à une dimension $u_t - u_{xx} + u^3 - \lambda u = 0$ dans $\mathbb{R}^+ \times (0, L)$, $u(t, 0) = u(t, L) = 0$ sur $\mathbb{R}^+$ telles que $\int_\Omega u(0, x)\, u(t, x)\, dx < 0$ pour un certain $t > 0$.

1 – A simple positivity property

Let Ω be a bounded, open subset of $\mathbb{R}^N$ with a Lipschitz continuous boundary and let us consider the linear parabolic equation

$$(1.1) \qquad u_t - \Delta u + a(t, x)\, u = 0 \ \text{ in } \mathbb{R}^+ \times \Omega\,, \qquad u = 0 \ \text{ on } \mathbb{R}^+ \times \partial\Omega\,,$$

where $a \in L^\infty(\mathbb{R}^+ \times \Omega)$. For any $u_0 \in L^\infty(\Omega)$, there is a unique global solution

$$u \in C\Big([0, \infty); L^\infty(\Omega)\Big) \cap C\Big((0, \infty); H_0^1(\Omega)\Big)$$

of (1.1) with initial datum $u(0, x) = u_0(x)$. It is well-known that (1.1) is positivity preserving in the sense that if $u_0 \geq 0$, then $u(t, x) \geq 0$ a.e. on $\mathbb{R}^+ \times \Omega$. For more

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general initial data, when $a = 0$ we know that the inner product $(u_0, u(t,\cdot))$ in the sense of $L^2(\Omega)$ is nonnegative (in fact, even positive if $u_0 \neq 0$) since the heat semi-group is the exponential of a self-adjoint operator. More generally we have the following

Proposition 1.1. *Let $L = L^*$ be a self-adjoint linear operator on a real Hilbert space H, generating a strongly continuous semi-group on H and $\alpha \in L^\infty(\mathbb{R}^+)$. Then for any $u_0 \in H$, the unique mild solution $u \in C([0,\infty); H)$ of*

$$u' = Lu + \alpha(t)\, u \tag{1.2}$$

such that $u(0) = u_0$ is such that $(u_0, u(t)) \geq 0$ for all $t \geq 0$.

Proof: Denoting by $A(t)$ the primitive of $\alpha(t)$ which vanishes at 0 we have

$$u(t) = \exp(A(t)) \exp(tL)\, u_0 \quad \text{ for all } \; t \geq 0\ .$$

The result follows immediately since $\exp(tL) = \exp[(t/2)L]\{\exp[(t/2)L]\}^* \geq 0$. ∎

Corollary 1.2. *If $a(t,x) = a_1(t) + a_2(x)$ with $a_1 \in L^\infty(\mathbb{R}^+)$ and $a_2 \in L^\infty(\Omega)$, then for any $u_0 \in L^\infty(\Omega)$, the unique global solution u of (1.1) with initial datum $u(0,x) = u_0(x)$ is such that $u(0) = u_0$ is such that $(u_0, u(t,\cdot))_H \geq 0$ for all $t \geq 0$, where $(\ ,\ )_H$ denotes the inner product in $H = L^2(\Omega)$.*

Proof: Just apply Proposition 1.2 with $L = \Delta - a_2(x) I$ with Dirichlet boundary conditions. ∎

2 – A counterexample

In the investigation of uniqueness of anti-periodic solutions to semi-linear parabolic equations (cf. e.g. [2, 5, 7, 8]) the question naturally arises of whether an equation such as (1.1) can have a non-trivial solution u with $u(\tau,\cdot) = -u(0,\cdot)$ for some $\tau > 0$. Such a possibilty would be excluded if we knew that Corollary 1.2 is valid for *any* potential $a \in L^\infty(\mathbb{R}^+ \times \Omega)$. As we shall see now, it is *not* the case. Consider the one-dimensional semilinear heat equation

$$u_t - u_{xx} + cu^3 - \lambda u = 0 \ \text{ in } \mathbb{R}^+ \times (0,L)\,, \qquad u(t,0) = u(t,L) = 0 \ \text{ on } \mathbb{R}^+\ , \tag{2.1}$$

with $c > 0$, $\lambda > 0$. All solutions of this problem are global and uniformly bounded on $\mathbb{R}^+ \times (0,L)$. For $(\pi/L)^2 = \lambda_1(0,L) < \lambda < \lambda_2(0,L) = 4(\pi/L)^2$, the stationary "elliptic problem"

$$\varphi \in H_0^1(0,L)\,, \qquad -\varphi_{xx} + c\,\varphi^3 - \lambda\,\varphi = 0 \tag{2.2}$$

has exactly 3 solutions, namely 0, the positive solution φ and the negative solution $(-\varphi)$. Setting $\Omega = (0, L)$, we shall establish

Theorem 2.1. *For any $c > 0$ and $\lambda_1 < \lambda < \lambda_2$, there is $u_0 \in L^\infty(\Omega)$ such that the unique global solution u of (2.1) with initial datum $u(0,x) = u_0(x)$ satisfies*

$$\int_\Omega u_0(x)\, u(t,x)\, dx < 0 \tag{2.3}$$

for some $t > 0$.

Proof: We proceed by contradiction. Assume, instead of (2.3), that for *all* $u_0 \in L^\infty(\Omega)$ we have

$$\forall\, t \geq 0\,, \qquad \int_\Omega u_0(x)\, u(t,x)\, dx \geq 0\ . \tag{2.4}$$

Since any solution u of (2.1) is well-known (cf. e.g. [4]) to converge at infinity to one of the 3 solutions of (2.2), let us investigate first what happens if $u(t,\cdot)$ converges to φ as $t \to \infty$. From (2.4) we deduce immediately, by passing to the limit

$$\int_\Omega u_0(x)\, \varphi(x)\, dx \geq 0\ . \tag{2.5}$$

At this stage, changing if necessary u to $(-u)$, we have obtained the following properties:

– If $u(t,\cdot)$ converges to φ as $t \to \infty$, then $\int_\Omega u_0(x)\, \varphi(x)\, dx \geq 0$.

– Similarly if $u(t,\cdot)$ converges to $(-\varphi)$ as $t \to \infty$, then $\int_\Omega u_0(x)\, \varphi(x)\, dx \leq 0$.

To derive a contradiction, we shall prove the following

Lemma 2.2. *Assuming $\int_\Omega u_0(x)\, \varphi(x)\, dx > 0$, we have $u(t) \to \varphi$ as $t \to \infty$, and*

$$\forall\, t \geq 0\,, \qquad \int_\Omega u(t,x)\, \varphi(x)\, dx \geq 0\ . \tag{2.6}$$

Proof: Since by the previous results $u(t)$ cannot tend to $(-\varphi)$ as $t \to \infty$, we must have either $u(t) \to 0$ or $u(t) \to \varphi$ as $t \to \infty$. Now let u_ε be the solution of equation (2.1) such that $u_\varepsilon(0) = u_0 - \varepsilon\varphi$ with $\varepsilon > 0$. For $\varepsilon > 0$ small enough, we have $\int_\Omega (u_0(x) - \varepsilon\varphi(x))\, \varphi(x)\, dx > 0$, and the solution u_ε of equation (2.1) such that $u_\varepsilon(0) = u_0 - \varepsilon\varphi$ also tends either to 0 or φ at infinity while $w := u - u_\varepsilon \geq 0$.

Now if $u(t) \to 0$ as $t \to \infty$, we also must have $u_\varepsilon(t)$ as $t \to \infty$, both convergences being uniform on $[0, L]$. Since $\lambda > \lambda_1(0, L) = (\pi/L)^2$, an immediate calculation now shows that, as a consequence of the equation

$$w_t - w_{xx} + c(u^2 + u\, u_\varepsilon + u_\varepsilon^2)\, w = \lambda w \quad \text{in } \mathbb{R}^+ \times (0, L)\,, \quad w(t, 0) = w(t, L) = 0 \quad \text{on } \mathbb{R}^+$$

there exists $T > 0$ and $\eta > 0$ for which

$$\forall\, t \geq T\,, \qquad \frac{d}{dt} \int_\Omega w(t, x)\, \psi(x)\, dx \geq \eta \int_\Omega w(t, x)\, \psi(x)\, dx$$

with $\psi(x) := \sin(\pi/L)\, x$ on $[0, L]$. Of course this implies that either $w = 0$ for $t \geq T$, excluded by backward uniqueness (cf. e.g. [1, 3]) or w is unbounded as $t \to \infty$, a contradiction. Consequently we must have $u(t) \to \varphi$ as $t \to \infty$. Then (2.6) follows from the fact that for each $\tau > 0$, $v(t, \cdot) = u(t + \tau, \cdot)$ is a solution of (2.1) with $v(t) \to \varphi$ as $t \to \infty$. ∎

Proof of Theorem 2.1 (continued): We now turn our attention to those initial data u_0 *orthogonal* to φ in H, which means

$$\int_\Omega u_0(x)\, \varphi(x)\, dx = 0\;. \tag{2.7}$$

Considering v_ε be the solution of equation (2.1) such that $v_\varepsilon(0) = u_0 + \varepsilon\varphi$ with $\varepsilon > 0$, we remark that as $\varepsilon \to 0$, $v_\varepsilon(t, \cdot)$ converges to $u(t, \cdot)$ uniformly for each $t \geq 0$ fixed. By Lemma 2.2 we have

$$\forall\, t \geq 0\,, \qquad \int_\Omega v_\varepsilon(t, x)\, \varphi(x)\, dx \geq 0$$

and by letting $\varepsilon \to 0$, we deduce:

$$\forall\, t \geq 0\,, \qquad \int_\Omega u(t, x)\, \varphi(x)\, dx \geq 0\;.$$

Changing u_0 to $(-u_0)$, from (2.7) we also deduce

$$\forall\, t \geq 0\,, \qquad \int_\Omega u(t, x)\, \varphi(x)\, dx \leq 0\;.$$

Hence finally (2.7) *implies*

$$\forall\, t \geq 0\,, \qquad \int_\Omega u(t, x)\, \varphi(x)\, dx = 0\;. \tag{2.8}$$

The fact that (2.7) implies (2.8) is contradictory with direct properties of (2.1). Since $\varphi(x)$ is not constant, there is $h(x) \in L^2(\Omega)$ such that, for instance

$$\int_\Omega h(x)\,\varphi(x)\,dx = 0\,, \qquad \int_\Omega h(x)\,\varphi^3(x)\,dx > 0\ . \tag{2.9}$$

Let $h_n(x)$ be a sequence of C^∞ functions with compact support converging to h in $L^2(\Omega)$. For n large we have

$$\frac{\displaystyle\int_\Omega h_n(x)\,\varphi(x)\,dx}{\displaystyle\int_\Omega \varphi^2(x)\,dx} = c_n \to 0$$

while $\int_\Omega (h_n(x) - c_n\,\varphi(x))\,\varphi(x)\,dx = 0$ and $\int_\Omega (h_n(x) - c_n\,\varphi(x))\,\varphi^3(x)\,dx > 0$ for n large. Therefore we can find $h(x) \in L^\infty(\Omega)$ (and even a C^∞ function) satisfying (2.9). Picking $u_0 = \alpha h$ with $\alpha > 0$ small enough, we now find

$$\int_\Omega u_0(x)\,\varphi(x)\,dx = 0\,, \qquad \int_\Omega u_0(x)\,\varphi(x)\,\Big(\varphi^2(x) - u_0^2(x)\Big)\,dx > 0\ . \tag{2.10}$$

On the other hand for $t > 0$ we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(t,x)\,\varphi(x)\,dx &= \int_\Omega u_t(t,x)\,\varphi(x)\,dx = \int_\Omega \{u_{xx} - u^3 + \lambda u)\,\varphi\,dx \\ &= \int_\Omega \{\varphi_{xx} + \lambda\varphi)\,u\,dx - \int_\Omega u^3\,\varphi\,dx = \int_\Omega u\,\varphi(\varphi^2 - u^2)\,dx\ .\end{aligned}$$

By considering *small* values of t, we see that $\int_\Omega u(t,x)\,\varphi(x)\,dx$ is increasing on a small time interval $[t', t'']$. Since $\int_\Omega u_0(x)\,\varphi(x)\,dx = 0$, this contradicts (2.8). The proof of Theorem 2.1 is now complete. ∎

Corollary 2.3. *The conclusion of Corollary 1.2 is not valid for a general potential* $a \in (\mathbb{R}^+ \times \Omega)$.

Proof: We choose $\Omega = (0, L)$, $u_0 \in L^\infty(\Omega)$ such that the unique global solution u of (2.1) with initial datum $u(0,x) = u_0(x)$ satisfies (2.3), and $a(t,x) := cu^2 - \lambda$. ∎

REFERENCES

[1] Bardos, C. and Tartar, L. – Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines, *Arch. Rat. Mech. Anal.,* 59 (1973), 10–25.

[2] Beaulieu, A. – Etude de l'unicité des solutions antipériodiques pour des équations d'évolution non linéaires, *Portugaliae Mathematica,* 49(4) (1992).

[3] Ghidaglia, J.M. – Some backward uniqueness results, *J. Nonlinear Anal*, T.M.A.
[4] Haraux, A. – *Systèmes dynamiques dissipatifs et applications*, R.M.A. nº 17, Collection dirigée par P.G. Ciarlet et J.L. Lions, Masson, Paris, 1991.
[5] Haraux, A. – Anti-periodic solutions of some nonlinear evolution equations, *Manuscripta Math.,* 63 (1989), 479–505.
[6] Haraux, A. and Kirane, M. – Estimations C^1 pour des problèmes paraboliques semilinéaires, *Ann. Fac. Sci. Toulouse,* 5 (1983), 265–280.
[7] Okochi, H. – On the existence of periodic solutions to some nonlinear abstract parabolic equations, *J. Math. Soc. Japan,* 40 (1988), 451–553.
[8] Souplet, P. – Une condition optimale d'unicité pour les solutions antipériodiques d'équations d'évolution paraboliques, *CRAS Série I,* t.319, 1037–1041.

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