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ON AN ANALOGUE OF COMPLETELY MULTIPLICATIVE FUNCTIONS

P. Haukkanen and P. Ruokonen

Abstract: We introduce exponentially A-multiplicative functions which serve as an analogue of completely multiplicative functions in the setting of exponential A-convolution, where A is Narkiewicz's regular convolution. We show that exponentially A-multiplicative functions under exponential A-convolution possess properties similar to the familiar properties of completely multiplicative functions under the Dirichlet convolution.

1 – Introduction

The Dirichlet convolution of two arithmetical functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)\, g(n/d) \ .$$

There is a large number of analogues and generalizations of the Dirichlet convolution in the literature (for general accounts, see e.g. [7], [12], [14]). One such analogue is the exponential A-convolution. In 1972, Subbarao [14] defined the exponential convolution by

$$(1.1)\quad \begin{cases} (f \odot g)(1) = f(1)\, g(1) \ , \\ (f \odot g)(n) = \displaystyle\sum_{d_1|n_1} \sum_{d_2|n_2} \cdots \sum_{d_u|n_u} f(p_1^{d_1}\, p_2^{d_2} \cdots p_u^{d_u})\, g(p_1^{n_1/d_1}\, p_2^{n_2/d_2} \cdots p_u^{n_u/d_u}) \ , \end{cases}$$

where $n\,(>1)$ has the canonical factorization

$$(1.2)\qquad n = p_1^{n_1}\, p_2^{n_2} \cdots p_u^{n_u} \ , \qquad n_1, n_2, ..., n_u > 0 \ .$$

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The exponential A-convolution was introduced in 1977 independently by Hanumanthachari [6] and Shindo [10] (see also [14, §8]). It is obtained by generalizing the exponential convolution using the idea of Narkiewicz's [8] A-convolution (for definition, see Section 2).

An arithmetical function f is said to be *multiplicative* if $f(1) = 1$ and

$$f(mn) = f(m)\,f(n)\ , \tag{1.3}$$

whenever $(m, n) = 1$. There are several analogues and generalizations of multiplicativity in the literature, see e.g. [7], [12], [14]. An analogue closely related to exponential convolution is the concept of exponentially multiplicative functions. Let $\gamma(n)$ be the product of distinct prime divisors of n with $\gamma(1) = 1$. We define a multiplicative function f to be *exponentially multiplicative* if $f(\gamma(n)) \neq 0$ for each positive integer n and

$$f(p^n) = f(p^d)\,f(p^{n/d}) \tag{1.4}$$

for all primes p and all positive integers d and n with $d|n$ and $(d, n/d) = 1$. It is easy to see that if f is exponentially multiplicative, then $f(\gamma(n)) = 1$ for each positive integer n.

A multiplicative function is said to be *completely multiplicative* if (1.3) holds for all m and n. It is well known that completely multiplicative functions possess a large number of properties with respect to the Dirichlet convolution (see [1], [2, Chapter 2], [3], [7, Chapter 1], [11], [12]).

Yocom [15] defined A-multiplicative functions as an analogue of completely multiplicative functions in the setting of Narkiewicz's A-convolution and showed that A-multiplicative functions under A-convolution share many properties with completely multiplicative functions under Dirichlet convolution.

In this paper we define an analogue of completely multiplicative functions in the setting of exponential A-convolution. We refer to these functions as exponentially A-multiplicative functions (for definition, see Section 3). We derive a large number of properties for exponentially A-multiplicative functions. These properties are similar in character to the properties of completely multiplicative functions. In Section 4 we present characterizations of exponentially A-multiplicative functions which may be referred to as characterizations of Apostol [1] type. In Section 5 we introduce an analogue of Rearick's [9] logarithm transformation and give a characterization of exponentially A-multiplicative functions involving this transformation (for a characterization in the classical case, see Carroll [5]). In Section 6 we apply the logarithm transformation in finding solutions for the functional equations $f \circ f \circ \cdots \circ f = g$ and $f \circ f \circ \cdots \circ f = fg$, where $\circ$ is the exponential

A-convolution. Note that Hanumanthachari [6] studied the functional equation $f \circ f \circ \cdots \circ f = g$. This type of functional equation has also been studied under other convolutions in the literature (see e.g. [4], [13]).

We assume that the reader is familiar with the elements of arithmetical functions. General background material on arithmetical functions can be found in most texts on number theory and more specialized material in the books by Apostol [2], McCarthy [7] and Sivaramakrishnan [12].

2 – Exponential A-convolution

Let A be a mapping from the set $\mathbb{N}$ of positive integers to the set of subsets of $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $A(n)$ is a subset of the set of the positive divisors of n. Then *the A-convolution* of two arithmetical functions f and g is defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)\, g(n/d) \ . \tag{2.1}$$

Narkiewicz [8] defined an A-convolution to be *regular* if

a) the set of arithmetical functions forms a commutative ring with respect to the ordinary addition and the A-convolution,

b) the A-convolution of multiplicative functions is multiplicative,

c) the function E, where $E(n) = 1$ for all n, has an inverse μ_A with respect to the A-convolution, and $\mu_A(n) = 0$ or -1 whenever n is a prime power.

It can be proved (cf. [8]) that an A-convolution is regular if, and only if,

i) $A(mn) = \{de\colon\ d \in A(m),\ e \in A(n)\}$ whenever $(m,n) = 1$,

ii) for each prime power $p^a > 1$ there exists a divisor t of a such that

$$A(p^a) = \{1, p^t, p^{2t}, ..., p^{rt}\} \ ,$$

where $rt = a$, and

$$A(p^{it}) = \{1, p^t, p^{2t}, ..., p^{it}\} \,, \qquad 0 \le i \le r \ .$$

The divisor t of a is said to be the A-type of p^a, and it is denoted by $t = t_A(p^a)$.

For example, if $D(n)$ is the set of all positive divisors of n and $U(n)$ is the set of unitary divisors of n (that is, $U(n) = \{d > 0\colon\ d|n,\ (d, n/d) = 1\}$), then the *$D$-convolution* and the *$U$-convolution* are regular. Note that D-convolution and U-convolution are the classical Dirichlet convolution and the unitary convolution (see e.g. [7, Chapter 4], [12, Section I.2]), respectively.

We assume throughout this paper that A-convolution is an arbitrary but fixed regular A-convolution.

A positive integer n is said to be A-primitive if $A(n) = \{1, n\}$. The generalized Möbius function μ_A is the multiplicative function given by

$$\mu_A(p^a) = \begin{cases} -1 & \text{if } p^a\,(>1) \text{ is } A\text{-primitive,} \\ 0 & \text{if } p^a \text{ is non-}A\text{-primitive .} \end{cases}$$

The A-analogue $\tau_A(n)$ of the divisor function is defined as the number of A-divisors of n, that is $\tau_A = E *_A E$. The A-analogue $\varphi_A(n)$ of Euler's function is given by $\varphi_A(n) = (N *_A \mu_A)(n)$, where $N(n) = n$ for all n. For further information on Narkiewicz's A-convolution we refer to [7, Chapter 4] and [8].

The exponential A-convolution of two arithmetical functions f and g is defined by

$$\text{(2.2)} \qquad \begin{cases} (f \circ g)(1) = f(1)\, g(1) \ , \\ (f \circ g)(n) = \displaystyle\sum_{d_1 \in A(n_1)} \cdots \sum_{d_u \in A(n_u)} f(p_1^{d_1} \cdots p_u^{d_u})\, g(p_1^{n_1/d_1} \cdots p_u^{n_u/d_u}) \ , \end{cases}$$

where $n\,(>1)$ has the canonical factorization (1.2). This definition is due to Hanumanthachari [6] and Shindo [10] (see also [14, §8]). In the sense of the notation $*_A$ it would be natural to denote the exponential A-convolution by the symbol $\odot_A$. We, however, adopt the brief notation $\circ$ by Hanumanthachari [6].

It is known [6] that the set of all arithmetical functions forms a commutative semigroup under the exponential A-convolution with identity $|\mu|$ where μ is the Möbius function. Units are functions f for which $f(\gamma(n)) \neq 0$ for each positive integer n. The condition $f(\gamma(n)) \neq 0$ for each positive integer n means that $f(1) \neq 0$ and $f(p_1 p_2 \cdots p_u) \neq 0$ whenever $p_1, p_2, ..., p_u$ are distinct primes. The inverse of f under the exponential A-convolution is denoted by f^{-1}, that is, $f \circ f^{-1} = f^{-1} \circ f = |\mu|$.

Further, it can be verified that the set of all exponentially multiplicative functions forms an Abelian group under the exponential A-convolution (cf. [6], Lemmas 1.1–1.3). For example, the inverse of the function E is given by $E^{-1} = \mu_A^{(e)}$, where $\mu_A^{(e)}$ is the exponential analogue of the Möbius function defined by

$$\text{(2.3)} \qquad \begin{cases} \mu_A^{(e)}(1) = 1 \ , \\ \mu_A^{(e)}(n) = \mu_A(n_1) \cdots \mu_A(n_u) \ , \end{cases}$$

where $n\,(>1)$ has the canonical factorization (1.2). Clearly, $\mu_A^{(e)}$ is a multiplicative function and also exponentially multiplicative.

3 – Exponentially A-multiplicative functions

Definition. An arithmetical function f is said to be *exponentially A-multiplicative* if $f(\gamma(n)) \neq 0$ for each positive integer n and (1.4) holds for all primes p and positive integers d and n with $d \in A(n)$.

Note that exponentially U-multiplicative functions are exponentially multiplicative functions, and exponentially D-multiplicative functions are functions for which (1.4) holds for all d and n with $d|n$. Exponentially A-multiplicative functions have not previously been defined in the literature.

The definition of exponentially multiplicative functions (or exponentially U-multiplicative functions) is attributable to Subbarao [14]. It should be noted that Subbarao does not require that each exponentially multiplicative function f possess the property $f(\gamma(n)) \neq 0$ for all positive integers n. Thus the exponentially multiplicative functions of this paper are unit exponentially multiplicative functions of Subbarao. However, since we use the inverse under the exponential A-convolution in our characterizations of exponentially A-multiplicative functions and transformations of arithmetical functions, it is practical to assume that exponentially multiplicative functions are units. Further, multiplicative-like functions are often defined to be units under the related convolution in the literature.

We are now in a position to present the properties of exponentially A-multiplicative functions mentioned in the introduction.

4 – Characterizations of the Apostol type

In this section we derive characterizations of exponentially A-multiplicative functions (see Theorems 4.1–4.11 below). Most of these characterizations are similar to those given for the classical completely multiplicative functions by Apostol and others (see [1], [2], [3], [7, Chapter 1], [11]) and two characterizations (Theorems 4.7 and 4.8) are similar to those introduced by Yocom [15] for A-multiplicative functions. All these characterizations (except for Theorem 4.1) are related to the exponential A-convolution. In the proofs of the characterizations we use the following observation. If f is an exponentially multiplicative function, then

$$f(p^n) = f(p^{q_1^{n_1} \cdots q_u^{n_u}}) = \prod_{i=1}^{u} f(p^{q_i^{n_i}}) \ .$$

Thus it is enough to consider f at p^{q^n}, where p and q are primes and n a positive integer. For the sake of brevity we omit the proofs of Theorems 4.1–4.7 and 4.10.

Theorem 4.1. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$f(p^{q^n}) = f(p^{q^t})^{n/t} \tag{4.1}$$

for all primes p and q and positive integers n, where $t = t_A(q^n)$.

Theorem 4.2. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$f^{-1} = \mu_A^{(e)} f \ , \tag{4.2}$$

where $\mu_A^{(e)}$ is as given in (2.3).

Theorem 4.3. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$f^{-1}(p^{q^n}) = 0 \tag{4.3}$$

for all primes p and all non-A-primitives q^n.

Theorem 4.4. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$f(g \circ h) = fg \circ fh \tag{4.4}$$

for all arithmetical functions g and h.

Theorem 4.5. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$(fg)^{-1} = fg^{-1} \tag{4.5}$$

for all unit functions g.

Theorem 4.6. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$f\,\tau_A^{(e)} = f \circ f \ , \tag{4.6}$$

where $\tau_A^{(e)}$ is the multiplicative function defined by $\tau_A^{(e)}(p^n) = \tau_A(n)$ for all prime powers p^n (> 1).

Theorem 4.7. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$f(g \circ g) = fg \circ fg \tag{4.7}$$

for some exponentially A-multiplicative function g which is never zero.

Theorem 4.8. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$f(E \circ g) = fE \circ fg \tag{4.8}$$

for some arithmetical function g which is strictly positive.

Proof: If f is exponentially A-multiplicative, then (4.8) is clear by Theorem 4.4.

Conversely, assume that (4.8) holds for some arithmetical function g which is strictly positive. We prove that (4.1) holds. We denote $A(q^n) = \{1, q^t, q^{2t}, ..., q^{rt}\}$, where $rt = n$, and proceed by induction on $\tau_A^{(e)}(p^{q^n})$, the number of elements in $A(q^n)$. If $\tau_A^{(e)}(p^{q^n}) = 2$, the result is clear.

Suppose that equation $f(p^{q^n}) = f(p^{q^t})^{n/t}$ holds for all $\tau_A^{(e)}(p^{q^n})$ with $2 \leq \tau_A^{(e)}(p^{q^n}) < r+1$. If $\tau_A^{(e)}(p^{q^n}) = r+1$, then by (4.8) and induction hypothesis

$$\begin{aligned} f(p^{q^{rt}}) \sum_{d \in A(q^{rt})} g(p^{q^{rt}/d}) &= \sum_{d \in A(q^{rt})} f(p^d)\, f(p^{q^{rt}/d})\, g(p^{q^{rt}/d}) \\ &= f(p^{q^{rt}}) \Big[g(p^{q^{rt}}) + g(p) \Big] + f(p^{q^t})^r \sum_{\substack{d \in A(q^{rt}) \\ d \neq 1, q^{rt}}} g(p^d) \end{aligned}$$

and thus

$$f(p^{q^{rt}}) \sum_{\substack{d \in A(q^{rt}) \\ d \neq 1, q^{rt}}} g(p^d) = f(p^{q^t})^r \sum_{\substack{d \in A(q^{rt}) \\ d \neq 1, q^{rt}}} g(p^d) \ .$$

Since g is strictly positive, the sum in above equation is always nonzero. Thus $f(p^{q^{rt}}) = f(p^t)^r$. This completes the proof. ∎

Theorem 4.9. *Suppose that g and G are two arithmetical functions such that $g = G \circ \mu_A^{(e)}$. An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$fG \circ f^{-1} = fg \tag{4.9}$$

for some G satisfying $G(p) = 1$ and $G(p^{q^n}) \neq 1$ for all prime numbers p and q and positive integers n.

Proof: If f is exponentially A-multiplicative, we get by Theorems 4.2 and 4.4 the equations

$$fG \circ f^{-1} = fG \circ f\mu_A^{(e)} = f(G \circ \mu_A^{(e)}) = fg\ .$$

Conversely, we show that if (4.9) holds, then (4.3) holds or $f^{-1}(p^{q^n}) = 0$ for all non-A-primitive q^n. Let $A(q^n) = \{1, q^t, q^{2t}, ..., q^{rt}\}$ and proceed by induction on the number of elements of $A(q^n)$. If $q^n = q^{2t}$, where q^t is A-primitive, the equation (4.9) gives

$$f^{-1}(p^{q^{2t}}) + f(p^{q^t})\, G(p^{q^t})\, f^{-1}(p^{q^t}) + f(p^{q^{2t}})\, G(p^{q^{2t}}) = f(p^{q^{2t}})\, g(p^{q^{2t}})\ .$$

Using $g = G \circ \mu_A^{(e)}$ this becomes

$$f^{-1}(p^{q^{2t}}) = G(p^{q^t}) \Big[-f(p^{q^{2t}}) - f(p^{q^t})\, f^{-1}(p^{q^t}) \Big]\ .$$

By equation $(f \circ f^{-1})(p^{q^{2t}}) = |\mu(p^{q^{2t}})| = 0$ the expression in the brackets reduces to $f^{-1}(p^{q^{2t}})$. Since $G(p^{q^t}) \neq 1$, we obtain $f^{-1}(p^{q^{2t}}) = 0$.

Suppose that $f^{-1}(p^{q^{jt}}) = 0$ for all j with $2 \leq j < r$, where q^t is A-primitive. If $q^n = q^{rt}$, then, by (4.9)

$$f(p^{q^{rt}})\, g(p^{q^{rt}}) = f^{-1}(p^{q^{rt}}) + f(p^{q^{(r-1)t}})\, G(p^{q^{(r-1)t}})\, f^{-1}(p^{q^t}) + f(p^{q^{rt}})\, G(p^{q^{rt}})\ .$$

On the other hand, we have

$$f(p^{q^{rt}})\, g(p^{q^{rt}}) = f(p^{q^{rt}}) \Big[-G(p^{q^{(r-1)t}}) + G(p^{q^{rt}}) \Big]\ .$$

Thus combining the above results we get

$$f^{-1}(p^{q^{rt}}) = G(p^{q^{(r-1)t}}) \Big[-f(p^{q^{(r-1)t}})\, f^{-1}(p^{q^t}) - f(p^{q^{rt}}) \Big]\ .$$

By equation $(f \circ f^{-1})(p^{q^{rt}}) = |\mu(p^{q^{rt}})| = 0$ the expression in the brackets reduces to $f^{-1}(p^{q^{rt}})$. Since $G(p^{q^{(r-1)t}}) \neq 1$, we obtain

$$f^{-1}(p^{q^{rt}}) = 0\ .$$

This completes the proof. ∎

Theorem 4.10. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$\sum_{d\in A(n)} f(p^d)\, f^{-1}(p^{n/d})\, d = f(p^n)\, \varphi_A^{(e)}(p^n) \;, \tag{4.10}$$

where $\varphi_A^{(e)}$ is the multiplicative function defined by $\varphi_A^{(e)}(p^n) = \varphi_A(n)$ for all prime powers p^n (> 1).

Theorem 4.11. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$f(\varphi_A^{(e)} \circ E) = f\, \varphi_A^{(e)} \circ fE \;. \tag{4.11}$$

Proof: The result is clear by Theorem 4.8, since $\varphi_A^{(e)}$ is a strictly positive arithmetical function. ∎

5 – Transformations

In this section we define Rearick's [9] logarithm transformation in the setting of exponential A-convolution and introduce its basic properties. We also give a characterization of exponentially A-multiplicative functions in terms of this transformation. The characterization is an analogue of Carroll's [5] characterization for completely multiplicative functions.

Let F_1 denote the set of arithmetical functions f such that $f(\gamma(n)) = 1$ for all positive integers n, and let F_0 denote the set of arithmetical functions f such that $f(\gamma(n)) = 0$ for all positive integers n. Let P denote the arithmetical function defined by $P(1) = 1$ and $P(n) = n_1\, n_2 \cdots n_u$ for $n\,(> 1)$ having the canonical factorization (1.2). For each arithmetical function f denote $f'(n) = f(n) \log P(n)$.

Definition. The logarithm transformation is a mapping $L\colon\ F_1 \to F_0$ defined by

$$Lf = f' \circ f^{-1} \;. \tag{5.1}$$

Remark 5.1. It is easy to verify that

$$(Lf)(n) = \begin{cases} 0 & \text{if } \gamma(n) = n, \\ (f' \circ f^{-1})(n) & \text{otherwise} \;. \end{cases}$$

Theorem 5.1. *The logarithm transformation is a one-to-one mapping of F_1 onto F_0.*

Proof: Let $g \in F_0$. We proceed inductively to define $f(n)$ uniquely such that $Lf = g$. Firstly, we define $f(\gamma(n)) = 1$ for all n. Secondly, we assume that the values $f(k)$ are defined for $k < n$ ($n \neq \gamma(n)$). The values $f^{-1}(k)$ are obtained from

$$(f \circ f^{-1})(k) = |\mu|(k) ,$$

and thus $f(n)$ can be found uniquely from

$$(f' \circ f^{-1})(n) = g(n) .$$

This completes the proof. ∎

Remark 5.2. By Theorem 5.1 we can define a transformation E of F_0 onto F_1 by

$$E(Lf) = f .$$

The transformation E may be referred to as *the exponential transformation*.

Theorem 5.2. *For $f, g \in F_1$,*

$$L(f \circ g) = Lf + Lg .$$

Proof: We have $(f \circ g)' = f' \circ g + f \circ g'$ and consequently

$$L(f \circ g) = (f' \circ g + f \circ g') \circ (f \circ g)^{-1} = f' \circ f^{-1} + g' \circ g^{-1} = Lf + Lg . \blacksquare$$

Corollary. *The groups $(F_1, \circ)$ and $(F_0, +)$ are isomorphic.*

Theorem 5.3. *If g is an exponentially A-multiplicative function, then*

$$L(fg) = g\, Lf$$

for all $f \in F_1$.

Proof: By Theorems 4.4 and 4.5,

$$L(fg) = (fg)' \circ (fg)^{-1} = f'g \circ f^{-1}g = (f' \circ f^{-1})\, g = g\, Lf . \blacksquare$$

Lemma 5.1. *A multiplicative function f is exponentially multiplicative if, and only if,*

$$(Lf)(p^a) = 0 , \tag{5.2}$$

for all primes p, whenever a is not a positive power of prime.

Proof: Assume that f is exponentially multiplicative. If $a = 1$, then (5.2) is clear. Suppose that a is not a positive prime power. Then a is of the form mn, where $m, n > 1$ and $(m, n) = 1$. Thus

$$\begin{aligned}
(Lf)(p^{mn}) &= \sum_{d \in A(mn)} f(p^d)\, f^{-1}(p^{mn/d}) \log d \\
&= \sum_{d_1 \in A(m)} \sum_{d_2 \in A(n)} f(p^{d_1 d_2})\, f^{-1}(p^{mn/d_1 d_2}) \log(d_1 d_2) \\
&= \sum_{d_1 \in A(m)} \sum_{d_2 \in A(n)} f(p^{d_1})\, f(p^{d_2})\, f^{-1}(p^{m/d_1})\, f^{-1}(p^{n/d_2}) [\log d_1 + \log d_2] \\
&= \sum_{d_1 \in A(m)} f(p^{d_1})\, f^{-1}(p^{m/d_1}) \log d_1 \sum_{d_2 \in A(n)} f(p^{d_2})\, f^{-1}(p^{n/d_2}) \\
&\quad + \sum_{d_2 \in A(n)} f(p^{d_2})\, f^{-1}(p^{n/d_2}) \log d_2 \sum_{d_1 \in A(m)} f(p^{d_1})\, f^{-1}(p^{m/d_1}) \\
&= (Lf)(p^m)\, |\mu(p^n)| + (Lf)(p^n)\, |\mu(p^m)| = 0\ .
\end{aligned}$$

Conversely, suppose that (5.2) holds. Define $g(1) = 1$ and

$$g(p_1^{a_1}\, p_2^{a_2} \cdots p_r^{a_r}) = \prod_{i=1}^{r} \prod_{q^k || a_i} f(p_i^{q^k})$$

for $n = p_1^{a_1}\, p_2^{a_2} \cdots p_r^{a_r} > 1$, where $a_1, ..., a_r > 0$. The function g is exponentially multiplicative and we shall show that $g = f$. Since g and f are multiplicative, it is enough to proof that $f = g$ at prime powers p^a.

If $a = 1$, then $g(p) = f(p) = 1$ and $(Lf)(p) = (Lg)(p) = 0$.

Suppose that $a = q^n = q^{rt} > 1$. Then $g(p^{q^n}) = \prod_{q^k || q^n} f(p^{q^k}) = f(p^{q^n})$. By the equation $f \circ f^{-1} = |\mu|$, the values of $f^{-1}(p^{q^{rt}})$ depend only on the values of $f(p^{q^{it}})$, where $0 \leq i \leq r$. Therefore also $f^{-1}(p^{q^n}) = g^{-1}(p^{q^n})$. Thus from the definition of L it follows that $(Lf)(p^{q^n}) = (Lg)(p^{q^n})$.

Suppose that a is not a positive prime power. Since g is exponentially multiplicative, the first part of this proof shows that then $(Lg)(p^a) = 0$ and by the hypothesis (5.2) also $(Lf)(p^a) = 0$. Therefore Lf and Lg agree for all prime powers, so $f = g$ at prime powers p^a by Theorem 5.1. This completes the proof. ∎

Lemma 5.2. *Let p and q be primes and n a positive integer. Suppose that f is an exponentially multiplicative function such that*

$$f(p^{q^{kt}}) = f(p^{q^t})^k \tag{5.3}$$

for all k with $2 \leq k \leq n/t$ and $t = t_A(q^n)$. Then

$$f^{-1}(p^{q^{kt}}) = 0 \tag{5.4}$$

for all k with $2 \leq k \leq n/t$ and $t = t_A(q^n)$.

Lemma 5.2 is similar to the first part of Theorem 4.2. Therefore we omit the proof.

Theorem 5.4. *An exponentially multiplicative function f is exponentially A-multiplicative if, and only if,*

$$(Lf)(p^{q^n}) = t\, f(p^{q^t})^{n/t} \log q \tag{5.5}$$

for all primes p and q and positive integers n, where $t = t_A(q^n)$.

Proof: Let $A(q^n) = \{1, q^t, q^{2t}, ..., q^{rt}\}$. Suppose that f is exponentially A-multiplicative. Thus (4.2) and (4.1) hold. Further, since $\mu_A^{(e)}(p^{q^{kt}}) \neq 0$ only if $k=0$ or $k=1$, we get

$$\begin{aligned}
(Lf)(p^{q^n}) &= \sum_{d\in A(q^n)} f(p^d)\, f(p^{q^{rt}/d})\, \mu_A^{(e)}(p^{q^{rt}/d}) \log d \\
&= f(p^{q^{rt}}) \sum_{d\in A(q^n)} \mu_A^{(e)}(p^{q^{rt}/d}) \log d \\
&= f(p^{q^t})^r \Big[-\log q^{(r-1)t} + \log q^{rt}\Big] \\
&= f(p^{q^t})^r\, t \log q \ .
\end{aligned}$$

Conversely, suppose that (5.5) holds and prove that then (4.1) holds. We proceed by induction on the number of elements of the set $A(q^n)$. For $q^n = q^{2t}$, we get by definition of L

$$\begin{aligned}
(Lf)(p^{q^{2t}}) &= \sum_{d\in A(q^{2t})} f(p^d)\, f^{-1}(p^{q^{2t}/d}) \log d \\
&= -t\, f(p^{q^t})^2 \log q + 2t\, f(p^{q^{2t}}) \log q \ .
\end{aligned}$$

By (5.5),

$$(Lf)(p^{q^{2t}}) = t\, f(p^{q^t})^2 \log q \ .$$

Combining these two results we obtain

$$2t\, f(p^{q^{2t}}) \log q = 2t\, f(p^{q^t})^2 \log q$$

and consequently

$$f(p^{q^{2t}}) = f(p^{q^t})^2 \ .$$

Now, suppose that (4.1) holds for all $q^n = q^{kt}$ with $2 \le k \le r-1$. By Lemma 5.2 $f^{-1}(p^{q^{kt}}) = 0$ for all k with $2 \le k \le r-1$. By definition of L and induction hypothesis we get

$$\begin{aligned}(Lf)(p^{q^{rt}}) &= \sum_{d\in A(q^{rt})} f(p^d)\, f^{-1}(p^{q^{rt/d}}) \log d \\ &= f(p^{q^{(r-1)t}})\, f^{-1}(p^{q^t}) \log q^{(r-1)t} + f(p^{q^{rt}}) \log q^{rt} \\ &= -(r-1)\, t\, f(p^{q^t})^r \log q + r\, t\, f(p^{q^{rt}}) \log q \ .\end{aligned}$$

By (5.5),

$$(Lf)(p^{q^{rt}}) = t\, f(p^{q^t})^r \log q \ .$$

Thus

$$r\, t\, f(p^{q^{rt}}) \log q = r\, t\, f(p^{q^t})^r \log q$$

and consequently

$$f(p^{q^{rt}}) = f(p^{q^t})^r \ .$$

Therefore (4.1) holds for $q^n = q^{rt}$. This completes the proof. ∎

6 – Applications to functional equations

In this section we apply the logarithm transformation and its basic properties to obtain solution for the functional equations $f^r = g$ and $f^r = fg$, where f^r is the r^{th} power of f under the exponential A-convolution, that is,

$$f^r = f \circ f \circ \cdots \circ f \qquad (r \text{ times}) \ .$$

Lemma 6.1. *Suppose that $f \in F_1$. Then f is exponentially multiplicative if, and only if, f^r is exponentially multiplicative.*

Theorem 6.1. *Let r be a positive integer and let $g \in F_1$. Then the equation $f^r = g$ has a unique solution $f \in F_1$ given by*

$$f = E\left(\frac{1}{r}\, Lg\right) \ . \tag{6.1}$$

The solution is exponentially multiplicative if, and only if, g is exponentially multiplicative.

Proof: By Theorem 5.2, we have $r\, Lf = Lg$. Thus, by Remark 5.2, we obtain (6.1). The result for exponentially multiplicative functions follows from Lemma 6.1. ∎

Theorem 6.2. *Let r be a positive integer, and let g be an exponentially A-multiplicative function such that $g(n) \neq r$ for all n with $n \neq \gamma(n)$. Then the functional equation $f^r = fg$ has exactly one solution in F_1, namely $f = |\mu|$.*

Proof: By Theorems 5.2 and 5.3, we have $r\,Lf = g\,Lf$. Thus $Lf \equiv 0$; hence $f = |\mu|$. This completes the proof of Theorem 6.2. ∎

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Pentti Haukkanen and Pia Ruokonen,
Department of Mathematical Sciences, University of Tampere,
P.O. Box 607, FIN-33101 Tampere – FINLAND