PORTUGALIAE MATHEMATICA
Vol. 55 Fasc. 2 – 1998

HOMOGENIZATION OF TWO RANDOMLY WEAKLY CONNECTED MATERIALS

M. Briane and L. Mazliak

Introduction

In this paper, we are interested in an homogenization problem of two disjoint ε-periodic materials $O_{1\varepsilon}$ and $O_{2\varepsilon}$ connected in each cell of size ε by a small bridge the size of which is random. We therefore extend the kind of deterministic problems first studied by Khruslov [8] and then in [3].

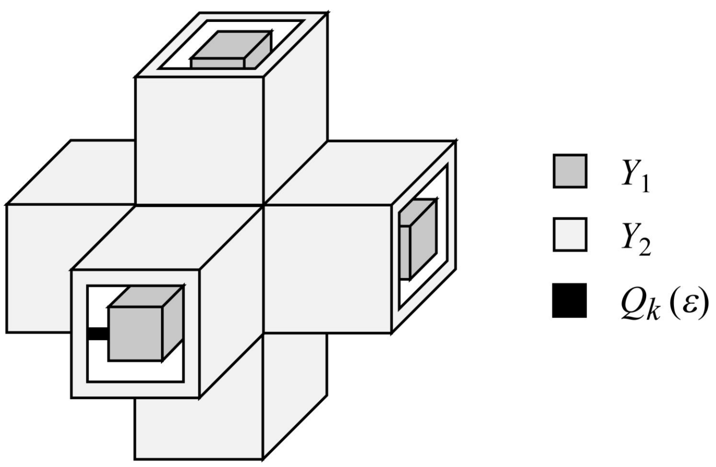

Fig. 1 – Period cell $Y_k(\varepsilon)$.

Received: October 3, 1996; *Revised*: April 3, 1997.

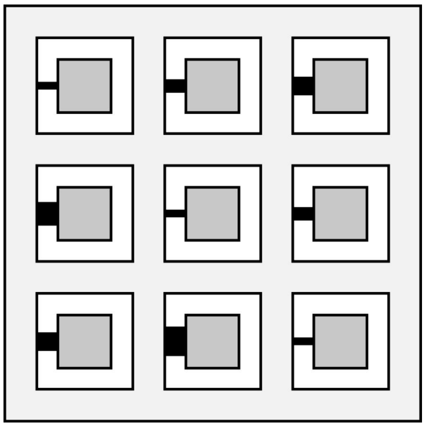

Fig. 2 – Section of the random set O_ε.

In [3], the first author studies the homogenization of the Neumann problem

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f & \text{in } O_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial O_\varepsilon \ , \end{cases}$$

where $O_\varepsilon = O_{1\varepsilon} \cup O_{2\varepsilon} \cup \mathcal{O}_\varepsilon$ where as above, in each cell of size ε, $\mathcal{O}_\varepsilon$ is a (deterministic) bridge of small size. He proves that the asymptotic behaviour depends on the size of the bridge. More precisely, it depends on whether the order ε^α of this size is such that $\alpha < 2$ (supercritical case), $\alpha = 2$ (critical case), $\alpha > 2$ (subcritical case). The proof is based on Tartar's energy method by finding good test functions in order to identify the limit problem.

Homogenization problems in a random setting have been already widely studied by: Kozlov [9], Varadhan and Papanicolaou [11], Bensoussan [2], Dal Maso and Modica [6] e.g. for general random coefficients, then by Zhikov [14] and [15] e.g. for randomly perforated domains. A survey of homogenization results in a random context can be found in the book of Jikov, Kozlov and Oleinik [7]. However, in these texts, the geometry of the system is always random and therefore cannot be specificied. The only central tool used consists in ergodic hypotheses on the coefficients in order to pass to the limit.

On the contrary, we are interested in keeping the geometrical setting (essential to solve the problem in the deterministic case), letting just the bridge size be random. The major difference with the deterministic case comes from the absence of strong convergences due to the randomness of the solutions. In particular, the imbedding of $L^2(\Omega; H^1(O))$ in $L^2(\Omega \times O)$ is not compact if Ω is not countable.

This makes it difficult for passing to the limit in the equation. This problem can be dealt with by averaging on the space variable and using the law of large numbers (see Lemma 3.4 and its proof below).

As in the deterministic case, we obtain three distinct cases following the mean size of the bridge. In particular, we prove that the system is not coupled when there are sufficiently large bridge with positive probability.

Notations

$(\Omega, \mathcal{F}, P)$ is a probability space such that the σ-algebra is countably generated. In particular, $L^p(\Omega)$, $1 \le p < \infty$ is a separable Banach space.

We denote by $E(\psi) = \displaystyle\int_\Omega \psi\, dP$, $\psi \in L^1(\Omega)$ the expectation of the random variable ψ.

O is a bounded open set in $\mathbb{R}^N$, $N \ge 3$, with a Lipschitz boundary.

$V = L^2(\Omega; H^1(O))$ is the space of random variables with values in $H^1(O)$. Taking an orthonormal basis $(e_n(x))_{n\in\mathbb{N}}$ of $H^1(O)$, V is the space of functions u that can be written as follows

$$u(\omega, x) = \sum_{n\ge 0} \hat{u}_n(\omega)\, e_n(x)$$

where $\hat{u}_n \in L^2(\Omega)$ and $\sum_{n\ge0} \|\hat{u}_n\|^2_{L^2(\Omega)} < \infty$. It is an Hilbert space, provided with the norm

$$\|u\|_V^2 = \sum_{n\ge 0} \|\hat{u}_n\|^2_{L^2(\Omega)} = E\Big(\|u(\omega,\cdot)\|^2_{H^1(O)}\Big)\ .$$

We denote also $V_{loc} = L^2(\Omega; H^1_{loc}(O))$.

$\mathcal{D}_\Omega(O)$ is the space of smooth random functions i.e. functions $\varphi(\omega,\cdot) \in \mathcal{D}(O)$ such that $\|\nabla_x^j \varphi\|_{L^\infty(\Omega\times O)} < \infty$, $\forall\, j \in \mathbb{N}$.

$L^\infty(\Omega; C^1(\overline{O}))$ is the space of smooth random functions i.e. functions $\varphi(\omega,\cdot) \in C^1(\overline{O})$ such that $\|\varphi\| + \|\nabla\varphi\|_{L^\infty(\Omega\times O)} < \infty$.

For X a given open subset of $\mathbb{R}^N$, $H^1_\#(X)$ is the space of $\mathbb{Z}^N$-periodic functions which belong to $H^1_{loc}(X)$.

Moreover, we denote by $\displaystyle⨍_Y f$ the normalized integral $\displaystyle\frac{1}{|Y|}\int_Y f$.

1 – The homogenization problem

1.1. Geometry of the problem

Let $Y = [0,1[^N$ be the unit cube of $\mathbb{R}^N$, $N \geq 3$. E_1 and E_2 are two Y-periodic open subsets of $\mathbb{R}^N$ with Lipschitz boundary, which are connected. It is also assumed that $\overline{E_1} \cap \overline{E_2} = \emptyset$. Observe that such sets exist since we assumed that $N \geq 3$.

We denote for $\varepsilon > 0$, $O_{i\varepsilon} = \varepsilon E_i \cap O$, $i = 1,2$ and $Y_i = E_i \cap Y$, $i = 1,2$.

On the probability space $(\Omega, \mathcal{F}, P)$, we consider an indexed by $k \in \mathbb{Z}^N$ family of independent and identically distributed random processes $(\alpha_k(\varepsilon))_{\varepsilon>0}$ such that $\forall\, \varepsilon > 0$, $0 < \alpha_k(\varepsilon) < a$ a.s. where a is a positive constant. In the sequel, the index k is omitted when only the distribution of α_k is used. Therefore, we shall write expressions such as $E(\alpha(\varepsilon))$.

For each $k \in \mathbb{Z}^N$, we define a cylindrical "bridge" $Q_k(\varepsilon)$ joining Y_1 and Y_2 such that $Y_k(\varepsilon) = Y_1 \cup Y_2 \cup Q_k(\varepsilon)$ is connected; the length of $Q_k(\varepsilon)$ is a positive constant $\ell > 0$ and its section area is equal to $\alpha_k(\varepsilon)$.

We set

$$\mathcal{O}_\varepsilon = \bigcup_{k \in \mathbb{Z}^N} (\varepsilon\, k + \varepsilon\, Q_k(\varepsilon)) \cap O$$

which is the union of the random bridges and

$$O_\varepsilon = O_{1\varepsilon} \cup O_{2\varepsilon} \cup \mathcal{O}_\varepsilon$$

is the (random) domain connected by bridges of random size.

1.2. Position of the problem

Our aim is to study the homogenization of the model, i.e. the asymptotic behaviour of the following Neumann problem in the random domain O_ε

$$\begin{cases} -\Delta_x u_\varepsilon + u_\varepsilon = f & \text{in } O_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial O_\varepsilon\ , \end{cases} \tag{1}$$

where f is a given (deterministic) function of $L^2(O)$.

We need to formulate the problem in a Hilbert setting in order to get a variational formulation. Since O_ε is a random set, we have to take care of measurability for solutions of (1). Instead of the "natural" space $L^2(\Omega; H^1(O_\varepsilon))$ (which is not well defined since O_ε is random), we consider a larger space. More precisely, consider $\widetilde{O}_\varepsilon$, the deterministic open set obtained by replacing in O_ε each cylinder $Q_k(\varepsilon)$ by a similar one $\widetilde{Q}_k$ but with a (deterministic) section equal to a. The Hilbert space $\widetilde{V}_\varepsilon = L^2(\Omega; H^1(\widetilde{O}_\varepsilon))$ is provided with the norm

$$\|u\|^2_{\widetilde{V}_\varepsilon} = E\Big(\|u(\omega,\cdot)\|^2_{H^1(\widetilde{O}_\varepsilon)}\Big) = \int_{\Omega\times O} \mathbf{1}_{\widetilde{O}_\varepsilon}(u^2 + |\nabla_x u|^2)\ .$$

Let us consider the space V_ε of the restrictions $u_{|O_\varepsilon}$ of functions $u \in \widetilde{V}_\varepsilon$, provided with the norm

$$\|u\|^2_{V_\varepsilon} = \int_{\Omega\times O} \mathbf{1}_{O_\varepsilon}(u^2 + |\nabla_x u|^2)\ .$$

We may then obtain the following result.

Lemma 1.1. *Assume that there exists a deterministic function $g(\varepsilon) > 0$ such that*

$$\alpha_k(\varepsilon) > g(\varepsilon)\,, \qquad \forall\, \varepsilon > 0,\ \ \forall\, k \in \mathbb{Z}^N\ . \tag{2}$$

Assume moreover that O is only composed of entire cells.
Then, $(V_\varepsilon, \|\cdot\|_{V_\varepsilon})$ is an Hilbert space.

The proof is given in Appendix.

We can now give a variational formulation of problem (1). From Lax–Milgram's theorem, there exists a unique $u_\varepsilon \in V_\varepsilon$ such that

$$\int_{\Omega\times O} \mathbf{1}_{O_\varepsilon}\, \nabla_x u_\varepsilon \cdot \nabla_x \varphi + \int_{\Omega\times O} \mathbf{1}_{O_\varepsilon}\, u_\varepsilon\, \varphi = \int_{\Omega\times O} \mathbf{1}_{O_\varepsilon}\, f\, \varphi\,, \qquad \forall\, \varphi \in V_\varepsilon\ . \tag{3}$$

2 – The results

2.1. The limiting behaviour

Our first result describes the limiting behaviour of problem (3). Of course, it is still a very imprecise result. We however emphasize that it requires very little on the random processes α_k.

Proposition 2.1. *Let $\chi_i^\lambda \in H^1_\#(E_i)$, $\lambda \in \mathbb{R}^N$ and $i = 1, 2$, be the unique (up to an additive constant) solution of*

$$\int_{Y_i} \nabla\chi_i^\lambda \cdot \nabla\varphi = \int_{Y_i} \lambda\cdot\nabla\varphi\,, \qquad \forall\, \varphi \in H^1_\#(E_i)\ , \tag{4}$$

and A_i be the positive definite matrix

$$A_i\,\lambda = \frac{1}{|Y|}\int_{Y_i}(\lambda - \nabla\chi_i^\lambda)\,, \qquad \lambda \in \mathbb{R}^N\ . \tag{5}$$

Assume that

$$\lim_{\varepsilon\to 0} E(\alpha(\varepsilon)) = 0\ . \tag{6}$$

Then, there exist two functions $u_{i\varepsilon} \in V_{loc} = L^2(\Omega; H^1_{loc}(O))$, $i = 1,2$ such that $\mathbf{1}_{O_{i\varepsilon}} u_{i\varepsilon} = \mathbf{1}_{O_\varepsilon} u_\varepsilon$ and a subsequence ε' such that $u_{i\varepsilon'} \rightharpoonup u_i$ in V_{loc} where u_i are solutions of the equation

$$\begin{aligned} \int_{\Omega\times O}(A_1\,\nabla_x u_1 + A_2\,\nabla_x u_2)\cdot\nabla_x\varphi + \int_{\Omega\times O}(\theta_1\,u_1 + \theta_2\,u_2)\,\varphi = \\ = \int_{\Omega\times O}\theta\,f\,\varphi\,, \qquad \forall\,\varphi \in V_{loc}\ , \end{aligned} \tag{7}$$

where $\theta_i = \frac{|Y_i|}{|Y|}$ and $\theta = \theta_1 + \theta_2$.

2.2. Identification of limiting behaviour parameters

To determine the homogenization of problem (3) completely, we have to find an additional equation to (7) in order to caracterize the functions u_i, $i = 1, 2$. Of course, we have to require more restrictive assumptions on the random processes $\alpha(\varepsilon)$.

Similarly to the deterministic case, we will distinguish three cases:

– the supercritical case where $\alpha(\varepsilon)$ is much larger than ε^2 with positive probability,

– the subcritical case where $\alpha(\varepsilon)$ is much smaller than ε^2,

– the critical case where $\alpha(\varepsilon)$ is of order ε^2.

The following results give a precise mathematical sense to these notions, and precise the homogenization equation (7).

Theorem 2.2. *Assume that the process $(\alpha(\varepsilon))_{\varepsilon>0}$ is such that there exists a random variable γ such that*

$$\frac{\alpha(\varepsilon)}{\varepsilon^2} \to \gamma \quad \text{strongly in}\ \ L^1(\Omega)\ . \tag{8}$$

Then, the sequence of functions $(\mathbf{1}_{O_\varepsilon} u_\varepsilon)$ *weakly converges in* $L^2(\Omega \times O)$ *to the function* $\theta_1 u_1 + \theta_2 u_2$ *where* u_1 *and* u_2 *are the (deterministic) solutions in* $H^1(O)$ *of the coupled system*

$$(9) \qquad \begin{cases} -\mathrm{div}_x(A_1 \nabla_x u_1) + \theta_1\, u_1 + \dfrac{1}{\ell}\, E(\gamma)\,(u_1 - u_2) = \theta_1\, f & \text{in } O, \\ -\mathrm{div}_x(A_2 \nabla_x u_2) + \theta_2\, u_2 + \dfrac{1}{\ell}\, E(\gamma)\,(u_2 - u_1) = \theta_2\, f & \text{in } O, \\ A_i \nabla_x u_i \cdot n = 0 & \text{in } \partial\Omega\ . \end{cases}$$

Remark 2.1.

1) Observe that hypothesis (8) implies (6).

2) The previous theorem deals with the critical case ($E(\gamma) > 0$) and the subcritical case ($E(\gamma) = 0$).

3) Of course, (8) implies that for every $k \in \mathbb{Z}^N$, there exists an independent sequence of random variables $(\gamma_k)_{k \in \mathbb{Z}^N}$ such that

$$(10) \qquad \frac{\alpha_k(\varepsilon)}{\varepsilon^2} \to \gamma_k \quad \text{strongly in} \quad L^1(\Omega)\ .$$

Theorem 2.3. *Assume that*

$$(11) \qquad P\Big(\Big\{\frac{\alpha(\varepsilon)}{\varepsilon^2} \to +\infty\Big\}\Big) > 0$$

and that there exists $\gamma_0 \in L^1(\Omega)$ *such that*

$$(12) \qquad \frac{\varepsilon^2}{\alpha(\varepsilon)}\, \mathbf{1}_{\{\frac{\alpha(\varepsilon)}{\varepsilon^2} \to \infty\}} \le \gamma_0 \quad \text{a.s.}\ .$$

Then, $u = u_1 = u_2$ *is the deterministic solution in* $H^1(O)$ *of the Neumann problem*

$$(13) \qquad \begin{cases} -\mathrm{div}_x(A_1 + A_2)\, \nabla_x u + \theta\, u = \theta\, f & \text{in } O, \\ (A_1 + A_2)\, \nabla_x u \cdot n = 0 & \text{on } \partial O\ . \end{cases}$$

Let us give two examples to illustrate these theorems.

Example 2.2: Suppose that $\alpha(\varepsilon) = a\, \varepsilon^\beta$ where β is an i.i.d. (independent and identically distributed) bounded and positive random variable. One can easily check that the subcritical case corresponds to $P(\beta > 2) = 1$, the supercritical case to $P(\beta < 2) > 0$ and the critical case to $P(\beta \geq 2) = 1$, $P(\beta = 2) > 0$.

Example 2.3: Suppose that $\alpha(\varepsilon) = \nu\, \varepsilon^{\delta_1} + (1-\nu)\, \varepsilon^{\delta_2}$ where ν is a Bernoulli random variable ($p = P(\nu = 1) = 1 - P(\nu = 0)$) and δ_1, δ_2 are two fixed positive numbers. Assume for example that $\delta_1 < 2$, $\delta_2 \geq 2$ and $p > 0$. Then (11) is satisfied and condition (12) is satisfied with $\gamma = \nu$. This kind of situation corresponds to a random mixing of two sizes of bridges ε^{δ_1} and ε^{δ_2}.

3 – Proofs of the results

3.1. Weak convergence and Extension property

We begin by stating two technical lemmas. The first one is just an observation about weak convergence.

Lemma 3.1. *Let $u_\varepsilon(\omega, x) \rightharpoonup u(\omega, x)$ weakly in $L^2(\Omega; H^1(O))_w$ and $v_\varepsilon(x) \rightharpoonup v(x)$ weakly in $L^2(O)_w$. Then*

$$\lim_{\varepsilon \to 0} \int_{\Omega \times O} u_\varepsilon\, v_\varepsilon = \int_{\Omega \times O} u\, v \ .$$

Proof: Consider $\psi \in L^2(\Omega)$. Observe that $E(\psi u_{i\varepsilon})$ is a bounded sequence in $H^1(O)$, which strongly converges in $L^2(O)$ up to a subsequence. Since $\mathbf{1}_{O_{i\varepsilon}} \rightharpoonup \theta_i$ weakly in $L^\infty(O)$ and $E(\psi u_{i\varepsilon}) \to E(\psi u_i)$ weakly in $H^1(O)$, we obtain thanks to Rellich's Theorem and up to a subsequence, $E(\psi u_{i\varepsilon}) \to E(\psi u_i)$ strongly in $L^2(O)$ and thus for the whole sequence $\mathbf{1}_{O_{i\varepsilon}} E(\psi u_{i\varepsilon}) \rightharpoonup \theta_i\, E(\psi u_i)$ weakly in $L^2(O)$. Since the tensor products $\varphi(x)\, \psi(\omega)$, $\varphi \in L^2(O)$ and $\psi \in L^2(\Omega)$, generate $L^2(\Omega \times O)$, we deduce from the latter that $\mathbf{1}_{O_{i\varepsilon}} u_{i\varepsilon} \rightharpoonup \theta_i\, u_i$ weakly in $L^2(\Omega \times O)$. ∎

We now state an extension result which will be useful in the proof of Proposition 2.1.

Lemma 3.2. *There exist extension operators $P_{i,\varepsilon}$, $i = 1, 2$, from $V_{i\varepsilon} = L^2(\Omega; H^1(O_{i\varepsilon}))$ into $V_{loc} = L^2(\Omega; H^1_{loc}(O))$ and constants $c_i > 0$ such that for any $u \in V_{i\varepsilon}$, $\mathbf{1}_{O_{i\varepsilon}} P_{i\varepsilon} u = \mathbf{1}_{O_{i\varepsilon}} u$ and*

$$\|P_{i\varepsilon} u\|_{V_{loc}} \leq c_i\, \|u\|_{V_{i\varepsilon}}\,, \qquad i = 1, 2\ , \tag{14}$$

and for any $u \in V_{i\varepsilon} \cap L^\infty(\Omega \times O_{i\varepsilon})$

$$\|P_{i\varepsilon} u\|_{L^\infty_{loc}(\Omega\times O)} \le c_i \, \|u\|_{L^\infty(\Omega\times O_{i\varepsilon})}\,, \qquad i = 1,2\ , \tag{15}$$

where the constants c_i *only depend on the open set* O*.*

Proof: E_i, $i = 1,2$, are connected open sets with Lipschitz boundary, we can then use an extension result due to Acerbi, Chiado–Piat, Dal Maso, Percivale [1]: there exist extension operators, $p_{i\varepsilon}$, $i = 1,2$, from $H^1(O_{i\varepsilon})$ into $H^1_{loc}(O)$ and constants $c_i > 0$ such that for any $u \in H^1(O_{i\varepsilon})$, $\mathbf{1}_{O_{i\varepsilon}} p_{i\varepsilon} u = \mathbf{1}_{O_{i\varepsilon}} u$ and

$$\|p_{i\varepsilon} u\|_{H^1_{loc}(O)} \le c_i \, \|u\|_{H^1(O_{i\varepsilon})}\,, \qquad i = 1,2\ , \tag{16}$$

and for any $u \in H^1(O_{i\varepsilon}) \cap L^\infty(O_{i\varepsilon})$,

$$\|u\|_{L^\infty_{loc}(O)} \le c_i \, \|u\|_{L^\infty(O_{i\varepsilon})}\,, \qquad i = 1,2\ , \tag{17}$$

where the constants c_i only depend on the set O.

Now, let $u \in V_{i\varepsilon}$, since $p_{i\varepsilon}$ is a continuous operator between the Banach spaces $H^1(O_{i\varepsilon})$ and $H^1_{loc}(O)$, one has that $\omega \mapsto p_{i\varepsilon} u(\omega,\cdot)$ is measurable and belongs to $L^1(\Omega; H^1_{loc}(O))$, from Bochner's integral theory, (see e.g. Yosida [13]) and also to V by (16).

We then define extension operators denoted by $P_{i\varepsilon}$ by $P_{i\varepsilon} u \colon \omega \mapsto p_{i\varepsilon} u(\omega,\cdot)$ which satisfy $\mathbf{1}_{O_{i\varepsilon}} P_{i\varepsilon} u = \mathbf{1}_{O_{i\varepsilon}} u$, $i = 1,2$.

Moreover, estimates (14) and (15) are direct consequences of estimates (16) and (17) respectively. ∎

3.2. Proof of Proposition 2.1

Taking function u_ε as test function in equation (3), one obtains using the boundedness of $(\|u_\varepsilon\|_{L^2})$ the following estimates $\|\mathbf{1}_{O_\varepsilon} u_\varepsilon\|_{L^2(\Omega\times O)} \le c$ and $\|\mathbf{1}_{O_\varepsilon} \nabla_x u_\varepsilon\|_{L^2(\Omega\times O)^N} \le c$ where c is a deterministic constant.

Then, there exist a subsequence, denoted ε for simplicity, $u \in L^2(\Omega \times O)$ and $\xi \in L^2(\Omega \times O)^N$ such that $\mathbf{1}_{O_\varepsilon} u_\varepsilon \rightharpoonup \theta\, u$ (recall that θ is the constant equal to the limit volume fraction of material) and $\xi_\varepsilon = \mathbf{1}_{O_\varepsilon} \nabla_x u_\varepsilon \rightharpoonup \xi$ weakly in $L^2(\Omega \times O)^N$. Then, using the extension operator $P_{i\varepsilon}$, there exist $u_i \in V$ such that $u_{i\varepsilon} = P_{i\varepsilon}(u_{\varepsilon|_{O_{i\varepsilon}}}) \rightharpoonup u_i$ weakly in V, $i = 1,2$.

3.2.1. First step: $\theta\, u = \theta_1\, u_1 + \theta_2\, u_2$

One has $\mathbf{1}_{O_\varepsilon} u_\varepsilon = \mathbf{1}_{O_\varepsilon} u_{1\varepsilon} + \mathbf{1}_{O_{2\varepsilon}} u_{2\varepsilon} + \mathbf{1}_{\mathcal{O}_\varepsilon} u_\varepsilon$ and by the Cauchy–Schwarz inequality

$$\|\mathbf{1}_{\mathcal{O}_\varepsilon} u_\varepsilon\|^2_{L^1(\Omega\times O)} \le c\,\|\mathbf{1}_{\mathcal{O}_\varepsilon}\|^2_{L^2(\Omega\times O)} = c\int_{\Omega\times O}\sum_{\varepsilon k\in O}\mathbf{1}_{\varepsilon k+\varepsilon Q_k(\varepsilon)} =$$
$$= c\,E\Big(\sum_{\varepsilon k\in O}|\varepsilon\, Q_k(\varepsilon)|\Big) = c\,E\Big(\sum_{\varepsilon k\in O}\ell\,\varepsilon^N\alpha_k(\varepsilon)\Big) \le c'\,E(\alpha(\varepsilon))$$

which implies that $\mathbf{1}_{\mathcal{O}_\varepsilon} u_\varepsilon \to 0$ strongly in $L^1(\Omega\times O)$ by (6).

Since $(\mathbf{1}_{\mathcal{O}_\varepsilon} u_\varepsilon)$ is bounded in the Hilbert space $L^2(\Omega\times O)$, we obtain that $\mathbf{1}_{\mathcal{O}_\varepsilon} u_\varepsilon \rightharpoonup 0$ weakly in $L^2(\Omega\times O)$.

It remains to prove that $\mathbf{1}_{O_{i\varepsilon}} u_{i\varepsilon} \rightharpoonup \theta_i\, u_i$, $i=1,2$ weakly in $L^2(\Omega\times O)$.

This convergence is clear in the deterministic case due to the compact imbedding of $H^1(O)$ in $L^2(O)$ (Rellich's theorem). Here, it is a direct consequence of Lemma 3.1 with $u_\varepsilon = \varphi\, u_{i\varepsilon}$ and $v_\varepsilon = \mathbf{1}_{O_{i\varepsilon}}$, where $\varphi\in\mathcal{D}_\Omega(O)$ which is dense in $L^2(\Omega\times O)$.

3.2.2. Second step: $\xi = A_1\nabla_x u_1 + A_2\nabla_x u_2$

We have $\xi_2 = \xi_{1\varepsilon} + \xi_{2\varepsilon} + \xi_{3\varepsilon}$ where $\xi_{i\varepsilon} = \mathbf{1}_{O_{i\varepsilon}}\nabla_x u_{i\varepsilon}$, $i=1,2$.

Proceeding as in first step, we have $\xi_{3\varepsilon} = \mathbf{1}_{\mathcal{O}_\varepsilon}\xi_\varepsilon \rightharpoonup 0$ weakly in $L^2(\Omega\times O)^N$. Let us prove that $\xi_{i\varepsilon}\rightharpoonup\xi_i$ weakly in $L^2(\Omega\times O)^N$ where $\xi_i = A_i\nabla_x u_i$, $i=1,2$. For that, as in the deterministic case of [3], using the fact that $\overline{E}_1\cap\overline{E}_2=\emptyset$, we consider the (deterministic) test function $w^\lambda_{i\varepsilon}$ of $W^{1,p}(O)$ for some $p>2$, defined by

$$\begin{cases} w^\lambda_{i\varepsilon}(x) \rightharpoonup \lambda\cdot x & \text{weakly in } W^{1,p}(O),\\ w^\lambda_{i\varepsilon}(x) = \lambda\cdot x - \varepsilon\,\chi^\lambda_i\Big(\dfrac{x}{\varepsilon}\Big) & \text{in } O_{i\varepsilon},\\ w^\lambda_{i\varepsilon}(x) = \lambda\cdot x & \text{in } O_{\varepsilon j},\quad j\neq i\ , \end{cases} \tag{18}$$

where χ^λ_i are solutions of (4).

Now, we apply Tartar's energy method [12] by plugging function $\varphi\, w^\lambda_{i\varepsilon}$ for $\varphi\in\mathcal{D}_\Omega(O)$ in equation (3). We obtain

$$\int_{\Omega\times O}\xi_\varepsilon\cdot\nabla w^\lambda_{i\varepsilon}\,\varphi + \int_{\Omega\times O}\xi_\varepsilon\cdot\nabla\varphi\, w^\lambda_{i\varepsilon} + \int_{\Omega\times O}\mathbf{1}_{O_\varepsilon}u_\varepsilon\,\varphi\, w^\lambda_{i\varepsilon} = \int_{\Omega\times O}\mathbf{1}_{O_\varepsilon}f\,\varphi\, w^\lambda_{i\varepsilon}$$

and then, using the strong convergence $w^\lambda_{i\varepsilon}\to\lambda\cdot x$ in $L^2(O)$,

$$\int_{\Omega\times O}\xi_\varepsilon\cdot\nabla w^\lambda_{i\varepsilon}\,\varphi \ \longrightarrow\ -\int_{\Omega\times O}\xi\cdot\nabla\varphi\,\lambda\cdot x - \int_{\Omega\times O}\theta\,u\,\varphi\,\lambda\cdot x + \int_{\Omega\times O}\theta\,f\,\varphi\,\lambda\cdot x$$

the latter is equal to $\int_{\Omega\times O} \xi\cdot\lambda\,\varphi$ by plugging $\varphi\,\lambda\cdot x$ in (3) and then passing to the limit.

Thus

$$\int_{\Omega\times O} \xi_\varepsilon\cdot\nabla w_{i\varepsilon}^\lambda\,\varphi \;\longrightarrow\; \int_{\Omega\times O} \xi\cdot\lambda\,\varphi\ .$$

On the other hand, using the definition (18) of $w_{i\varepsilon}^\lambda$,

$$\int_{\Omega\times O} \xi_\varepsilon\cdot\nabla w_{i\varepsilon}^\lambda\,\varphi = \int_{\Omega\times O} \xi_{i\varepsilon}\cdot\nabla w_{i\varepsilon}^\lambda\,\varphi + \int_{\Omega\times O} \xi_{\varepsilon j}\cdot\lambda\,\varphi + \int_{\Omega\times O} \xi_{3\varepsilon}\cdot\nabla w_{i\varepsilon}^\lambda\,\varphi\ .$$

Since $\nabla w_{i\varepsilon}^\lambda$ is bounded in $L^p(O)$ for some $p>2$, the third term on the right hand side is bounded by $c\,E(|\mathcal{O}_\varepsilon|^{\frac{1}{q}})$ where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, due to Hölder's inequality, and thus converges to 0 by (6) since $E(|\mathcal{O}_\varepsilon|^{\frac{1}{q}})\le E(|\mathcal{O}_\varepsilon|)^{\frac{1}{q}}$.

We have

$$\begin{aligned}\int_{\Omega\times O} \xi_{i\varepsilon}\cdot\nabla w_{i\varepsilon}^\lambda\,\varphi &= \int_{\Omega\times O} \mathbf{1}_{O_{i\varepsilon}}\nabla w_{i\varepsilon}^\lambda\cdot\nabla_x u_{i\varepsilon}\,\varphi\\ &= \int_{\Omega\times O} \mathbf{1}_{O_{i\varepsilon}}\nabla w_{i\varepsilon}^\lambda\cdot\nabla_x(\varphi\,u_{i\varepsilon}) - \int_{\Omega\times O} \mathbf{1}_{O_{i\varepsilon}}\nabla w_{i\varepsilon}^\lambda\cdot\nabla_x\varphi\,u_{i\varepsilon}\ ,\end{aligned}$$

the first term on the right hand side is equal to 0 since χ_i^λ is solution of (4) and the second term converges to $\int_{\Omega\times O} A_i\lambda\cdot\nabla_x\varphi\,u_i$ by Lemma 3.1 since $\mathbf{1}_{O_{i\varepsilon}}\nabla w_{i\varepsilon}^\lambda \rightharpoonup A_i\lambda$ weakly in $L^2(O)^N$.

Finally, we obtain

$$\begin{aligned}\int_{\Omega\times O} \xi\cdot\nabla\varphi &= -\int_{\Omega\times O} A_i\,\lambda\cdot\nabla_x\varphi\,u_i + \int_{\Omega\times O} \xi_j\cdot\lambda\,\varphi\\ &= \int_{\Omega\times O} A_i\,\nabla_x u_i\cdot\lambda\,\varphi + \int_{\Omega\times O} \xi_j\cdot\lambda\,\varphi\end{aligned}$$

since $\varphi\in\mathcal{D}_\Omega(O)$ and

$$\xi = A_i\,\nabla_x u_i + \xi_j = \xi_1+\xi_2\,,\qquad i\ne j\in\{1,2\}\ .$$

Hence, $\xi_i = A_i\nabla_x u_i$ and $\xi = A_1\nabla_x u_1 + A_2\nabla_x u_2$.

3.2.3. Conclusion of the proof

Plugging $\varphi\in V$ in (3) yields after passing to the limit

$$\int_{\Omega\times O} \xi\cdot\nabla_x\varphi + \int_{\Omega\times O} \theta\,u\,\varphi = \int_{\Omega\times O} \theta\,f\,\varphi$$

which gives (7) by steps 1 and 2. ∎

3.3. Proofs of the Theorems

Now the problem is to obtain another equation in order to determine u_i, $i = 1, 2$ which are already solutions to (3) and thus to determine the weak limit of $\mathbf{1}_{O_\varepsilon} u_\varepsilon$ which is equal to $\theta_1 u_1 + \theta_2 u_2$.

3.3.1. Energy cost and compactness results

We again state two preliminary lemmas. The first one yields an estimate of the energy cost due to each random bridge and is a simple adaptation of a result proved in [3] for the deterministic case.

Lemma 3.3. *Let $\widetilde{Y} = Y_1 \cup Y_2 \cup \widetilde{Q}$ where $\widetilde{Q}$ is the cylinder of length ℓ and of section area a (it therefore contains every bridge $Q_k(\varepsilon)$) and let $\hat{v}$ be the deterministic function in $H^1_\#(\widetilde{Y})$ (i.e. Y-periodic and locally in H^1 on the periodic open set obtained by Y-repetition of $\widetilde{Y}$) defined by*

$$\begin{cases} \hat{v}(y) = 1 & \text{in } Y_1, \\ \hat{v}(y) = 0 & \text{in } Y_2, \\ \hat{v} \text{ is affine} & \text{in } \widetilde{Q}\ . \end{cases} \tag{19}$$

Then, for any $k \in \mathbb{Z}^N$ and $v \in L^2(Y_k(\varepsilon))$, one has

$$\left| \frac{1}{\varepsilon^2} \fint_{Y_k(\varepsilon)} \nabla\hat{v} \cdot \nabla v - \delta_k(\varepsilon) \fint_{Y_1} v + \delta_k(\varepsilon) \fint_{Y_2} v \right| \le c\, \frac{(\alpha_k(\varepsilon))^r}{\varepsilon^2}\, \|\nabla v\|_{L^2(Y_k(\varepsilon))} \tag{20}$$

where $\delta_k(\varepsilon) = \dfrac{1}{\varepsilon^2} \fint_{Y_k(\varepsilon)} |\nabla\hat{v}|^2$ and $c > 0$, $r > \frac{1}{2}$ are two deterministic constants independent of k and ε.

The proof of this lemma is given in [3] with $\alpha_k(\varepsilon) = \varepsilon^2$. The second result is a compactness result since it allows us to pass to the limit in a product of weak convergences. It also provides the mean behaviour of the thin random bridges using a law of large numbers.

Lemma 3.4. *Let $(\gamma_k)_{k\in\mathbb{Z}^N}$ be a family of real random variables in $L^1(\Omega)$, let $\chi \in L^\infty(\mathbb{R}^N)$ which has a compact support in $\overline{Y}$, and let v_ε be a sequence of $V \cap L^\infty(\Omega \times O)$ such that $v_\varepsilon \rightharpoonup v$ weakly in V and v_ε is bounded in $L^\infty(\Omega \times O)$.*

Then, the following limit holds

$$\int_{\Omega\times O} \sum_{\varepsilon k \in O} \gamma_k(\omega)\, \chi\!\left(\frac{x}{\varepsilon} - k\right) v_\varepsilon(\omega, x) \;\longrightarrow\; \int_{\Omega\times O} E(\gamma)\, \overline{\chi}\, v(\omega, x)\ . \tag{21}$$

The previous sum is intended to be extended over the $k \in \mathbb{Z}^N$ such that

$$\varepsilon\, k + \varepsilon\, Y_k(\varepsilon) \subset O\ .$$

Proof: We first replace each $\gamma_k \in L^1(\Omega)$ by its truncature of size $n \in \mathbb{N}$, $T_n(\gamma_k)$ where $T_n(t) = \max(-n, \min(n,t))$, so that $(T_n(\gamma_k))_k$ is a sequence of bounded random variables.

Indeed, let

$$I_\varepsilon(v_\varepsilon) = \int_{\Omega\times O} \sum_{\varepsilon k\in O} \gamma_k(\omega)\, \chi\Big(\frac{x}{\varepsilon} - k\Big)\, v_\varepsilon(\omega, x)$$

and $I^n_\varepsilon(v_\varepsilon)$ similarly defined with $T_n(\gamma_k)$. Using boundedness of v_ε and independence, we obtain

$$\begin{aligned} |I^n_\varepsilon(v_\varepsilon) - I_\varepsilon(v_\varepsilon)| &\le c \int_{\Omega\times O} \sum_{\varepsilon k\in O} |T_n(\gamma_k) - \gamma_k|\, \mathbf{1}_Y\Big(\frac{x}{\varepsilon} - k\Big) \\ &\le c \int_{\Omega} \sum_{\varepsilon k\in O} |T_n(\gamma_k) - \gamma_k|\, \varepsilon^N \\ &\le c\, E(T_n(\gamma) - \gamma) \ \to\ 0 \end{aligned}$$

uniformly with respect to $\varepsilon > 0$.

We then may replace γ_k by $T_n(\gamma_k)$.

Consider a covering $(K_j)_{j\in\mathbb{Z}^N}$ of $\mathbb{R}^N$ by cubes K_j with no common interior point and with a length $h > 0$ ($h \gg \varepsilon$).

We shall replace v_ε by

$$\overline{v}_\varepsilon(\omega, x) = ⨍_{K_j} \mathbf{1}_O\, v_\varepsilon(\omega, y)\, dy \qquad \text{if} \ \ x \in \overset{\circ}{K}_j, \quad j \in \mathbb{Z}^N\ .$$

Using Poincaré–Wirtinger's inequality in each homothetic $\frac{1}{h} K_j$ and rescaling with respect to h, we obtain the following estimate

$$\|\overline{v}_\varepsilon - v_\varepsilon\|_{L^2(O_h)} \le c\, h\, \|\nabla_x v_\varepsilon\|_{L^2(O_h)} \ \ \text{a.s.}\ , \tag{22}$$

where $O_h = \bigcup_{K_j\subset O} K_j$ and $c > 0$ the deterministic constant corresponding to Poincaré–Wirtinger's inequality in any cube of $\mathbb{R}^N$ of side equal to 1.

Since O has a Lipschitz boundary, $|O - O_h| \le c\, h$ and then estimate (22) implies that

$$|I_\varepsilon(\overline{v}_\varepsilon) - I_\varepsilon(v_\varepsilon)| \le c \int_{\Omega\times O_h} |\overline{v}_\varepsilon - v_\varepsilon| + c\, |O - O_h| \le c\, h \tag{23}$$

since (γ_k), χ, (v_ε) are bounded in $L^\infty(\Omega\times O)$ and $(\nabla_x v_\varepsilon)$ is bounded in $L^2(\Omega\times O)$.

Let us compute the limit of $I_\varepsilon(\overline{v}_\varepsilon)$ at fixed h. We have

$$\begin{aligned} I_\varepsilon(\overline{v}_\varepsilon) &= \int_\Omega \sum_{j\in\mathbb{Z}^N} \overline{v}_{j\varepsilon} \int_{K_j} \sum_{\varepsilon k\in O} \gamma_k\, \chi\Big(\frac{x}{\varepsilon}-k\Big) \\ &= \int_\Omega \sum_{j\in\mathbb{Z}^N} \overline{v}_{j\varepsilon} \Big(\sum_{\varepsilon k\in K_j} \gamma_k\, \overline{\chi}\, \varepsilon^N + O(\varepsilon)\Big) \end{aligned}$$

where $\overline{v}_{j\varepsilon} = \fint_{K_j} \mathbf{1}_O v_\varepsilon$ and the term $O(\varepsilon)$ comes from the sets $\varepsilon k + \varepsilon Y$ that meet the boundary of K_j. Observe that the sum over j is finite since $\overline{v}_{j\varepsilon} = 0$ if $K_j \cap O = \emptyset$. The law of large numbers gives

$$\sum_{\varepsilon k\in K_j} \gamma_k\, \varepsilon^N \;\to\; |K_j|\, E(\gamma) \quad \text{in } \; L^2(\Omega) \; \text{ and for each } j\in\mathbb{Z}^N \; .$$

One has also $\overline{v}_{j\varepsilon} \rightharpoonup \overline{v}_j = \fint_{K_j} \mathbf{1}_O\, v$ in $L^2(\Omega)$. Then, passing to the limit in the definition of $I_\varepsilon(\overline{v}_\varepsilon)$, gives

$$I_\varepsilon(\overline{v}_\varepsilon) \;\to\; \int_\Omega \sum_{j\in\mathbb{Z}^N} \overline{v}_j\, |K_j|\, E(\gamma)\, \overline{\chi}^Y = \int_{\Omega\times O} E(\gamma)\, \overline{\chi}^Y\, \overline{v} \tag{24}$$

where $\overline{v}(\omega,x) = \overline{v}_j(\omega)$ for $x \in \overset{\circ}{K}_j$ and where $\overline{\chi}^Y$ denotes the mean of χ over Y.

Now, the lower semi-continuity of the $L^2(\Omega\times O)$ norm combined with estimate (22) yields

$$\|\overline{v} - v\|_{L^2(\Omega\times O_h)} \le \underline{\lim}\, \|\overline{v}_\varepsilon - v_\varepsilon\|_{L^2(\Omega\times O_h)} \le c\, h$$

and thus, since $\overline{v}$ is bounded in $L^\infty(\Omega)$ independently of h,

$$\Big|\int_{\Omega\times O} (\overline{v} - v)\Big| \le c\, h + |O - O_h|^{\frac{1}{2}}\, \|\overline{v} - v\|_{L^2(\Omega\times O)} \le c\sqrt{h} \; .$$

Then denoting $I_0(v) = \int_{\Omega\times O} E(\gamma)\, \overline{\chi}^Y v$, we obtain, due to estimate (23)

$$\begin{aligned} |I_\varepsilon(v_\varepsilon) - I_0(v)| &\le |I_\varepsilon(\overline{v}_\varepsilon) - I_\varepsilon(v_\varepsilon)| + |I_\varepsilon(\overline{v}_\varepsilon) - I_0(\overline{v})| + |I_0(\overline{v}) - I_0(v)| \\ &\le |I_\varepsilon(\overline{v}_\varepsilon) - I_0(\overline{v})| + c\sqrt{h} \; . \end{aligned}$$

Recall that by (24) $I_\varepsilon(\overline{v}_\varepsilon) \to I_0(\overline{v})$ for any $h \le 1$. Then, the latter estimate proves that $I_\varepsilon(v_\varepsilon) \to I_0(v)$ and concludes the proof of Lemma 3.4. ∎

3.3.2. Proof of Theorem 2.2

We first proceed as in the deterministic case by plugging in equation (3) satisfied by u_ε, the function $\hat{v}_\varepsilon(x) = \hat{v}(\frac{x}{\varepsilon})$ where $\hat{v}$ is defined by (19). The function $\hat{v}_\varepsilon$ separates materials 1 and 2.

Let $\varphi \in L^\infty(\Omega; C^1(\overline{O}))$, we have

$$\int_{\Omega\times O} \mathbf{1}_{O_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x(\varphi\, \hat{v}_\varepsilon) + \int_{\Omega\times O} u_\varepsilon\, \varphi\, \hat{v}_\varepsilon = \int_{\Omega\times O} f\, \varphi\, \hat{v}_\varepsilon \ .$$

Since $\hat{v}_\varepsilon = 1$ in $O_{1\varepsilon}$ and $\hat{v}_\varepsilon = 0$ in $O_{2\varepsilon}$, we obtain with the notations of the proof of Proposition 2.1, i.e. $\xi_{i\varepsilon} = \mathbf{1}_{O_{i\varepsilon}} \nabla_x u_\varepsilon$, $i = 1, 2$,

$$\int_{\Omega\times O} \mathbf{1}_{O_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \varphi\, \hat{v}_\varepsilon = \int_{\Omega\times O} \xi_{1\varepsilon} \cdot \nabla_x \varphi + \int_{\Omega\times O} \mathbf{1}_{\mathcal{O}_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \varphi\, \hat{v}_\varepsilon \ ,$$

the last term on the right hand side being bounded by $c\,\|\mathbf{1}_{\mathcal{O}_\varepsilon}\|_{L^2(\Omega\times O)} \leq c\, E(\alpha(\varepsilon))^{\frac{1}{2}} \to 0$ by condition (6).

We have $\xi_{1\varepsilon} = \mathbf{1}_{O_{1\varepsilon}} \nabla u_\varepsilon \rightharpoonup A_1 \nabla_x u_1$ in $L^2(\Omega\times O)$ from Proposition 2.1, then the definition (19) of $\hat{v}$ gives

$$\lim \int_{\Omega\times O} \mathbf{1}_{O_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \varphi\, \hat{v}_\varepsilon = \int_{\Omega\times O} A_1 \nabla_x u_1 \cdot \nabla_x \varphi \ .$$

Moreover, since $u_{1\varepsilon} = P_{1\varepsilon} u_\varepsilon \rightharpoonup u_1$ weakly in V_{loc} and $\mathbf{1}_{O_{1\varepsilon}}$ is deterministic and weakly converges to θ_1 in $L^2(O)$, we have by (6) and by Lemma 3.1

$$\int_{\Omega\times O} \mathbf{1}_{O_\varepsilon} \varphi\, \hat{v}_\varepsilon\, u_\varepsilon = \int_{\Omega\times O} \mathbf{1}_{O_{1\varepsilon}} \varphi\, u_{1\varepsilon} + \int_{\Omega\times O} \mathbf{1}_{\mathcal{O}_\varepsilon} \varphi\, \hat{v}_\varepsilon\, u_\varepsilon \ \to \ \int_{\Omega\times O} \varphi\, \theta_1\, u_1$$

and similarly

$$\int_{\Omega\times O} f\, \varphi\, \hat{v}_\varepsilon \ \to \ \int_{\Omega\times O} \theta_1\, f\, \varphi \ .$$

Finally, we obtain

$$\lim \int_{\Omega\times O} \mathbf{1}_{O_\varepsilon} \varphi\, \nabla_x u_\varepsilon \cdot \nabla_x \hat{v}_\varepsilon = = \int_{\Omega\times O} \varphi\, \theta_1\, f - \int_{\Omega\times O} \varphi\, \theta_1\, u_1 - \int_{\Omega\times O} A_1 \nabla_x u_1 \cdot \nabla_x \varphi \ . \tag{25}$$

It remains to find the limit of the left hand side of (25) in another way.

Similarly to the deterministic case, we are led to the case $f \in L^\infty(O)$, using a density argument. Now, let us observe that (3) can be written for any $\varphi \in H^1(O)$ and $\psi \in L^2(\Omega)$

$$\int_\Omega \psi(\omega) \left(\int_O \mathbf{1}_{O_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \varphi + \int_O \mathbf{1}_{O_\varepsilon} u_\varepsilon\, \varphi - \int_O \mathbf{1}_{O_\varepsilon} f\, \varphi \right) dP(\omega) = 0$$

which implies that a.s.

$$\int_{O_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \varphi + \int_{O_\varepsilon} u_\varepsilon \, \varphi = \int_{O_\varepsilon} f \, \varphi$$

i.e. $u_\varepsilon(\omega, \cdot)$ is solution of the Neumann problem (1). Then, the maximum principle implies that $\|u_\varepsilon(\omega,\cdot)\|_{L^\infty(O_\varepsilon)} \le \|f\|_{L^\infty(O)}$ a.s.. Considering a subsequence ε, we obtain that $\mathbf{1}_{O_\varepsilon} u_\varepsilon \in L^\infty(\Omega \times O)$ and $\|\mathbf{1}_{O_\varepsilon} u_\varepsilon\|_{L^\infty(\Omega\times O)} \le \|f\|_{L^\infty(O)}$ for any ε. Using an integration by parts,

$$\int_{\Omega\times O} \mathbf{1}_{O_\varepsilon} \varphi \, \nabla_x u_\varepsilon \cdot \nabla_x \hat{v}_\varepsilon = \int_{\Omega\times O} \mathbf{1}_{\mathcal{O}_\varepsilon} \nabla_x \hat{v}_\varepsilon \cdot \nabla_x(\varphi \, u_\varepsilon) - \int_{\Omega\times O} \mathbf{1}_{\mathcal{O}_\varepsilon} \nabla_x \varphi \cdot \nabla_x \hat{v}_\varepsilon \, u_\varepsilon \ .$$

The last term on the right hand side is bounded by $\dfrac{c}{\varepsilon} \|\mathbf{1}_{\mathcal{O}_\varepsilon}\|_{L^1(\Omega\times O)} \le c' \, E\Big(\dfrac{\alpha(\varepsilon)}{\varepsilon}\Big) \to 0$ since by (8) $\dfrac{\alpha(\varepsilon)}{\varepsilon^2} \to \gamma$ strongly in $L^1(\Omega)$.

We will now compute the limit of $\displaystyle\int_{\Omega\times O} \mathbf{1}_{\mathcal{O}_\varepsilon} \nabla_x \hat{v}_\varepsilon \cdot \nabla_x(\varphi u_\varepsilon)$.

We proceed as in the deterministic case using estimate (20). However, we have to use it cell by cell since the bridge is different in each cell. Denote $Y_{\varepsilon k} = \varepsilon k + \varepsilon Y_k(\varepsilon)$, $Y^i_{\varepsilon k} = \varepsilon k + \varepsilon Y_i$, $i = 1, 2$ and $\check{O}_\varepsilon$, resp. $\check{O}_{i\varepsilon}$, the set obtained as the union of the $Y_{\varepsilon k} \subset O_\varepsilon$, resp. $Y^i_{\varepsilon k} \subset O_{i\varepsilon}$, $i = 1, 2$ and denote $v_\varepsilon = \varphi \, u_\varepsilon$. Then, by rescaling (8) with respect to ε and summing over k such that $Y_{\varepsilon k} \subset \check{O}_\varepsilon$, namely $\varepsilon k \in O$, we obtain the estimate

$$\left| \int_{\Omega\times O} \mathbf{1}_{\check{O}_\varepsilon} \nabla_x \hat{v}_\varepsilon \cdot \nabla v_\varepsilon - \int_{\Omega\times O} \sum_{\varepsilon k \in O} \delta_k(\varepsilon) \, (\theta_1^{-1} \, \mathbf{1}_{Y^1_{\varepsilon k}} - \theta_2^{-1} \, \mathbf{1}_{Y^2_{\varepsilon k}}) \, v_\varepsilon \right| \le$$
$$\le c \int_\Omega \sum_{\varepsilon k \in O} \frac{\alpha_k(\varepsilon)^r}{\varepsilon} \, \varepsilon^{\frac{N}{2}} \, \|\nabla_x v_\varepsilon\|_{L^2(Y_{\varepsilon k})} \ .$$

By the Cauchy–Schwarz inequality applied in O, the right-hand side is bounded by

$$c \int_\Omega \Big(\sum_{\varepsilon k \in O} \frac{\alpha_k(\varepsilon)^{2r}}{\varepsilon^2} \, \varepsilon^N \Big)^{\frac{1}{2}} \|\nabla_x v_\varepsilon\|_{L^2(O_\varepsilon)}$$

and still by Cauchy–Schwarz applied in Ω, it is bounded by

$$c \Big[E\Big(\sum_{\varepsilon k \in O_\varepsilon} \frac{\alpha_k(\varepsilon)^{2r}}{\varepsilon^2} \, \varepsilon^N \Big) \Big]^{\frac{1}{2}} \|\mathbf{1}_{O_\varepsilon} \nabla_x v_\varepsilon\|_{L^2(\Omega\times O)} \le c \, E\Big(\frac{\alpha(\varepsilon)^{2r}}{\varepsilon^2} \Big)^{\frac{1}{2}} =$$
$$= c \, E\Big[\gamma \, \alpha(\varepsilon)^{2r-1} + \Big(\frac{\alpha(\varepsilon)}{\varepsilon^2} - \gamma \Big) \alpha(\varepsilon)^{2r-1} \Big]^{\frac{1}{2}}$$

which tends to 0 by (6) and (8).

Therefore, we have

$$\int_{\Omega\times O} \varphi\, \mathbf{1}_{\check{O}_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \hat{v}_\varepsilon - \int_{\Omega\times O} \sum_{\varepsilon k\in O} \delta_k(\varepsilon)\, (\theta_1^{-1}\, \mathbf{1}_{Y^1_{\varepsilon k}} - \theta_2^{-1}\, \mathbf{1}_{Y^2_{\varepsilon k}})\, v_\varepsilon \;\longrightarrow\; 0 \;. \tag{26}$$

Now, observe that

$$\delta_k(\varepsilon) = \frac{1}{\ell}\, \frac{\alpha_k(\varepsilon)}{\varepsilon^2}$$

which implies, because of the boundedness of $\mathbf{1}_{O_\varepsilon} v_\varepsilon$,

$$\begin{aligned} &\Big| \int_{\Omega\times O} \sum_{\varepsilon k\in O} \delta_k(\varepsilon)\, (\theta_1^{-1}\, \mathbf{1}_{Y^1_{\varepsilon k}} - \theta_2^{-1}\, \mathbf{1}_{Y^2_{\varepsilon k}})\, v_\varepsilon - \\ &\qquad - \int_{\Omega\times O} \sum_{\varepsilon k\in O} \gamma_k\, (\ell^{-1}\, \theta_1^{-1}\, \mathbf{1}_{Y^1_{\varepsilon k}} - \ell^{-1}\, \theta_2^{-1}\, \mathbf{1}_{Y^2_{\varepsilon k}})\, v_\varepsilon \Big| \le \\ &\qquad \le E\Big(\Big| \frac{\alpha(\varepsilon)}{\varepsilon^2} - \gamma \Big| \Big) \;\to\; 0 \quad \text{by (8)} \;. \end{aligned}$$

Using Lemma 3.4 with $\chi = \mathbf{1}_{Y_i}$ and $v_\varepsilon = \varphi\, u_{i\varepsilon}$, (26) implies that

$$\int_{\Omega\times O} \varphi\, \mathbf{1}_{\check{O}_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \hat{v}_\varepsilon \;\longrightarrow\; \int_{\Omega\times O} \frac{1}{\ell}\, E(\gamma)\, (u_1 - u_2)\, \varphi \;.$$

Finally, by definition of $\check{O}_\varepsilon$ and by Cauchy–Schwarz inequality, we have

$$\begin{aligned} \Big| \int_{\Omega\times O} \varphi\, (\mathbf{1}_{O_\varepsilon} - \mathbf{1}_{\check{O}_\varepsilon})\, \nabla_x u_\varepsilon \cdot \nabla_x \hat{v}_\varepsilon \Big|^2 &\le c \int_\Omega \sum_{Y_{\varepsilon k}\cap\partial O\neq\emptyset} \int_{Y_{\varepsilon k}} |\nabla \hat{v}_\varepsilon|^2 \\ &\le c\, \varepsilon^N\, E\Big(\frac{\alpha_k(\varepsilon)}{\varepsilon^2}\Big)\, \#\Big\{k,\; Y_{\varepsilon k}\cap\partial O \neq \emptyset\Big\} \\ &\le c\, \varepsilon\, E\Big(\frac{\alpha_k(\varepsilon)}{\varepsilon^2}\Big) \;\to\; 0 \;, \end{aligned}$$

since the regularity of O implies that the number of $Y_{\varepsilon k}\cap\partial O \neq \emptyset$ is of order of ε^{1-N}, and thus

$$\int_{\Omega\times O} \mathbf{1}_{O_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \hat{v}_\varepsilon \;\longrightarrow\; \int_{\Omega\times O} \frac{1}{\ell}\, E(\gamma)\, (u_1 - u_2)\, \varphi \;. \tag{27}$$

Combining (25) and (27) yields

$$\int_{\Omega\times O} A_1 \nabla_x u_1 \cdot \nabla_x \varphi + \int_{\Omega\times O} \Big[\theta_1\, u_1 + \frac{1}{\ell}\, E(\gamma)\, (u_1 - u_2) \Big]\, \varphi = \int_{\Omega\times O} \theta_1\, f\, \varphi \tag{28}$$

for any $\varphi \in L^\infty(\Omega; C^1(\overline{O}))$ and by density for any $\varphi \in V$. The coupled system composed by (7) and (28) has a unique solution in V by the Lax–Milgram theorem and it is clear that $E(u_i)$, $i = 1, 2$ are solutions of this system since all the coefficients are deterministic. This gives (9) and concludes the proof of the theorem. ∎

3.3.3. Proof of Theorem 2.3

The key of the proof in the deterministic case for the supercritical case is the Poincaré–Wirtinger inequality applied to the basic cell of the material (see [4] for the general framework). Here, we cannot apply this method since the cells are all different. However, the key-ingredient is still estimate (20).

We have by the Cauchy–Schwarz inequality and for any $v \in H^1(Y_k(\varepsilon))$

$$\left|\frac{1}{\varepsilon^2}\int_{Y_k(\varepsilon)} \nabla\hat{v}\cdot\nabla v\right| \le c\,\frac{\alpha_k(\varepsilon)^{\frac12}}{\varepsilon^2}\,\|\nabla v\|_{L^2(Y_k(\varepsilon))} \quad \text{and} \quad \delta_k(\varepsilon) \ge c\,\frac{\alpha_k(\varepsilon)}{\varepsilon^2}\ ,$$

which, combined with estimate (20), yields

$$\left|\fint_{Y_1} v - \fint_{Y_2} v\right| \le c\left[\frac{1}{\alpha_k(\varepsilon)^{\frac12}} + \alpha_k(\varepsilon)^{r-1}\right]\|\nabla v\|_{L^2(Y_k(\varepsilon))}\ .$$

We thus obtain, since $r > \frac12$ and $\alpha_k(\varepsilon) \le a$, the new estimate

$$\left|\fint_{Y_1} v - \fint_{Y_2} v\right| \le \frac{c}{\alpha_k(\varepsilon)^{\frac12}}\,\|\nabla v\|_{L^2(Y_k(\varepsilon))}\,,\qquad \forall\, v \in H^1(Y_k(\varepsilon))\ . \tag{29}$$

Let $E_k = \left\{\frac{\alpha_k(\varepsilon)}{\varepsilon^2} \to \infty\right\}$ and let $\varphi \in \mathcal{D}_\Omega(O)$. We proceed similarly to the proof of Theorem 2.2 with estimate (20), i.e. we plug the function $\mathbf{1}_{E_k} v_\varepsilon$, where $v_\varepsilon = \varphi\, u_\varepsilon$, in the estimate obtained from (29) by rescaling with respect to ε, and we sum over each cell $Y_{k\varepsilon}$ such that $\varepsilon k \in O$, which yields

$$\left|\int_{\Omega\times O}\sum_{\varepsilon k\in O}\mathbf{1}_{E_k}\,(\theta_1^{-1}\,\mathbf{1}_{Y^1_{\varepsilon k}} - \theta_2^{-1}\,\mathbf{1}_{Y^2_{\varepsilon k}})\,v_\varepsilon\right| \le c\int_\Omega\sum_{\varepsilon k\in O}\frac{\varepsilon\,\mathbf{1}_{E_k}}{\alpha_k(\varepsilon)^{\frac12}}\,\varepsilon^{\frac N2}\,\|\nabla v_\varepsilon\|_{L^2(Y_{\varepsilon k})}\ .$$

By the Cauchy–Schwarz inequality applied to the sum over k and later on to the integral on Ω, the right hand side of the inequality is bounded by

$$c\left[E\Big(\sum_{\varepsilon k\in O}\varepsilon^n\,\frac{\varepsilon^2}{\alpha_k(\varepsilon)}\,\mathbf{1}_{E_k}\Big)\right]^{\frac12}\|\nabla v_\varepsilon\|_{L^2(\Omega\times O)} \le c\left[E\Big(\frac{\varepsilon^2}{\alpha_k(\varepsilon)}\,\mathbf{1}_{E_k}\Big)\right]^{\frac12}$$

which converges to 0 by (12) combined with Lebesgue dominated convergence Theorem.

On the other hand, from Lemma 3.4 applied with $\chi = \mathbf{1}_{Y_i}$, $i = 1, 2$ and $v_\varepsilon = \varphi\, u_\varepsilon$, we have

$$\int_{\Omega\times O} \sum_{\varepsilon k\in O} \mathbf{1}_{E_k}\, (\theta_1^{-1}\, \mathbf{1}_{Y^1_{\varepsilon k}} - \theta_2^{-1}\, \mathbf{1}_{Y^2_{\varepsilon k}})\, v_\varepsilon \;\longrightarrow\; \int_{\Omega\times 0} P(E_k)\, (u_1 - u_2)\, \varphi \;.$$

Finally, we obtain

$$\int_{\Omega\times O} P(E_k)\, (u_1 - u_2)\, \varphi = 0\,, \qquad \forall\, \varphi \in \mathcal{D}_\Omega(O)\;,$$

which implies $u_1 = u_2$ since $P(E_k) > 0$ by (11). The latter, combined with equation (7) yields (13) since the function $E(u_1) = E(u_2)$ is also solution of (11), the matrices A_i, $i = 1, 2$, being deterministic. This concludes the proof. ∎

4 – Appendix: proof of Lemma 1.1

Let $u \in V_\varepsilon$. We will a.s. extend $u(\omega, \cdot) \in H^1(O_\varepsilon)$ to a function $\widetilde{u}(\omega, \cdot) \in H^1(\widetilde{O}_\varepsilon)$, such that $\mathbf{1}_{O_\varepsilon} \widetilde{u} = \mathbf{1}_{O_\varepsilon} u$ and $\|\widetilde{u}(\omega, \cdot)\|_{H^1(\widetilde{O}_\varepsilon)} \le c(\varepsilon)\, \|u(\omega, \cdot)\|_{H^1(O_\varepsilon)}$ where $c(\varepsilon)$ is a deterministic constant.

For that purpose, let us first construct such an extension for each $U \in H^1(Y_k(\varepsilon))$. Let $\widetilde{Y}_k(\varepsilon)$ be an open subset of Y obtained by replacing $Y_k(\varepsilon)$ by a cylinder of same length ℓ the area section of which is equal to $a > \alpha_k(\varepsilon)$. Then, by using the usual technics of extension by reflection, we can get a function $\widetilde{U} \in H^1(\widetilde{Y}_k(\varepsilon))$ such that $\widetilde{U}_{|Y_k(\varepsilon)} = U$ and $\|\widetilde{U}\|_{H^1(\widetilde{Y}_k(\varepsilon))} \le c(\varepsilon)\, \|U\|_{H^1(Y_k(\varepsilon))}$ where $c(\varepsilon)$ is a deterministic constant which depends on $g(\varepsilon)$ defined in (2). By repeating the same procedure in each cell $\varepsilon k + \varepsilon Y_k(\varepsilon)$ with $u(\omega, x) = U(\frac{x}{\varepsilon} - k)$ we obtain $\widetilde{u}$ since Ω is only composed of entire cells.

By construction, $\widetilde{u} \in H^1(\widetilde{O}_\varepsilon)$, $\omega \mapsto \widetilde{u}(\omega, \cdot)$ is measurable and the following estimate holds a.s.

$$\|\widetilde{u}(\omega, \cdot)\|_{H^1(\widetilde{O}_\varepsilon)} \le c(\varepsilon)\, \|u(\omega, \cdot)\|_{H^1(O_\varepsilon)} \;.$$

Then, we can define an extension operator P from V_ε into $\widetilde{V}_\varepsilon$ such that $Pu = \widetilde{u}$ which satisfies $\|Pu\|_{\widetilde{V}_\varepsilon} \le c(\varepsilon)\, \|u\|_{V_\varepsilon}$ and $\mathbf{1}_{O_\varepsilon} Pu = \mathbf{1}_{O_\varepsilon} u$.

From the estimates

$$\frac{1}{c(\varepsilon)}\, \|Pu\|_{\widetilde{V}_\varepsilon} \le \|u\|_{V_\varepsilon} \le \|Pu\|_{\widetilde{V}_\varepsilon} \;,$$

we deduce that $(V_\varepsilon, \|\cdot\|_{V_\varepsilon})$ is an Hilbert space since $(\widetilde{V}_\varepsilon, \|\cdot\|_{\widetilde{V}_\varepsilon})$ is an Hilbert space. ∎

REFERENCES

[1] Acerbi, E., Chiado Piat, V., Dal Maso, G. and Percivale, D. – An extension theorem from connected sets, and homogenization in general periodic domains, *Nonlinear Analysis, Theory Methods and Applications,* 18(5) (1992), 481–495.

[2] Bensoussan, A. – *Homogenization for non linear elliptic equations with random highly oscillatory coefficients*, in "Partial Differential Equations and the Calculus of Variations", Essays in honor of E. De Giorgi (F. Colombini *and al.*, Eds.), Birkhauser, 1989.

[3] Briane, M. – Homogenization in some weakly connected materials, to appear in *Ricerche di Matematica.*

[4] Briane, M. – Poincaré–Wirtinger's inequality for the homogenization in perforated domains, to appear in *Bol. Uni. Mat. Ita.*

[5] Cioranescu, D. and Saint Jean Paulin, J. – Homogenization in open sets with holes, *J. Math. Anal. Appl.,* 71 (1979), 590–607.

[6] Dal Maso, G. and Modica, L. – Nonlinear stochastic homogenization and ergodic theory, *J. Rei. Ang. Math.,* B. 368 (1986), 27–42.

[7] Jikov, V.V., Kozlov, S.M. and Oleinik, O.A. – *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, Heidelberg, 1994.

[8] Khruslov, E.Ya. – The asymptotic behavior of solutions of the second boundary value problems under fragmentation of the boundary of the domain, *Maths. USSR Sbornik,* 35(2) (1979).

[9] Kozlov, S.M. – Averaging of random operators, *Maths. USSR Sbornik,* 37 (1980), 167–180.

[10] Murat, F. – *H-convergence,* Séminaire d'Analyse Fonctionnelle et Numérique, 1977–78, Université d'Alger. English translation in Murat, F., Tartar, L., *H*-convergence, *Topics in the Mathematical Modelling of Composite Materials*, (R.V. Kohn, Ed.), *Progress in Nonlinear Differential Equations and their Applications*, Birkaüser, Boston 1995, to appear.

[11] Papanicolaou, G.C. and Varadhan, S.R.S. – Boundary value problems with rapidly oscillating random coefficients, *Proceed. of Colloq. on Random Fields, Rigorous Results in Statistichal Mechanics and Quantum Field Theory* (J. Fritz, J.L. Lebowitz, D. Szasz, Eds.), Colloquia Mathematica Societ. Jànos Bolyai 10, Amsterdam, pp. 835–873, 1979.

[12] Tartar, L. – *Cours Peccot*, Collège de France, 1977, unpublished, partially written in [10].

[13] Yosida, K. – *Functional Analysis*, Springer, New-York, 1965.

[14] Zhikov, V.V. – On homogenization in random perforated domains of general type, *Matem. Zametki,* 53(1) (1993), 41–58.

[15] Zhikov, V.V. – On the homogenization of nonlinear variational problems in perforated domains, *Russian Journal of Mathematical Physics,* 2(3) (1994), 393–408.

Marc Briane,
Laboratoire d'Analyse Numérique t. 55-65 5è étage, Université Paris VI,
4, place Jussieu, 75252 Paris Cedex 05 – FRANCE
and
Département de Mathématiques, Université Paris XII,
61, ave. Général de Gaulle, 94010 Créteil Cedex – FRANCE

and

Laurent Mazliak,
Laboratoire de Probabilités t. 46-56 3è étage, Université Paris VI,
4, place Jussieu, 75252 Paris Cedex 05 – FRANCE