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SOME EMBEDDINGS OF THE SPACE OF PARTIALLY COMPLEX STRUCTURES

C. Ferreira and A. Machado

Abstract: Let E be a Euclidean n-dimensional vector space. A partially complex structure with dimension k in E is a couple (F, J), where $F \subset E$ is a real vector subspace, with dimension $2k$, and $J\colon F \to F$ is a complex structure in F, compatible with the induced inner product. The space of all such structures can be identified with the holomorphic homogeneous non symmetric space $O(n)/(U(k) \times O(n-2k))$. We study a family $(\mathcal{G}_{kt}(E))_{t\in[0,\pi[}$ of equivariant models of this homogeneous space inside the orthogonal group $O(E)$, from the viewpoint of its extrinsic geometry. The metrics induced by the biinvariant metric of $O(E)$ correspond to an interval of the one-parameter family of invariant compatible metrics of this homogeneous space, including the Kähler and the naturally reductive ones. The manifolds $\mathcal{G}_{kt}(E)$ are $(2,0)$-geodesic inside $O(E)$; some of them are minimal inside $O(E)$ and others are minimal inside a suitable sphere. We show also that the model $\mathcal{F}_k(E)$ inside the Lie algebra $o(E)$, corresponding to the compatible f-structures of Yano, is $(2,0)$-geodesic and minimal inside a sphere.

1 – Introduction

It has been known for a long time that a large class of compact symmetric homogeneous spaces can be isometric and equivariantly embedded in a convenient Euclidean space with an image that is minimal on a sphere (cf., for example Kobayashi [3] and Takeushi & Kobayashi [7]). These examples have been regarded later by Ferus [1] from the viewpoint of the notion of symmetric submanifold.

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More recently Uhlenbeck [8] used a slight modification of this method in order to embed the complex Grassmann manifold as a totally geodesic submanifold of the unitary group.

In this article we look to some interesting equivariant embeddings of a compact non symmetric homogeneous space, the space of partially complex structures. Following Salamon [6], given a Euclidean n-dimensional space E, a *partially complex structure* on E, with complex dimension k, is a couple (F, J), where F is a $2k$-dimensional real vector subspace and J is a complex structure on F compatible with the inner product. There is a natural transitive action of the orthogonal group $O(E)$ on the set of partially complex structures that realizes it as a homogeneous manifold $O(n)/U(k)\times O(n-2k)$ and it is well known that this manifold can be equivariantly embedded into the Lie algebra $o(E)$ by associating to each couple (F, J) the skew-adjoint linear map $\lambda\colon E \to E$ that equals J on F and is 0 on $F^{\perp}$. The image $\mathcal{F}_k(E)$ of this embedding is the set of all skew-adjoint linear maps λ such that $\lambda^3 = -\lambda$ a set whose origin goes back to the notion of f-structure of Yano [9]. The set of all partially complex structures on E, with complex dimension k, is a complex manifold in a natural way (cf., for example, Rawnsley [4]) and carries a one-parameter family of non homothetic invariant metrics, all of them compatible with the complex structure; among these ones there are two specially important, the Kähler metric and the naturally reductive one. The metric induced in $\mathcal{F}_k(E)$ by the ambient space is of course one of these invariant metrics but it is not any of the two we have referred.

Although the embedding that we denote by $\mathcal{F}_k(E)$ has been considered by several authors, it seems that its extrinsic geometry has not received sufficient attention. In section 3 we will show that this manifold, with its complex structure (J_λ), has some interesting extrinsic properties: it is minimal inside a sphere and its second fundamental form h_λ verifies $h_\lambda \circ (J_\lambda \times J_\lambda) = h_\lambda$, a relation that is opposed to the usual condition of circularity or pluriharmonicity and that is referred as "$(2,0)$-geodesic" in Rigoli & Tribuzy [5]. Moreover, and although this has of course an intrinsic nature, we remark also that $\mathcal{F}_k(E)$ has a traceless covariant derivative of the complex structure, in other words, it is a semi-Kähler manifold in the sense of Gray [2] or cosymplectic as in Salamon [6].

We obtain in section 4 a new family of equivariant embeddings of our manifold, now into the orthogonal group, by taking the image $\mathcal{G}_{kt}(E)$ of the homothetic manifold $t\mathcal{F}_k(E)$ by the exponential map of this group, for each real $t \neq 0 \bmod \pi$. In fact, for each such t, the map $\mathcal{F}_k(E) \to \mathcal{G}_{kt}(E)$, $\lambda \mapsto \exp(t\lambda)$, is a non homothetic equivariant diffeomorphism that can be easily described explicitly. The complex structure of $\mathcal{F}_k(E)$ induces a complex structure on each $\mathcal{G}_{kt}(E)$ that is compatible with the Riemannian metric that comes from $O(E)$, a metric that belongs to the one-parameter family referred above. The metrics of the one-parameter family that are associated to each one of the manifolds $\mathcal{G}_{kt}(E)$, modulo homothety, constitute a one side unbounded interval and include the two most important ones, the naturally reductive metric, for $t = \pm\frac{2\pi}{3} \bmod 2\pi$, and the Kähler metric, for $t = \pm\frac{\pi}{2} \bmod 2\pi$. Similarly to $\mathcal{F}_k(E)$, each manifold $\mathcal{G}_{kt}(E)$ is semi-Kähler and $(2,0)$-geodesic inside $O(E)$ (not as a submanifold of the whole Euclidean space $L(E;E)$). We constat the existence here of some phenomena of minimality: For each value of n and k, such that $n \geq 2k > 2$, $\mathcal{G}_{kt}(E)$ is minimal inside $O(E)$, when $\cos(t) = -\frac{n-2k}{n-k-1}$, and $\mathcal{G}_{kt}(E)$ is minimal inside a sphere centered at $a\,Id_E$, when $\cos(t) = -\frac{k+1}{2k}$ and $a = \frac{n-3k-1}{n}$. We remark also that, when $n = 3k-1$, the values of t such that $\mathcal{G}_{kt}(E)$ is minimal inside $O(E)$ are the ones that correspond to the naturally reductive metric.

2 – Notations and prerequisites

In this section we will fix some notations and recall some well known facts that we will use latter.

If E and F are Euclidean or Hermitian spaces, we will consider in the vector space $L(E;F)$, of all linear maps $E \to F$, the Hilbert–Schmidt inner product. If $\lambda \in L(E;F)$, the adjoint linear map of λ will be denoted by λ^*. The vector subspaces of $L(E;E)$ whose elements are the self-adjoint linear maps and the skew-adjoint ones will be denoted by $L_{sa}(E;E)$ and $L_{-sa}(E;E)$, respectively.

Let E be a Euclidean space. The intrinsic geometry of the orthogonal group $O(E) \subset L(E;E)$, with its natural biinvariant metric, can be deduced from the extrinsic one; in fact, this biinvariant metric is precisely the one that is induced on $O(E)$ as a Riemannian submanifold of $L(E;E)$. We will use several times the fact that, for each $\xi \in O(E)$, the orthogonal projection from $L(E;E)$ onto the tangent space $T_\xi(O(E))$ is given by $\pi_\xi(\alpha) = \frac{\alpha - \xi\circ\alpha^*\circ\xi}{2}$.

We will now recall a well known isometric embedding of the Grassmann manifold into $L_{sa}(E;E)$ (cf., for example, Ferus [1]). The set $\mathcal{G}r_k(E) \subset L_{sa}(E;E)$ $(0 \leq k \leq \dim(E))$,

$$\mathcal{G}r_k(E) = \Big\{\lambda \in L_{sa}(E;E) \;|\;\; \lambda^2 = \lambda,\; \operatorname{Tr}(\lambda) = k\Big\}\;,$$

can be regarded as a model for the Grassmannian of the k-dimensional vector subspaces of E, by means of the identification of each vector subspace F with the orthogonal projection π_F from E onto F, and, for each $\lambda \in \mathcal{G}r_k(E)$, the tangent vector space is

$$T_\lambda(\mathcal{G}r_k(E)) = \Big\{\alpha \in L_{sa}(E;E) \mid \ \alpha\circ\lambda + \lambda\circ\alpha = \alpha\Big\} \ .$$

When E is a Hermitian vector space, the Grassmannian $\mathcal{G}r_k(E)$ is only a real submanifold of $L(E;E)$ but it has a natural complex structure, namely the one for which the action of the general linear group $GL(E)$ is holomorphic. This complex structure is defined by the structure linear maps $J_\lambda\colon\ T_\lambda(\mathcal{G}r_k(E)) \to T_\lambda(\mathcal{G}r_k(E))$, $J_\lambda(\alpha) = i\,\alpha(2\lambda - Id)$.

3 – The first embedding of the space of partially complex structures

Let E be a Euclidean n-dimensional space. By a *partially complex structure* on E we mean a couple (F,J), where $F \subset E$ is an even dimensional vector subspace and $J:\ F \to F$ is a complex structure, compatible with the inner product (cf. Salamon [6]). This compatibility can be characterized by the fact that $J\colon\ F \to F$ is an orthogonal isomorphism or, equivalently, by the equality $J^* = -J$. We will associate, to each partially complex structure (F,J), the linear map $\lambda \in L(E;E)$ that equals J on F and vanishes on $F^\perp$, in other words, the one that has matrix $\begin{bmatrix} J & 0\\ 0 & 0\end{bmatrix}$ (here and henceforth, whenever a vector subspace $F \subset E$ is implicitly associated to a situation, we will assume that a matrix of linear maps must be interpreted in terms of the orthogonal direct sum $E = F \oplus F^\perp$). Then λ, like J, is skew-adjoint and $\lambda^3 = -\lambda$; in fact $-\lambda^2$ is the orthogonal projection onto F and J is the restriction of λ to F. Reciprocally, given $\lambda \in L(E;E)$ such that $\lambda^3 = -\lambda$ and λ is skew-adjoint, then $-\lambda^2$ is self-adjoint and idempotent, hence the orthogonal projection onto a vector subspace $F \subset E$, and the restriction J, of λ to F, maps F onto F and is easily seen to be a complex structure on F such that λ corresponds to the couple (F,J). This allows us to identify the set of partially complex structures with the subset $\mathcal{F}(E) \subset L(E;E)$,

$$\mathcal{F}(E) = \Big\{\lambda \in L(E;E) \mid \ \lambda^3 = -\lambda,\ \lambda^* = -\lambda\Big\} \ ,$$

whose elements are the compatible f-structures in the sense of Yano [9], and the set of those whose associated vector subspace has real dimension $2k$ with

$$\mathcal{F}_k(E) = \Big\{\lambda \in L(E;E) \mid \ \lambda^3 = -\lambda,\ \lambda^* = -\lambda,\ \mathrm{Tr}(\lambda^2) = -2k\Big\} \ ,$$

that is of course open and closed in $\mathcal{F}(E)$. There is a natural transitive action of the Lie group $O(E)$ on each $\mathcal{F}_k(E)$, that associates, to each $\xi \in O(E)$ and to each linear map λ, corresponding to the couple (F,J), the linear map λ' that corresponds to the couple (F',J'), such that $F'=\xi(F)$ and $\xi_{/F}: F\to F'$ is complex linear, with respect to the complex structures J and J'; in fact, as it is readily seen, we have $\lambda'=\xi\,\lambda\,\xi^*$ (from now on, we will often omit the composition sign). The fact that this transitive action is smooth and the compactness of $O(E)$ allow us to deduce:

Proposition 1. *For each integer k such that $0\le 2k\le n$, $\mathcal{F}_k(E)$ is a compact submanifold of $L(E;E)$.*

Each $\mathcal{F}_k(E)$ is open and closed in the union $\mathcal{F}(E)$, that is hence also a compact manifold. In fact, the $\mathcal{F}_k(E)$ with $2k<n$ are connected components of $\mathcal{F}(E)$, because the action of the connected subgroup $SO(E)$ on each of them is also transitive, and, by a similar reason, if $0<2k=n$, $\mathcal{F}_k(E)$ is the union of two connected components.

Proposition 2. *The manifold $\mathcal{F}_k(E)$ has dimension $2nk-3k^2-k$ and, for each $\lambda\in\mathcal{F}_k(E)$, corresponding to the couple (F,J), the elements of the tangent vector space $T_\lambda(\mathcal{F}_k(E))$ are the linear maps $\alpha\in L(E;E)$ that verify any of the following two equivalent conditions:*

a) $\alpha^*=-\alpha$ *and* $\alpha\,\lambda^2+\lambda\,\alpha\,\lambda+\lambda^2\,\alpha=-\alpha$;

b) *The matrix of α is $\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & 0\end{bmatrix}$, with $\alpha_{1,1}\colon F\to F$ skew-adjoint and anti-linear (with respect to J) and $\alpha_{1,2}=-\alpha^*_{2,1}$.*

Proof: The isotropy subgroup at λ is the set of elements $\xi\in O(E)$ whose matrix is $\begin{bmatrix}\xi_{1,1} & 0\\ 0 & \xi_{2,2}\end{bmatrix}$, with $\xi_{1,1}: F\to F$ unitary, with respect to J, and $\xi_{2,2}: F^\perp\to F^\perp$ orthogonal; its dimension is hence $k^2+\frac{(n-2k)\,(n-2k-1)}{2}$. The dimension of the orthogonal group being $\frac{n(n-1)}{2}$, we deduce that the dimension of the manifold $\mathcal{F}_k(E)$ is equal to $2nk-3k^2-k$. The fact that each $\lambda\in\mathcal{F}_k(E)$ verifies $\lambda^*=-\lambda$ and $\lambda^3=-\lambda$ implies, by differentiation that each $\alpha\in T_\lambda(\mathcal{F}_k(E))$ verifies condition a). If $\alpha\in L(E;E)$ verifies condition a) and has matrix $\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & \alpha_{2,2}\end{bmatrix}$, then one deduces, from the first equality, that $\alpha_{1,1}$ and $\alpha_{2,2}$ are skew-symmetric and that $\alpha_{1,2}=-\alpha^*_{2,1}$ and, from the second one, that $\alpha_{2,2}=0$ and $-\alpha_{1,1}+J\,\alpha_{1,1}\,J=0$, hence $J\,\alpha_{1,1}=-\alpha_{1,1}\,J$, so that we have condition b). We remark now that the space of linear maps $\alpha_{1,1}\colon F\to F$ that are skew-adjoint and anti-linear has dimension k^2-k, because the space of all skew-adjoint linear maps has dimension $2k^2 - k$ and the space of those that are skew-adjoint and $\mathbb{C}$-linear has dimension k^2. Hence the dimension of the space of linear maps $\alpha\colon E \to E$ that verify condition b) is $2nk - 3k^2 - k$, the same as the dimension of the tangent space $T_\lambda(\mathcal{F}_k(E))$ and this terminates our proof. ∎

It is well known that $\mathcal{F}(E)$ is diffeomorphic to the closed complex submanifold of the complex Grassmannian $\mathcal{G}r(E_\mathbb{C})$, of the complexified $E_\mathbb{C}$ of E (with the associated Hermitian inner product), whose elements correspond to the complex vector subspaces that are orthogonal to their conjugates (the isotropic subspaces) and that, as such, $\mathcal{F}(E)$ inherits a complex structure, that is invariant under the action of $O(E)$ (Rawnsley [4], Proposition 2.1). This diffeomorphism Ψ associates, to each $\lambda \in \mathcal{F}(E)$, corresponding to the couple (F, J), the orthogonal projection from $E_\mathbb{C}$ onto the i eigenspace of the complex linear extension $J : F_\mathbb{C} \to F_\mathbb{C}$. For our purposes, it will be enough to exhibit Ψ as a smooth immersion from $\mathcal{F}(E)$ into $\mathcal{G}r(E_\mathbb{C})$, in order to characterize the linear maps $J_\lambda\colon T_\lambda(\mathcal{F}(E)) \to T_\lambda(\mathcal{F}(E))$ that define the complex structure of $\mathcal{F}(E)$.

Proposition 3. *The map $\Psi\colon \mathcal{F}(E) \to \mathcal{G}r(E_\mathbb{C})$ is defined by $\Psi(\lambda) = -\frac{1}{2}(\lambda^2 + i\lambda)$, where, as usual, we use the same letter to denote a linear map $E \to E$ and its complex linear extension $E_\mathbb{C} \to E_\mathbb{C}$. The map Ψ is hence a smooth immersion.*

Proof: Let $\lambda \in \mathcal{F}(E)$ correspond to the couple (F, J). The fact that the complex structure J is compatible with the inner product of E implies that the complexified $F_\mathbb{C}$ is the orthogonal direct sum of the subspaces $F'_\mathbb{C}$ and $F''_\mathbb{C}$, whose elements are respectively the i eigenvectors and the $-i$ eigenvectors of the complex linear extension $J\colon F_\mathbb{C} \to F_\mathbb{C}$. Hence, the orthogonal projection from $F_\mathbb{C}$ onto $F'_\mathbb{C}$ is $\frac{1}{2}(Id - i\,J)$ and the fact that $-\lambda^2\colon E_\mathbb{C} \to F_\mathbb{C}$ is the orthogonal projection form $E_\mathbb{C}$ onto $F_\mathbb{C}$ implies that the orthogonal projection of $E_\mathbb{C}$ onto $F'_\mathbb{C}$ is $-\frac{1}{2}(Id - i\,J)\,\lambda^2$ *i.e.*, $-\frac{1}{2}(\lambda^2 + i\,\lambda)$, because $J\lambda^2 = -\lambda$, whence our characterization of $\Psi(\lambda)$. By differentiation we obtain $D\Psi_\lambda(\alpha) = -\frac{1}{2}(\alpha\,\lambda + \lambda\,\alpha + i\,\alpha)$, and, looking to the imaginary part of the second member, we deduce that Ψ is an immersion. ∎

Proposition 4. *For each $\lambda \in \mathcal{F}(E)$, corresponding to the couple (F, J), there exists a complex structure J_λ of the tangent vector space $T_\lambda(\mathcal{F}(E))$, defined by $J_\lambda(\alpha) = (Id_E + \lambda^2)\,\alpha\,\lambda - \lambda\,\alpha$ or, in matricial terms,*

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & 0 \end{bmatrix} \longmapsto \begin{bmatrix} -J\,\alpha_{1,1} & -J\,\alpha_{1,2} \\ \alpha_{2,1}\,J & 0 \end{bmatrix} = \begin{bmatrix} \alpha_{1,1}\,J & -J\,\alpha_{1,2} \\ \alpha_{2,1}\,J & 0 \end{bmatrix}.$$

The family (J_λ) defines $\mathcal{F}(E)$ as a complex manifold and is, in fact, the only one that makes the smooth immersion $\Psi : \mathcal{F}(E) \to \mathcal{G}r(E_\mathbb{C})$ holomorphic. The *complex structure on $\mathcal{F}(E)$ is compatible with the inner product that comes from the ambient space $L_{sa}(E;E)$ and with the action of $O(E)$.*

Proof: Recalling the matricial characterization of the tangent vector space in Proposition 2 and the fact that $J^* = -J$, we deduce that the matricial characterization of J_λ defines indeed a linear map $J_\lambda\colon\ T_\lambda(\mathcal{F}(E)) \to T_\lambda(\mathcal{F}(E))$ and that $J_\lambda \circ J_\lambda = -Id$. The intrinsic characterization of $J_\lambda(\alpha)$ is a simple consequence of the matricial one. We have hence an almost complex structure in $\mathcal{F}(E)$ and, in order to conclude that it is indeed a complex structure, it will be enough to prove that the immersion Ψ is holomorphic. In order to do that, we use matrices relative to the direct sums $E = F \oplus F^\perp$ and $E_{\mathbb{C}} = F_{\mathbb{C}} \oplus F_{\mathbb{C}}^\perp$, as well as the intrinsic characterization of the complex structure of the Grassmann manifold in section 2: If $\alpha \in T_\lambda(\mathcal{F}(E))$ has matrix $\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & 0 \end{bmatrix}$, then $D\Psi_\lambda(\alpha) = -\frac{1}{2}(\alpha\,\lambda + \lambda\,\alpha + i\,\alpha)$ has matrix $-\frac{1}{2}\begin{bmatrix} i\alpha_{1,1} & J\,\alpha_{1,2} + i\,\alpha_{1,2} \\ \alpha_{2,1}\,J + i\,\alpha_{2,1} & 0 \end{bmatrix}$ and $\Psi(\lambda) = -\frac{1}{2}(\lambda^2 + i\,\lambda)$ and $2\,\Psi(\lambda) - Id$ have matrices $\frac{1}{2}\begin{bmatrix} Id - i\,J & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -i\,J & 0 \\ 0 & -Id \end{bmatrix}$ so that $J_{\Psi(\lambda)}(D\Psi_\lambda(\alpha)) = i\,D\Psi_\lambda(\alpha) \circ (2\,\Psi(\lambda) - Id)$ has matrix $-\frac{1}{2}\begin{bmatrix} i\,\alpha_{1,1}\,J & -i\,J\,\alpha_{1,2} + \alpha_{1,2} \\ -\alpha_{2,1} + i\,\alpha_{2,1}\,J & 0 \end{bmatrix}$, that is also the matrix of $D\Psi_\lambda(J_\lambda(\alpha))$, hence $J_{\Psi(\lambda)}(D\Psi_\lambda(\alpha)) = D\Psi_\lambda(J_\lambda(\alpha))$, as we want. The fact that the complex structure is compatible with the inner product that comes from the ambient space follows easily from its matricial characterization and its compatibility with the action of $O(E)$ comes from its intrinsic characterization. ∎

Let us remark that some authors (cf. Rigoli & Tribuzy [5]) use the $-i$ eigenspace of the complex extension of J, instead of the i eigenspace, and obtain hence the conjugate complex structure in $\mathcal{F}(E)$.

It is well known that there are two specially interesting Riemannian metrics in $\mathcal{F}(E)$: The first one, the *naturally reductive metric*, can be characterized by the condition that, for each fixed element $\lambda \in \mathcal{F}(E)$, the map $R_\lambda\colon\ O(E) \to \mathcal{F}(E)$, $\xi \mapsto \xi\,\lambda\,\xi^*$, should be a Riemannian submersion; the second one, the *Kähler metric*, can be obtained from the metric of the complex Grassmann manifold $\mathcal{G}r(E_{\mathbb{C}})$ by the requirement that $\Psi :\ \mathcal{F}(E) \to \mathcal{G}r(E_{\mathbb{C}})$ should be an isometric immersion. Unfortunately, with the exception of some degenerate cases, the Riemannian metric in $\mathcal{F}(E)$ that comes from the ambient space $L_{sa}(E;E)$ is neither of these two, even modulo homothety. Let, in fact $\lambda \in \mathcal{F}(E)$ correspond to the couple (F, J) and let us use the corresponding direct sum $E = F \oplus F^\perp$

to exhibit matricial characterizations. We have then, in what concerns each of these metrics:

The linear map $D(R_\lambda)_{Id}: L_{-sa}(E;E) \to T_\lambda(\mathcal{F}(E))$ applies each α, with matrix $\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix}$, into $\alpha\,\lambda - \lambda\,\alpha$, with matrix $\begin{bmatrix} \alpha_{1,1}\,J - J\,\alpha_{1,1} & -J\,\alpha_{1,2} \\ \alpha_{2,1}\,J & 0 \end{bmatrix}$, so that its kernel $\mathcal{H}_\lambda$ is the set of such α with $\alpha_{1,1}$ a $\mathbb{C}$-linear map, $\alpha_{1,2} = 0$ and $\alpha_{2,1} = 0$, and the orthogonal complement $\mathcal{M}_\lambda$ of $\mathcal{H}_\lambda$ is the set of skew-adjoint linear maps α whose matrix is $\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & 0 \end{bmatrix}$, with $\alpha_{1,1}$ anti-linear. The image of such $\alpha \in \mathcal{M}_\lambda$ has matrix $\begin{bmatrix} 2\,\alpha_{1,1}\,J & -J\,\alpha_{1,2} \\ \alpha_{2,1}\,J & 0 \end{bmatrix}$, so that there are elements in $\mathcal{M}_\lambda$ whose image have the norm multiplied by different positive constants.

We have seen, in the course of the proof of Proposition 4, how the linear maps $D\Psi_\lambda\colon T_\lambda(\mathcal{F}(E)) \to T_{\Psi(\lambda)}(\mathcal{G}r(E_\mathbb{C}))$ are defined in matricial terms and this shows, as before, that there are elements in $T_\lambda(\mathcal{F}(E))$ whose image have the norm multiplied by different positive constants.

Another way to verify that the induced metric in $\mathcal{F}(E)$ is not a Kähler metric is to compute the covariant derivative of the complex structure. The formula that we will obtain for the referred covariant derivative and the computation of the second fundamental form of $\mathcal{F}(E)$ inside $L_{-sa}(E;E)$ will show the existence, after all, of some relationships between the metric and complex structures.

First of all, the matricial characterization of the tangent vector spaces obtained in Proposition 2 implies readily that, for each $\lambda \in \mathcal{F}(E)$, corresponding to the couple (F,J), the orthogonal projection $\pi_\lambda: L_{-sa}(E;E) \to T_\lambda(\mathcal{F}(E))$ associates, to each $\beta \in L_{-sa}(E;E)$ with matrix $\begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{bmatrix}$, the element with matrix $\begin{bmatrix} \frac{1}{2}(\beta_{1,1} + J\,\beta_{1,1}\,J) & \beta_{1,2} \\ \beta_{2,1} & 0 \end{bmatrix}$ and this allows us to deduce the following intrinsic characterization of π_λ:

Proposition 5. *If $\lambda \in \mathcal{F}(E)$, the orthogonal projection $\pi_\lambda\colon L_{-sa}(E;E) \to T_\lambda(\mathcal{F}(E))$ is defined by $\pi_\lambda(\beta) = \frac{1}{2}(\lambda\,\beta\,\lambda - 3\,\lambda^2\,\beta\,\lambda^2 - 2\,\lambda^2\,\beta - 2\,\beta\,\lambda^2)$.*

The second fundamental form $h_\lambda\colon T_\lambda(\mathcal{F}(E)) \times T_\lambda(\mathcal{F}(E)) \to T_\lambda(\mathcal{F}(E))^\perp$ can be computed through the use of the formula $h_\lambda(\alpha,\beta) = D\pi_\lambda(\alpha)(\beta)$. This leads to the intrinsic characterization of $h_\lambda(\alpha,\beta)$ in next proposition, from which the matricial one follows through a long, but straightforward, computation.

Proposition 6. *The second fundamental form of $\mathcal{F}(E)$ inside $L_{-sa}(E;E)$ is* *defined by*

$$h_\lambda(\alpha,\beta) = \frac{1}{2}\Big(\alpha\,\beta\,\lambda + \lambda\,\beta\,\alpha - 3\,\alpha\,\lambda\,\beta\,\lambda^2 - 3\,\lambda\,\alpha\,\beta\,\lambda^2 - 3\,\lambda^2\,\beta\,\alpha\,\lambda - 3\,\lambda^2\,\beta\,\lambda\,\alpha - \\ - 2\,\alpha\,\lambda\,\beta - 2\,\lambda\,\alpha\,\beta - 2\,\beta\,\alpha\,\lambda - 2\,\beta\,\lambda\,\alpha\Big)$$

and, in matricial terms,

$$\left(\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & 0\end{bmatrix},\ \begin{bmatrix}\beta_{1,1} & \beta_{1,2}\\ \beta_{2,1} & 0\end{bmatrix}\right) \longmapsto \begin{bmatrix}\mu_{1,1} & 0\\ 0 & \mu_{2,2}\end{bmatrix},$$

$$\mu_{1,1} = \frac{1}{2}\Big(-\alpha_{1,1}\,J\,\beta_{1,1} - \beta_{1,1}\,J\,\alpha_{1,1} + \alpha_{1,2}\,\beta_{2,1}\,J + J\,\beta_{1,2}\,\alpha_{2,1} + \\ + J\,\alpha_{1,2}\,\beta_{2,1} + \beta_{1,2}\,\alpha_{2,1}\,J\Big)\ ,$$

$$\mu_{2,2} = -\alpha_{2,1}\,J\,\beta_{1,2} - \beta_{2,1}\,J\,\alpha_{1,2}\ .$$

As horrible as the previous formulas may seem, they allow us, nevertheless, to conclude:

Proposition 7. *The second fundamental form h_λ of $\mathcal{F}(E)$ verifies the condition*

$$h_\lambda\Big(J_\lambda(\alpha),\, J_\lambda(\beta)\Big) = h_\lambda(\alpha,\beta)\ .$$

Proof: Use the matricial formulas for h_λ and J_λ in Propositions 6 and 4. ∎

Let us remark that the previous condition, that is the opposite of the usual condition of circularity or pluriharmonicity, translates the fact that the embedding of $\mathcal{F}(E)$ into $L_{-sa}(E;E)$ is $(2,0)$-geodesic, in the sense of Rigoli & Tribuzy [5]. This condition plays also an important role in Ferus [1], where it is referred as "equation (4)". It is also straightforward to compute the trace of the second fundamental form from its matricial characterization.

Proposition 8. *For each $\lambda\in\mathcal{F}(E)$, corresponding to the couple (F,J), the trace of the second fundamental form $h_\lambda\colon\ T_\lambda(\mathcal{F}(E))\times T_\lambda(\mathcal{F}(E))\to T_\lambda(\mathcal{F}(E))^\perp$ is $-\frac{2n-3k-1}{2}\,\lambda$. Hence, the manifold $\mathcal{F}_k(E)$ is minimal on the sphere centered on 0 and with radius $\sqrt{2k}$.*

Proposition 9. *Let $\lambda\in\mathcal{F}(E)$ correspond to the couple (F,J). The covariant derivative $\nabla J_\lambda\colon\ T_\lambda(\mathcal{F}(E))\to L(T_\lambda(\mathcal{F}(E));T_\lambda(\mathcal{F}(E)))$ of the complex structure* *of $\mathcal{F}(E)$ is defined in matricial terms by*

$$\left(\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & 0\end{bmatrix}, \begin{bmatrix}\beta_{1,1} & \beta_{1,2}\\ \beta_{2,1} & 0\end{bmatrix}\right) \longmapsto \begin{bmatrix}0 & -\alpha_{1,1}\,\beta_{1,2}\\ \beta_{2,1}\,\alpha_{1,1} & 0\end{bmatrix}.$$

Proof: For each $\lambda \in \mathcal{F}(E)$, let $\overline{J}_\lambda\colon\, L_{-sa}(E;E) \to L_{-sa}(E;E)$ be the linear map defined by $\overline{J}_\lambda(\beta) = (Id_E + \lambda^2)\,\beta\,\lambda - \lambda\,\beta$, a linear map that extends the complex structure $J_\lambda\colon\, T_\lambda(\mathcal{F}(E)) \to T_\lambda(\mathcal{F}(E))$, by the intrinsic characterization of this one given in Proposition 4. In matricial terms, $\overline{J}_\lambda$ is defined by $\begin{bmatrix}\beta_{1,1} & \beta_{1,2}\\ \beta_{2,1} & \beta_{2,2}\end{bmatrix} \mapsto \begin{bmatrix}-J\,\beta_{1,1} & -J\,\beta_{1,2}\\ \beta_{2,1}\,J & 0\end{bmatrix}$ and, differentiating the intrinsic formula, $D\overline{J}_\lambda(\alpha)(\beta) = (\alpha\,\lambda + \lambda\,\alpha)\,\beta\,\lambda + (Id + \lambda^2)\,\beta\,\alpha - \alpha\,\beta$, whence, if $\alpha, \beta \in T_\lambda(\mathcal{F}(E))$ have matrices $\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & 0\end{bmatrix}$ and $\begin{bmatrix}\beta_{1,1} & \beta_{1,2}\\ \beta_{2,1} & 0\end{bmatrix}$, then $D\overline{J}_\lambda(\alpha)(\beta)$ has matrix

$$\begin{bmatrix}J\,\alpha_{1,2}\,\alpha_{2,1}\,J - \alpha_{1,1}\,\beta_{1,1} - \alpha_{1,2}\,\beta_{2,1} & -\alpha_{1,1}\,\beta_{1,2}\\ \beta_{2,1}\,\alpha_{1,1} & \beta_{2,1}\,\alpha_{1,2} - \alpha_{2,1}\,\beta_{1,2}\end{bmatrix}$$

(recall that $\alpha_{1,1}$ and $\beta_{1,1}$ are anti-linear, with respect to J). Using the formula

$$\nabla J_\lambda(\alpha)(\beta) = D\overline{J}_\lambda(\alpha)(\beta) + \overline{J}_\lambda(h_\lambda(\alpha,\beta)) - h_\lambda(\alpha, J_\lambda(\beta))$$

as well as the matricial characterizations of $\overline{J}_\lambda$ and h_λ, one obtains the stated matricial formula for $\nabla J_\lambda(\alpha)(\beta)$. ∎

One of the consequences of the preceding formula is that we have $\nabla J_\lambda(\alpha)(\alpha) = 0$, when the matrix of $\alpha \in T_\lambda(\mathcal{F}(E))$ has one of the following forms $\begin{bmatrix}\alpha_{1,1} & 0\\ 0 & 0\end{bmatrix}$ or $\begin{bmatrix}0 & \alpha_{1,2}\\ \alpha_{2,1} & 0\end{bmatrix}$. The fact that, as one verifies easily, $T_\lambda(\mathcal{F}(E))$ has an orthonormal basis with each element of one of these forms implies:

Proposition 10. *The bilinear map $T_\lambda(\mathcal{F}(E)) \times T_\lambda(\mathcal{F}(E)) \to T_\lambda(\mathcal{F}(E))$ defined by $(\alpha,\beta) \mapsto \nabla J_\lambda(\alpha)(\beta)$ is traceless, hence $\mathcal{F}(E)$ is a semi-Kähler manifold (Gray [2]).*

Proof: The condition that the above bilinear map is traceless is known to be equivalent to the fact that the Kähler form is coclosed (this is essentially [2], formula 4.6). ∎

4 – A family of embeddings into the orthogonal group

Let again E be a Euclidean n-dimensional space. In section 3 we embedded the space of partially complex structures into the vector space of skew-adjoint maps $E \to E$, obtaining as image a submanifold $\mathcal{F}(E) \subset L_{-sa}(E;E)$. We will verify now that, by taking the image of the homothetic copies $t\,\mathcal{F}(E)$ by the exponential map of the orthogonal group, we obtain a family of non isometric embeddings into $O(E)$, whose induced metrics include the Kähler metric and the naturally reductive one.

For each partially complex structure (F, J) and each $t \in \mathbb{R}$, we will denote $J_t \colon F \to F$ the orthogonal isomorphism defined by

$$J_t = \cos(t)\, Id_F + \sin(t)\, J \ ,$$

whose adjoint map, $J_t^* = \cos(t)\, Id_F - \sin(t)\, J$, is the inverse of J_t, and we will denote $\xi_t :\ E \to E$ the orthogonal isomorphism whose matrix, relative to the orthogonal decomposition $E = F \oplus F^\perp$ is $\begin{bmatrix} J_t & 0 \\ 0 & Id_{F^\perp} \end{bmatrix}$. The fact that, by differentiation, we obtain $J'_t = J_t \circ J$ implies that $J_t = \exp(t\,J)$ so that, in particular, $J_{s+t} = J_s \circ J_t$. We will also use often, without further reference, the formulas $J_t + J_t^* = 2\cos(t)\, Id_F$, $J_t - J_t^* = 2\sin(t)\, J$. In the same spirit, if $\lambda \in \mathcal{F}(E)$, with matrix $\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$, is the element corresponding to the couple (F, J), we verify readily that $\xi_t = Id_E + \sin(t)\,\lambda + (1-\cos(t))\,\lambda^2$, a formula that implies, by differentiation, that $\xi'_t = \xi_t \circ \lambda$ so that $\xi_t = \exp(t\,\lambda)$.

Proposition 11. *For each $t \in \mathbb{R}$, such that $\sin(t) \neq 0$, there exists a diffeomorphism Λ_t from $\mathcal{F}(E)$ onto a submanifold $\mathcal{G}_t(E) \subset O(E)$ defined by*

$$\Lambda_t(\lambda) = \exp(t\,\lambda) = \xi_t = Id_E + \sin(t)\,\lambda + (1-\cos(t))\,\lambda^2 \ ,$$

whose inverse $\Lambda_t^{-1} \colon \mathcal{G}_t(E) \to \mathcal{F}(E)$ is defined by $\Lambda_t^{-1}(\xi) = \frac{\xi - \xi^*}{2\sin(t)}$.

Proof: The matricial characterization of $\Lambda_t(\lambda) = \xi_t$ implies that, if we denote $\mathcal{G}_t(E)$ the image of the map $\Lambda_t \colon \mathcal{F}(E) \to O(E)$, we have a well defined smooth map $\mathcal{G}_t(E) \to \mathcal{F}(E)$ defined by $\xi \mapsto \frac{1}{2\sin(t)}(\xi - \xi^*)$ and that this map is a left inverse to Λ_t. ∎

We will denote $\mathcal{G}_{kt}(E)$ the subset of $\mathcal{G}_t(E)$ whose elements correspond to the couples (F, J) such that F has real dimension $2k$; each $\mathcal{G}_{kt}(E)$ is open and closed in $\mathcal{G}_t(E)$ and, by restriction of Λ_t, we obtain diffeomorphisms $\mathcal{F}_k(E) \to \mathcal{G}_{kt}(E)$, so that $\mathcal{G}_{kt}(E)$, like $\mathcal{F}_k(E)$, is a compact submanifold with dimension $2nk - 3k^2 - k$.

The formula for Λ_t shows that it is equivariant so that, as with $\mathcal{F}_k(E)$, the natural transitive action of $O(E)$ in each $\mathcal{G}_{kt}(E)$ is the action by conjugation. We will now identify intrinsically the elements of $\mathcal{G}_{kt}(E)$:

Proposition 12. *We have*

$$\begin{aligned}\mathcal{G}_t(E) &= \Big\{\xi \in O(E) \mid \ \xi^2 - \xi^* = (2\cos(t)+1)\,(\xi - Id_E)\Big\} \\ &= \Big\{\xi \in O(E) \mid \ \xi^3 - Id_E = (2\cos(t)+1)\,(\xi^2 - \xi)\Big\}\end{aligned}$$

and $\mathcal{G}_{kt}(E)$ *is the subset of* $\mathcal{G}_t(E)$ *whose elements have trace* $n - 2k\,(1-\cos(t))$.

Proof: The equivalence between the two conditions comes by composing both members of the equalities either with ξ or with ξ^*. The fact that each $\xi \in \mathcal{G}_{kt}(E)$ belongs to $O(E)$, verifies $\xi^2 - \xi^* = (2\cos(t)+1)\,(\xi - Id_E)$ and has trace $n-2k(1-\cos(t))$ is a trivial consequence of the matricial characterization of the linear map ξ that corresponds to a couple (F, J), if we recall that a skew-adjoint map is traceless. Let us assume now that $\xi \in O(E)$ verifies $\xi^2 - \xi^* = (2\cos(t)+1)\,(\xi - Id_E)$. Taking the adjoint to both members we obtain $\xi^{*2} - \xi = (1+2\cos(t))\,(\xi^* - Id_E)$ and defining then $\lambda = \frac{1}{2\sin(t)}\,(\xi - \xi^*)$ we deduce that $\lambda^* = -\lambda$ and $\lambda^3 = -\lambda$, so that $\lambda \in \mathcal{F}(E)$; computing then $\Lambda_t(\lambda) = Id + \sin(t)\,\lambda + (1-\cos(t))\,\lambda^2$, we obtain ξ, hence $\xi \in \mathcal{G}_t(E)$. ∎

A specially simple case is the one with $t = \pm\frac{2\pi}{3} \bmod 2\pi$; the manifold $\mathcal{G}_t(E)$ is just the set of cubic roots of the identity in the orthogonal group. Also, for $t = \pm\frac{\pi}{2} \bmod 2\pi$, another important case as will be seen later, the manifold $\mathcal{G}_t(E)$ is contained (in general strictly) in the set of fourth roots of the identity in this group.

Proposition 13. *Let* $\lambda \in \mathcal{F}(E)$ *correspond to the couple* (F, J) *and let* $\xi = \Lambda_t(\lambda) \in \mathcal{G}_t(E)$. *The isomorphism* $D(\Lambda_t)_\lambda \colon T_\lambda(\mathcal{F}(E)) \to T_\xi(\mathcal{G}_t(E))$ *is defined then by*

$$D(\Lambda_t)_\lambda(\alpha) = \sin(t)\,\alpha + (1-\cos(t))\,(\alpha\,\lambda + \lambda\,\alpha)$$

or, in matricial terms, by

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & 0 \end{bmatrix} \ \longmapsto\ 2\sin\!\Big(\frac{t}{2}\Big) \begin{bmatrix} \cos(\frac{t}{2})\,\alpha_{1,1} & J_{t/2}\,\alpha_{1,2} \\ \alpha_{2,1}\,J_{t/2} & 0 \end{bmatrix}.$$

Proof: The intrinsic formula comes readily by differentiation of $\Lambda_t(\lambda) = Id_E + \sin(t)\,\lambda + (1-\cos(t))\,\lambda^2$ and the matricial one follows if we recall the fact that $\alpha_{1,1}$ is anti-linear as well as the identity $\sin(t)\, Id_F + (1-\cos(t))\, J = 2\sin(\frac{t}{2})\cos(\frac{t}{2})\, Id_F + 2\sin^2(\frac{t}{2})\, J = 2\sin(\frac{t}{2})\, J_{t/2}$. ∎

Proposition 14. *Let $\xi \in \mathcal{G}_t(E)$ correspond to the couple (F, J). The tangent vector space $T_\xi(\mathcal{G}_t(E))$ is then the set of all $\beta \in L(E; E)$ whose matrix relative to the direct sum $E = F \oplus F^\perp$ has the form $\begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & 0 \end{bmatrix}$, with $\beta_{1,1} \colon F \to F$ anti-linear and skew-adjoint and $\beta_{1,2}^* = -\beta_{2,1}\, J_t^*$ (this last equality being equivalent to $\beta_{2,1}^* = -J_t^*\, \beta_{1,2}$).*

Proof: If $\beta \in T_\xi(\mathcal{G}_t(E))$, let α, with matrix $\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & 0 \end{bmatrix}$, be the element of $T_\lambda(\mathcal{F}(E))$ whose image by the isomorphism $D(\Lambda_t)_\lambda$ is β. Then $\beta_{1,1} = \sin(t)\, \alpha_{1,1}$ is anti-linear and skew-adjoint, because this happens to $\alpha_{1,1}$, and

$$\begin{aligned} \beta_{1,2}^* &= \left(2\sin\left(\frac{t}{2}\right) J_{t/2}\, \alpha_{1,2}\right)^* = -2\sin\left(\frac{t}{2}\right) \alpha_{2,1}\, J_{t/2}^* = \\ &= -2\sin\left(\frac{t}{2}\right) \alpha_{2,1}\, J_{t/2}\, J_t^* = -\beta_{2,1}\, J_t^* \;. \end{aligned}$$

All we have to remark now is that, if $2k$ is the real dimension of F, the fact that the vector space of all anti-linear and skew-adjoint maps $F \to F$ has real dimension $k^2 - k$ implies that the space of all linear maps $\beta \in L(E; E)$ whose matrix verify the conditions above has dimension $2nk - 3k^2 - k$, equal to the dimension of the manifold $\mathcal{G}_t(E)$ at ξ. ∎

Remark 1. The matricial characterization of the tangent vector space $T_\lambda(\mathcal{F}_k(E))$ referred in Proposition 2 implies that this space is the orthogonal direct sum of two vector subspaces, invariant by the isotropy subgroup of $O(E)$, namely the ones that correspond to matrices with each of the two forms $\begin{bmatrix} \alpha_{1,1} & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \alpha_{1,2} \\ \alpha_{2,1} & 0 \end{bmatrix}$. This fact implies the existence of a one-parameter family of (homothety classes of) invariant Riemannian metrics on $\mathcal{F}_k(E)$, namely those that are obtained by maintaining the orthogonality of the two subspaces and the inner product in the second one and multiplying by a positive factor the inner product in the first subspace. One of the consequences of the matricial characterization of the isomorphism $D(\Lambda_t)_\lambda$ in Proposition 13 is that the original metric in $\mathcal{F}_k(E)$ and the ones that are induced by the non isometric diffeomorphisms Λ_t are, roughly speaking, half the spectrum of these invariant ones, namely those that correspond to a multiplicative factor less or equal to one. It would be interesting to find other natural embeddings of $\mathcal{F}_k(E)$, inducing the other half of the spectrum.

Proposition 15. *When $t = \pm\frac{2\pi}{3} \bmod 2\pi$, the Riemannian metric induced in $\mathcal{G}_t(E)$ by the ambient space is the naturally reductive one, modulo homothety.*

Proof: Let $\xi \in \mathcal{G}_t(E)$ correspond to the couple (F, J) and let $\lambda \in \mathcal{F}(E)$ be the corresponding element. In section 3 we identified the orthogonal complement $\mathcal{M}_\lambda$ of the isotropy Lie algebra $\mathcal{H}_\lambda \subset o(E)$ as the set of skew-adjoint maps whose matrix has the form $\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & 0 \end{bmatrix}$, with $\alpha_{1,1}$ anti-linear and we showed that the image of such element in $T_\lambda(\mathcal{F}(E))$ had matrix $\begin{bmatrix} 2\alpha_{1,1}\,J & -J\,\alpha_{1,2} \\ \alpha_{2,1}\,J & 0 \end{bmatrix}$. By composition with the isomorphism $D(\Lambda_t)_\lambda \colon T_\lambda(\mathcal{F}(E)) \to T_\xi(\mathcal{G}_t(E))$ we obtain an isomorphism $\mathcal{M}_\xi = \mathcal{M}_\lambda \to T_\xi(\mathcal{G}_t(E))$ associating, to each α with matrix $\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & 0 \end{bmatrix}$, the element with matrix $2\sin(\frac{t}{2}) \begin{bmatrix} 2\cos(\frac{t}{2})\,\alpha_{1,1}\,J & -J_{t/2}\,J\alpha_{1,2} \\ \alpha_{2,1}\,J\,J_{t/2} & 0 \end{bmatrix}$, so that, recalling that J and $J_{t/2}$ are orthogonal isomorphisms, we conclude that this isomorphism is homothetic when $|\cos(\frac{t}{2})| = \frac{1}{2}$. ∎

Like $\mathcal{F}(E)$, the manifold $\mathcal{G}_t(E)$ is also a complex manifold in a natural way that may be characterized by the condition that the diffeomorphism $\Lambda_t \colon \mathcal{F}(E) \to \mathcal{G}_t(E)$ should be holomorphic. Using the matricial characterization of the derivative of this diffeomorphism in Proposition 13 and the matricial characterization of the structure linear maps of $\mathcal{F}(E)$ in Proposition 4, we deduce immediately the following matricial characterization of the structure linear maps of the complex manifold $\mathcal{G}_t(E)$:

Proposition 16. *For each $\xi \in \mathcal{G}_t(E)$, corresponding to the couple (F, J), the structure map $J_\xi \colon T_\xi(\mathcal{G}_t(E)) \to T_\xi(\mathcal{G}_t(E))$ is defined, in matricial terms, by*

$$\begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & 0 \end{bmatrix} \longmapsto \begin{bmatrix} -J\,\beta_{1,1} & -J\,\beta_{1,2} \\ \beta_{2,1}\,J & 0 \end{bmatrix} = \begin{bmatrix} \beta_{1,1}\,J & -J\,\beta_{1,2} \\ \beta_{2,1}\,J & 0 \end{bmatrix}.$$

By equivariance, we know that the complex structure of $\mathcal{G}_t(E)$ is invariant by the action of $O(E)$. The matricial characterization of the structure maps in the preceding proposition shows that this complex structure is also compatible with the Riemannian structure of $\mathcal{G}_t(E)$ that comes from the ambient space.

In order to calculate the second fundamental form of $\mathcal{G}_t(E)$ inside $L(E;E)$, we need a formula for the orthogonal projections onto the tangent vector spaces:

Proposition 17. *For each $\xi \in \mathcal{G}_t(E)$, corresponding to a couple (F, J), the orthogonal projection $\pi_\xi \colon L(E;E) \to T_\xi(\mathcal{G}_t(E))$ is defined, in matricial terms rel-*

ative to the orthogonal direct sum $E = F \oplus F^\perp$, *by* $\begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{bmatrix} \mapsto \begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & 0 \end{bmatrix}$, *where* $\beta_{1,1} = \frac{1}{4}(\gamma_{1,1} - \gamma_{1,1}^* + J\,\gamma_{1,1}\,J - J\,\gamma_{1,1}^*\,J)$, $\beta_{1,2} = \frac{1}{2}(\gamma_{1,2} - J_t\,\gamma_{2,1}^*)$, $\beta_{2,1} = \frac{1}{2}(\gamma_{2,1} - \gamma_{1,2}^*\,J_t)$.

Proof: The matricial characterization of $T_\xi(\mathcal{G}_t(E))$ in Proposition 14 implies easily that $\pi_\xi(\gamma)$ should have a matrix of the referred type, with $\beta_{1,1}$ the orthogonal projection of $\gamma_{1,1}$ onto the vector subspace of $L(F;F)$ whose elements are the skew-adjoint anti-linear maps and with the couple $(\beta_{1,2}, \beta_{2,1})$ the orthogonal projection of the couple $(\gamma_{1,2}, \gamma_{2,1})$ onto the vector subspace of $L(F^\perp;F) \times L(F;F^\perp)$ whose elements are the couples $(\beta_{1,2}, \beta_{2,1})$ with $\beta_{1,2}^* = -\beta_{2,1}\,J_t^*$. The fact that $T_\xi(\mathcal{G}_t(E)) \subset T_\xi(O(E))$ implies that $\pi_\xi(\gamma)$ is also the orthogonal projection onto $T_\xi(\mathcal{G}_t(E))$ of the orthogonal projection of γ onto $T_\xi(O(E))$, this last projection with matrix $\frac{1}{2}\begin{bmatrix} \gamma_{1,1} - J_t\,\gamma_{1,1}^*\,J_t & \gamma_{1,2} - J_t\,\gamma_{2,1}^* \\ \gamma_{2,1} - \gamma_{1,2}^*\,J_t & \gamma_{2,2} - \gamma_{2,2}^* \end{bmatrix}$, by the formula referred in section 2. The fact that we have already $(\gamma_{1,2} - J_t\,\gamma_{2,1}^*)^* = \gamma_{1,2}^* - \gamma_{2,1}\,J_t^* = -(\gamma_{2,1} - \gamma_{1,2}^*\,J_t)\,J_t^*$ implies now, by our initial remarks, the correctness of the stated formulas for $\beta_{1,2}$ and $\beta_{2,1}$. In what concerns $\beta_{1,1}$, we remark that the orthogonal projection of $\frac{1}{2}(\gamma_{1,1} - J_t\,\gamma_{1,1}^*\,J_t)$ onto the space of anti-linear maps is

$$\frac{1}{4}\Big((\gamma_{1,1} - J_t\,\gamma_{1,1}^*\,J_t) + J(\gamma_{1,1} - J_t\,\gamma_{1,1}^*\,J_t)\,J\Big) =$$

$$= \frac{1}{4}\Bigg(\gamma_{1,1} + J\,\gamma_{1,1}\,J - \Big(\cos(t)\,Id + \sin(t)\,J\Big)\,\gamma_{1,1}^*\,\Big(\cos(t)\,Id + \sin(t)\,J\Big) -$$

$$- \Big(\cos(t)\,J - \sin(t)\,Id\Big)\,\gamma_{1,1}^*\Big(\cos(t)\,J - \sin(t)\,Id\Big)\Bigg) =$$

$$= \frac{1}{4}\Big(\gamma_{1,1} - \gamma_{1,1}^* + J\,\gamma_{1,1}\,J - J\,\gamma_{1,1}^*\,J\Big)$$

and the fact that this projection is readily seen to be also skew-adjoint implies that it is also the orthogonal projection onto the subspace of anti-linear skew-adjoint maps. ∎

Proposition 18. *For each* $\xi \in \mathcal{G}_t(E)$, *corresponding to the couple* (F, J), *the second fundamental form of* $\mathcal{G}_t(E)$ *inside* $L(E;E)$, $h_\xi \colon T_\xi(\mathcal{G}_t(E)) \times T_\xi(\mathcal{G}_t(E)) \to T_\xi(\mathcal{G}_t(E))^\perp$, *is defined, in matricial terms, by*

$$\left(\begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & 0 \end{bmatrix}, \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & 0 \end{bmatrix}\right) \longmapsto \begin{bmatrix} \mu_{1,1} & \mu_{1,2} \\ \mu_{2,1} & \mu_{2,2} \end{bmatrix},$$

$$\mu_{1,1} = \frac{1}{4\sin(t)}\Bigg(2\Big(\beta_{1,1}\,\gamma_{1,1}\,J + \gamma_{1,1}\,\beta_{1,1}\,J\Big) + 2\sin(t)\,\Big(\beta_{1,2}\,\gamma_{2,1} + \gamma_{1,2}\,\beta_{2,1}\Big) + \\ + (1+\cos(t))\,\Big(J\,\beta_{1,2}\,\gamma_{2,1} + \beta_{1,2}\,\gamma_{2,1}\,J + J\,\gamma_{1,2}\,\beta_{2,1} + \gamma_{1,2}\,\beta_{2,1}\,J\Big)\Bigg)$$

$$\mu_{1,2} = \frac{1}{2}\,\Big(\beta_{1,1}\,J_t^*\,\gamma_{1,2} + \gamma_{1,1}\,J_t^*\,\beta_{1,2}\Big)$$

$$\mu_{2,1} = \frac{1}{2}\,\Big(\beta_{2,1}\,J_t^*\,\gamma_{1,1} + \gamma_{2,1}\,J_t^*\,\beta_{1,1}\Big)$$

$$\mu_{2,2} = \frac{1}{2(1-\cos(t))}\,\Big(-\beta_{2,1}\,\gamma_{1,2} - \gamma_{2,1}\,\beta_{1,2} + \beta_{2,1}\,J_t^*\,\gamma_{1,2} + \gamma_{2,1}\,J_t^*\,\beta_{1,2}\Big)\;.$$

Proof: First of all, and in order to be able to differentiate, we write down a non matricial formula for the orthogonal projection computed in Proposition 17. In fact, using the matricial characterizations:

$$\frac{2\,Id_E - \xi - \xi^*}{2(1-\cos(t))} = \begin{bmatrix} Id_F & 0 \\ 0 & 0 \end{bmatrix}, \qquad \frac{\xi + \xi^* - 2\cos(t)\,Id_E}{2(1-\cos(t))} = \begin{bmatrix} 0 & 0 \\ 0 & Id_{F^\perp} \end{bmatrix},$$

$$\frac{\xi - \xi^*}{2\sin(t)} = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{(1-2\cos(t))\,\xi - \xi^* + 2\cos(t)\,Id_E}{2(1-\cos(t))} = \begin{bmatrix} J_t & 0 \\ 0 & 0 \end{bmatrix},$$

we can write

$$\pi_\xi(\gamma) \;=\; \frac{1}{16(1-\cos(t))^2}\,(2\,Id_E - \xi - \xi^*)\,(\gamma-\gamma^*)\,(2\,Id_E - \xi - \xi^*) \;+ \\ + \frac{1}{16\sin(t)^2}\,(\xi-\xi^*)\,(\gamma-\gamma^*)\,(\xi-\xi^*) + \\ + \frac{1}{8(1-\cos(t))^2}\,(2\,Id_E - \xi - \xi^*)\,\gamma\,\Big(\xi + \xi^* - 2\cos(t)\,Id_E\Big) + \\ + \frac{1}{8(1-\cos(t))^2}\,\Big(\xi + \xi^* - 2\cos(t)\,Id_E\Big)\,\gamma\,(2\,Id_E - \xi - \xi^*) - \\ - \frac{1}{8(1-\cos(t))^2}\Big((1-2\cos(t))\,\xi - \xi^* + 2\cos(t)\,Id_E\Big)\,\gamma^*\,\Big(\xi + \xi^* - 2\cos(t)\,Id_E\Big) - \\ - \frac{1}{8(1-\cos(t))^2}\,\Big(\xi + \xi^* - 2\cos(t)\,Id_E\Big)\,\gamma^*\,\Big((1-2\cos(t))\,\xi - \xi^* + 2\cos(t)\,Id_E\Big)\;.$$

By using the formula $h_\xi(\beta,\gamma) = D\pi_\xi(\beta)(\gamma)$, we can now write

$$\begin{aligned}
h_\xi(\beta,\gamma) \;=\; & -\frac{1}{16(1-\cos(t))^2}\,(\beta+\beta^*)\,(\gamma-\gamma^*)\,(2\,Id_E-\xi-\xi^*)\;-\\
&-\frac{1}{16(1-\cos(t))^2}\,(2\,Id_E-\xi-\xi^*)\,(\gamma-\gamma^*)\,(\beta+\beta^*)+\\
&+\frac{1}{16\sin(t)^2}\,(\beta-\beta^*)\,(\gamma-\gamma^*)\,(\xi-\xi^*)+\\
&+\frac{1}{16\sin(t)^2}\,(\xi-\xi^*)\,(\gamma-\gamma^*)\,(\beta-\beta^*)-\\
&-\frac{1}{8(1-\cos(t))^2}\,(\beta+\beta^*)\,\gamma\,\Big(\xi+\xi^*-2\cos(t)\,Id_E\Big)+\\
&+\frac{1}{8(1-\cos(t))^2}\,(2\,Id_E-\xi-\xi^*)\,\gamma\,(\beta+\beta^*)+\\
&+\frac{1}{8(1-\cos(t))^2}\,(\beta+\beta^*)\,\gamma\,(2\,Id_E-\xi-\xi^*)-\\
&-\frac{1}{8(1-\cos(t))^2}\,\Big(\xi+\xi^*-2\cos(t)\,Id_E\Big)\,\gamma\,(\beta+\beta^*)-\\
&-\frac{1}{8(1-\cos(t))^2}\Big((1-2\cos(t))\,\beta-\beta^*\Big)\,\gamma^*\,\Big(\xi+\xi^*-2\cos(t)\,Id_E\Big)-\\
&-\frac{1}{8(1-\cos(t))^2}\Big((1-2\cos(t))\,\xi-\xi^*+2\cos(t)\,Id_E\Big)\,\gamma^*\,\Big(\beta+\beta^*\Big)-\\
&-\frac{1}{8(1-\cos(t))^2}\,(\beta+\beta^*)\,\gamma^*\,\Big((1-2\cos(t))\,\xi-\xi^*+2\cos(t)\,Id_E\Big)-\\
&-\frac{1}{8(1-\cos(t))^2}\,\Big(\xi+\xi^*-2\cos(t)\,Id_E\Big)\,\gamma^*\,\Big((1-2\cos(t))\,\beta-\beta^*\Big)
\end{aligned}$$

and the matricial characterization follows by a long but straightforward computation, using the identities $3-4\cos(t)+\cos^2(t) = (1-\cos(t))\,(3-\cos(t))$ and $\frac{\sin(t)}{1-\cos(t)} = \frac{1+\cos(t)}{\sin(t)}$. ∎

The preceding formula for the second fundamental form of the manifold $\mathcal{G}_t(E)$ inside the Euclidean space $L(E;E)$ will be useful, in particular, in calculating the covariant derivative of the complex structure:

Proposition 19. *For each $\xi \in \mathcal{G}_t(E)$, the covariant derivative of the complex structure $\nabla J_\xi\colon T_\xi(\mathcal{G}_t(E)) \to L(T_\xi(\mathcal{G}_t(E)); T_\xi(\mathcal{G}_t(E)))$ is defined by the condition that, whenever $\beta,\gamma \in T_\xi(\mathcal{G}_t(E))$ have matrices $\begin{bmatrix}\beta_{1,1} & \beta_{1,2}\\ \beta_{2,1} & 0\end{bmatrix}$ and $\begin{bmatrix}\gamma_{1,1} & \gamma_{1,2}\\ \gamma_{2,1} & 0\end{bmatrix}$,*

$\nabla J_\xi(\beta)(\gamma)$ *has matrix* $\dfrac{\cos(t)}{\sin(t)}\begin{bmatrix} 0 & -J_t\,\beta_{11}\,\gamma_{1,2} \\ \gamma_{2,1}\,\beta_{1,1}\,J_t & 0 \end{bmatrix}$. *As a consequence, the manifold* $\mathcal{G}_t(E)$ *is always semi-Kähler and, in the case where* $t = \pm\frac{\pi}{2} \bmod 2\pi$, $\mathcal{G}_t(E)$ *is even a Kähler manifold.*

Proof: For each $\xi \in \mathcal{G}_t(E)$, corresponding to the couple (F, J), let $\overline{J}_\xi : L(E;E) \to L(E;E)$ be defined, in matricial terms, by $\begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{bmatrix} \mapsto \begin{bmatrix} -J\,\gamma_{1,1} & -J\,\gamma_{1,2} \\ \gamma_{2,1}\,J & 0 \end{bmatrix}$, a linear map that is hence a prolongation of the structure map $J_\xi : T_\xi(\mathcal{G}_t(E)) \to T_\xi(\mathcal{G}_t(E))$. Using the second and third matricial characterizations referred in the proof of the preceding result, we get the intrinsic characterization

$$\overline{J}_\xi(\gamma) = \frac{1}{4\sin(t)\,(1-\cos(t))}\Big(\xi + \xi^* - 2\cos(t)\,Id\Big)\,\gamma\,(\xi - \xi^*) - \frac{1}{2\sin(t)}\,(\xi - \xi^*)\,\gamma\;,$$

that allows us to write, by differentiation,

$$\begin{aligned} D\overline{J}_\xi(\beta)(\gamma) = {}& \frac{1}{4\sin(t)\,(1-\cos(t))}\,(\beta + \beta^*)\,\gamma\,(\xi - \xi^*) + \\ &+ \frac{1}{4\sin(t)\,(1-\cos(t))}\Big(\xi + \xi^* - 2\cos(t)\,Id\Big)\,\gamma\,(\beta - \beta^*) - \frac{1}{2\sin(t)}\,(\beta - \beta^*)\,\gamma\;. \end{aligned}$$

Using the formula $\nabla J_\xi(\beta)(\gamma) = D\overline{J}_\xi(\beta)(\gamma) + \overline{J}_\xi(h_\xi(\beta,\gamma)) - h_\xi(\beta, J_\xi(\gamma))$, as well as the matricial characterization of the second fundamental form in Proposition 18, we obtain the stated matricial formula, after a long but straightforward calculation. This matricial formula implies, in particular, that we $\nabla J_\xi(\beta)(\beta) = 0$, whenever the matrix of $\beta \in T_\xi(\mathcal{G}_t(E))$ has one of the two forms $\begin{bmatrix} \beta_{1,1} & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{bmatrix}$, and the fact that $T_\xi(\mathcal{G}_t(E))$ has an orthonormal basis with each element of one of these forms implies that the bilinear map $(\beta,\gamma) \mapsto \nabla J_\xi(\beta)(\gamma)$ is traceless, so that we have a semi-Kähler manifold. It is clear that we have even a Kähler manifold whenever $\cos(t) = 0$, that is to say, whenever $t = \pm\frac{\pi}{2} \bmod 2\pi$. ∎

We compute next the second fundamental form of $\mathcal{G}_t(E)$ when we take the orthogonal group $O(E)$ as ambient space, instead of the whole space $L(E;E)$.

Proposition 20. *Let* $\xi \in \mathcal{G}_t(E)$ *correspond to the couple* (F, J). *The second fundamental form* $\widehat{h}_\xi : T_\xi(\mathcal{G}_t(E)) \times T_\xi(\mathcal{G}_t(E)) \to T_\xi(O(E))$, *of* $\mathcal{G}_t(E)$ *inside* $O(E)$, *is then defined, in matricial terms, by*

$$\left(\begin{bmatrix}\beta_{1,1} & \beta_{1,2}\\ \beta_{2,1} & 0\end{bmatrix}, \begin{bmatrix}\gamma_{1,1} & \gamma_{1,2}\\ \gamma_{2,1} & 0\end{bmatrix}\right) \longmapsto \begin{bmatrix}\widehat{\mu}_{1,1} & 0\\ 0 & \widehat{\mu}_{2,2}\end{bmatrix},$$

$$\widehat{\mu}_{1,1} = \frac{1}{4\sin(t)}\Bigg(\Big(1+\cos(t)\Big)\Big(J\beta_{1,2}\,\gamma_{2,1} + \beta_{1,2}\,\gamma_{2,1}\,J + J\,\gamma_{1,2}\,\beta_{2,1} + \gamma_{1,2}\,\beta_{2,1}\,J\Big) +$$
$$+\,2\cos(t)\Big(\beta_{1,1}\,\gamma_{1,1}\,J\,J_t + \gamma_{1,1}\,\beta_{1,1}\,J\,J_t\Big)\Bigg)$$

$$\widehat{\mu}_{2,2} = \frac{1}{4(1-\cos(t))}\Big(\beta_{2,1}(J_{2t}^{*} - Id)\,\gamma_{1,2} + \gamma_{2,1}(J_{2t}^{*} - Id)\,\beta_{1,2}\Big)\ .$$

Hence, $\mathcal{G}_t(E)$ *is* $(2,0)$*-geodesic inside* $O(E)$*, in the sense that* $\widehat{h}_\xi(J_\xi(\beta), J_\xi(\gamma)) = \widehat{h}_\xi(\beta,\gamma)$.

Proof: The fact that $\widehat{h}_\xi(\beta,\gamma)$ is the orthogonal projection of $h_\xi(\beta,\gamma)$ onto the tangent space $T_\xi(O(E))$ implies, by the formula referred in section 2, that $\widehat{h}_\xi(\beta,\gamma) = \frac{1}{2}(h_\xi(\beta,\gamma) - \xi \circ h_\xi(\beta,\gamma)^* \circ \xi)$ so that, with the notations of Proposition 18, $\widehat{h}_\xi(\beta,\gamma)$ has matrix $\begin{bmatrix}\widehat{\mu}_{1,1} & \widehat{\mu}_{1,2}\\ \widehat{\mu}_{2,1} & \widehat{\mu}_{2,2}\end{bmatrix}$, with $\widehat{\mu}_{1,1} = \frac{1}{2}(\mu_{1,1} - J_t\,\mu_{1,1}^{*}\,J_t)$, $\widehat{\mu}_{1,2} = \frac{1}{2}(\mu_{1,2} - J_t\,\mu_{2,1}^{*})$, $\widehat{\mu}_{2,1} = \frac{1}{2}(\mu_{2,1} - \mu_{1,2}^{*}\,J_t)$ and $\widehat{\mu}_{2,2} = \frac{1}{2}(\mu_{2,2} - \mu_{2,2}^{*})$. The stated formulas follow in a straightforward way, if we recall the characterization of the tangent space $T_\xi(\mathcal{G}_t(E))$ in Proposition 14. The equality $\widehat{h}_\xi(J_\xi(\beta), J_\xi(\gamma)) = \widehat{h}_\xi(\beta,\gamma)$ has a trivial verification, if we recall the matricial characterization of the structure maps J_ξ in Proposition 16. ∎

By a straightforward computation, we can determine the trace of the preceding second fundamental form:

Proposition 21. *Let* E *have dimension* n *and let* $\xi \in \mathcal{G}_{kt}(E)$ *correspond to the couple* (F,J)*. The trace of the second fundamental form* $\widehat{h}_\xi$*, of* $\mathcal{G}_{kt}(E)$ *inside* $O(E)$*, has then a matrix* $\begin{bmatrix}\widehat{\rho}_{1,1} & 0\\ 0 & 0\end{bmatrix}$, $\widehat{\rho}_{1,1} = -\frac{n-2k+(n-k-1)\cos(t)}{2\sin(t)}\,J\,J_t$*. Hence, if* $k>1$*, there exists* t *such* $\mathcal{G}_{kt}(E)$ *is minimal inside* $O(E)$*, namely the one defined by the condition* $\cos(t) = -\frac{n-2k}{n-k-1}$.

As a special case, we realize that, when $k>1$ and $n=3k-1$, the manifolds $\mathcal{G}_{kt}(E)$ that are minimal inside $O(E)$ are the ones with $t = \pm\frac{2\pi}{3} \bmod 2\pi$, hence the ones that have the naturally reductive metric.

It is easily seen that, for each $a \in \mathbb{R}$, the manifolds $\mathcal{G}_{kt}(E)$ are contained in spheres centered at $a\,Id_E$ so that it is natural to ask when are these manifolds minimal in such spheres. This is known to be equivalent to the fact that the trace of the second fundamental form h_ξ, of $\mathcal{G}_{kt}(E)$ inside $L(E;E)$, is a multiple of $\xi - a\,Id_E$ and our last proposition, whose proof is again straightforward, shows that the answer is "sometimes yes".

Proposition 22. *Let E have dimension n and let $\xi \in \mathcal{G}_{kt}(E)$ correspond to the couple (F,J). The trace of the second fundamental form h_ξ, of $\mathcal{G}_{kt}(E)$ inside $L(E;E)$, has then a matrix* $\begin{bmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{2,2} \end{bmatrix}$,

$$\rho_{1,1} = \frac{n-2k}{2}\,Id_F + \frac{-(n-k-1)-(n-2k)\cos(t)}{2\sin(t)}\,J\,, \qquad \rho_{2,2} = -k\,Id_{F^\perp}\,.$$

As a consequence, if $k>1$, the manifold $\mathcal{G}_{kt}(E)$ is minimal in a sphere centered at $a\,Id_E$, for $a = \frac{n-3k-1}{n}$ and $\cos(t) = -\frac{k+1}{2k}$.

REFERENCES

[1] Ferus, D. – Symmetric submanifolds of Euclidean space, *Math. Ann.,* 247 (1980), 81–93.

[2] Gray, A. – Minimal varieties and almost Hermitian submanifolds, *Michigan Math. J.,* 12 (1965), 273–287.

[3] Kobayashi, S. – On isometric imbeddings of compact symmetric spaces, *Tôhoku Math. J.,* 20 (1968), 21–25.

[4] Rawnsley, J.H. – *f-structures, f-twistor spaces and harmonic maps.* In: "Geometry Seminar L. Bianchi II (1984)", Lecture Notes in Math., vol. 1164, Springer Verlag, 1985, 86–159.

[5] Rigoli, M. and Tribuzy, R. – The Gauss map for Kählerian submanifolds of $\mathbb{R}^n$, *Transactions AMS,* 332 (1992), 515–528.

[6] Salamon, S. – *Harmonic and holomorphic maps.* In: "Geometry Seminar L. Bianchi II (1984)", Lecture Notes in Math., vol. 1164, Springer Verlag, 1985, 161–224.

[7] Takeushi and Kobayashi – Minimal imbeddings of R-spaces, *J. Differential Geom.,* 2 (1968), 203–215.

[8] Uhlenbeck, K. – Harmonic maps into Lie groups (classical solutions of the chiral model), *J. Differential Geom.,* 30 (1989), 1–50.

[9] Yano, K. – On a structure defined by a tensor field f of type $(1,1)$ satisfying $f^3+f=0$, *Tensor,* 14 (1963) 99–109.

Cecília Ferreira and Armando Machado,
CMAF da Universidade de Lisboa,
Av. Prof. Gama Pinto 2, 1699 Lisboa Codex – PORTUGAL
E-mail: `cecilia@ptmat.lmc.fc.ul.pt` and `armac@ptmat.lmc.fc.ul.pt`