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EXISTENCE RESULTS FOR SOME QUASILINEAR ELLIPTIC PROBLEMS WITH RIGHT HANDSIDE IN L^1

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Abstract: We study the existence of unbounded renormalized solutions, for quasilinear elliptic equations in a bounded domain. In a first part, we introduce the symmetrized problem, and we get an existence result assuming the existence of a renormalized supersolution of the symmetrized problem. Afterwards, we get a sub-super solution theorem for an equation with a more general right handside.

1 – Introduction

Let Ω be an open bounded set of R^N with $N \geq 1$. We consider the following problem:

$$\begin{cases} -\operatorname{div} A(x,u,Du) = F(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \,. \end{cases} \tag{1.1}$$

We assume that:

$$A(x,s,\xi) \text{ is a Caratheodory function: } \Omega \times R^{N+1} \to R^N \;, \tag{1.2}$$

$$\Big\langle A(x,s,\xi) - A(x,s,\xi'),\, \xi-\xi' \Big\rangle > 0 \quad \text{a.e. } x \in \Omega, \;\; \forall\, s \in R, \;\; \forall\, \xi,\xi' \in R^N, \;\; \xi \neq \xi' \;, \tag{1.3}$$

$$\alpha\, |\xi|^p \leq \langle A(x,s,\xi),\, \xi\rangle \quad \text{a.e. } x \in \Omega, \;\; \forall\, s \in R, \;\; \forall\, \xi \in R^N \;, \tag{1.4}$$

$$|A(x,s,\xi)| \leq \beta(|s|)\, (|\xi|^{p-1} + b(x)) \quad \text{a.e. } x \in \Omega, \;\; \forall\, s \in R, \;\; \forall\, \xi \in R^N \tag{1.5}$$

where β is a function: $[0,+\infty[\; \to [0,+\infty[$ defined everywhere and bounded on the bounded intervalls and where b is a positive function of $L^{p'}(\Omega)$,

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(1.6) $F(x,s)$ is a Caratheodory function: $\Omega \times R \to R^+$,

(1.7)
$$0 \le F(x,s) \le \sum_{i=0}^{m} f_i(x) \times g_i(s)$$
where $m \in N$ and $f_i(x) \in L^1(\Omega)$, $f_i(x) \ge 0$, $0 \le i \le m$ and, for $0 \le i \le m$, $g_i\colon R \to]0,+\infty[$, continous, nondecreasing .

We shall denote by $f^\star(s)$ the unidimensional decreasing rearrangement of f, that is to say, the unique decreasing function such that $|f > t| = |f^\star > t|$ for every t. We shall denote by $\tilde{f}(x)$ the spherical decreasing rearrangement of f, that is to say $\tilde{f}(x) = f^\star(\omega_N\,|x|^N)$ for every x in $\tilde{\Omega}$, where $\tilde{\Omega}$ is the ball of R^N centered at the origin, such that $|\tilde{\Omega}| = |\Omega|$, and where ω_N is the measure of the unit ball in R^N. For all the definitions and properties concerning symetrization see [5].

Let us consider the symmetrized problem:

$$\begin{cases} -\alpha\,\Delta_p u = \displaystyle\sum_{i=0}^{m} \tilde{f}_i(x)\, g_i(u) & \text{in } \tilde{\Omega}, \\ u = 0 & \text{on } \partial\tilde{\Omega}\ , \end{cases} \tag{1.8}$$

where $\Delta_p u = \mathrm{Div}(|Du|^{p-2} Du)$. We shall use the following notations and definitions:

We note:

$$T_k\,u = \begin{cases} k & \text{if } u \ge k, \\ u & \text{if } -k < u < k, \\ -k & \text{if } u \le -k\ , \end{cases}$$

and $L^0(\Omega)$, the space of measurable functions wich are finite a.e. in Ω. Let us recall the definition of [7]:

Definition 1.1. We call renormalized solution of (1.1) a function u such that:

$$u \in L^0(\Omega)\ ,$$

$$T_k u \in W_0^{1,p}(\Omega), \quad \forall\, k \in R^+\ ,$$

$$\frac{1}{k} \int_{k \le |u| \le 2k} |Du|^p\, dx \;\to\; 0 \quad \text{when } k \to +\infty\ ,$$

$$\int_\Omega A(x,u,Du)\, Du\, h'(u)\, w\, dx + \int_\Omega A(x,u,Du)\, Dw\, h(u)\, dx = \int_\Omega F(x,u)\, h(u)\, w\, dx\ ,$$

$$\forall\, w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \ \text{ and } \ \forall\, h \in C^1(R) \ \text{ or piecewise affine}$$

and with compact support.

In the same way, we define a renormalized supersolution:

Definition 1.2. We call renormalized supersolution of (1.1) a function ψ such that:

$$\psi \in L^0(\Omega) \ ,$$

$$T_k\psi \in W^{1,p}(\Omega), \quad \forall\, k \in R^+ \ ,$$

$$\exists\, C_\psi \in R^+ \ \text{ such that, } \ \forall\, k \in R^+, \ \ 0 \le \psi \le C_\psi \ \text{on } \partial\Omega \ ,$$

$$\frac{1}{k}\int_{k\le\psi\le 2k} |D\psi|^p \, dx \to 0 \quad \text{ when } \ k \to +\infty \ ,$$

$$\int_\Omega A(x,\psi,D\psi)\, D\psi\, h'(\psi)\, w\, dx + \int_\Omega A(x,\psi,D\psi)\, Dw\, h(\psi)\, dx \ge \int_\Omega F(x,\psi)\, h(\psi)\, w\, dx \ ,$$

$$\forall\, w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \ \text{ and } \ \forall\, h \in C^1(R) \ \text{ or piecewise affine}$$

and with compact support.

The definition of a renormalized subsolution is obtained exchanging $\ge$ by $\le$. Let us remark that if a renormalized solution u is in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then u is an ordinary weak solution, that is to say u verifies:

$$\int_\Omega A(x,u,Du)\, D\varphi = \int_\Omega F(x,u)\, \varphi \quad \forall\, \varphi \in W_0^{1,p}(\Omega) \ .$$

This is also true for sub and supersolutions. The main result of this work is the following:

Theorem 1.1. *We suppose that A satisfies (1.2), (1.3), (1.4), (1.5), and that F verifies (1.6), (1.7). If there exists a supersolution $\psi \ge 0$ for the problem (1.8), then there exists a renormalized nonnegative solution u for problem (1.1) such that $|u > t| \le |\psi > t|$.*

Theorem 1.1 is a generalization of ([6]). In this paper the functions f_i are supposed to be in $L^q(\Omega)$ with $q \ge \max(p', N/p)$ and ψ in $L^\infty(\Omega)$ and of course u is also in $L^\infty(\Omega)$, moreover in [6], A is roughly independent of u. Notice that $q \ge \max(p', N/p)$ insure that the problem:

$$\text{(1.9)} \qquad \begin{cases} -\alpha\, \Delta_p u = f(x) & \text{in } \ \Omega, \\ u = 0 \quad \text{on } \ \partial\Omega \ , \end{cases}$$

has a solution in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ if $f \in L^q(\Omega)$. Here, f is in $L^1(\Omega)$, and then the solution of (1.9) is no more in $L^\infty(\Omega)$. Such problems with right handside in L^1 have been studied in [1] and in [7] in which renormalized solutions are introduced.

To prove this theorem, we shall first get a comparison result with the symmetrized problem, and in a second time we shall prove a sub-super solution theorem.

2 – Comparison with the symmetrized problem

Let us consider the following problem:

$$\tag{2.1} \begin{cases} -\operatorname{div} A(x,u,Du) = \displaystyle\sum_{i=0}^{m} f_i(x)\, g_i(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega\ . \end{cases}$$

Theorem 2.1. *We suppose that A satisfies (1.2), (1.3), (1.4), (1.5), and that the functions f_i and g_i satisfy (1.7). If problem (1.8) has a renormalized supersolution $\psi \geq 0$, then problem (2.1) has a nonnegative renormalized solution u such that $|u>t| \leq |\psi>t|$, for all $t \geq 0$.*

Proof: Let $n \in N$, we set, for $0 \leq i \leq m$, $f_{i,n}(x) = \inf(f(x), n)$.

Let $v \in L^\infty(\Omega)$, we consider the problem:

$$\tag{2.2} \begin{cases} -\operatorname{Div} A(x,u,Du) = \displaystyle\sum_{i=0}^{m} f_{i,n}(x)\, g_i(v) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega\ . \end{cases}$$

Recall that a weak subsolution of (2.2), is a function $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ which verifies:

$$\tag{2.3} \begin{cases} \displaystyle\int_\Omega A(x,v,Dv)\, D\varphi\, dx \leq \int_\Omega \sum_{i=0}^{m} f_{i,n}(x)\, g_i(v)\, dx\, \varphi \quad \forall\, \varphi \in W_0^{1,p}(\Omega), \\ v \leq 0 \quad \text{on } \partial\Omega\ . \end{cases}$$

We prove the following lemma:

Lemma 2.1. *We suppose that A satisfies (1.2), (1.3), (1.4), (1.5), and that the functions f_i and g_i satisfy (1.7). Moreover we suppose that $v \geq 0$ verifies (2.3), then there exists a nonnegative weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of (2.2) such that $u \geq v$.*

Let $M > 0$ such that: $0 \le v(x) \le M$. We set:

$$\bar{A}_M\Big(x, u(x), Du(x)\Big) = \begin{cases} A\Big(x, M, Du(x)\Big) & \text{if } \ u(x) \ge M, \\ A\Big(x, u(x), Du(x)\Big) & \text{if } \ v(x) \le u(x) \le M, \\ A\Big(x, v(x), Du(x)\Big) & \text{if } \ u(x) \le v(x) \ ; \end{cases}$$

then the problem:

$$\text{(2.4)} \qquad \begin{cases} -\operatorname{div} \bar{A}(x,u,Du) = \displaystyle\sum_{i=0}^{m} f_{i,n}(x)\, g_i(v) \quad \text{in } \ \Omega, \\ u = 0 \quad \text{on } \ \partial\Omega \ , \end{cases}$$

has at least one nonnegative weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty$, such that:

$$\|u\|_\infty \le \int_0^{|\Omega|} \alpha^{-p/p'} N^{-p'} \omega_N^{-p'/N} s^{-p'+p'/N} \left(\int_0^s \Big(\sum_{i=0}^{m} f_{i,n}\, g_i(v) \Big)^* (\sigma)\ d\sigma \right)^{p'/p} ds$$
$$= C_n \ .$$

The existence comes from the theorem of [4, p. 180], moreover u is nonnegative because the right handside is nonnegative, and L^∞ estimate can be proved by symmetrization techniques (see for instance [5] and the demonstrations below). Remark that C_n is independent of M, and then we can choose M such that:

$$\text{(2.5)} \qquad M > C_n \ .$$

We are now going to prove that $u \ge v$. We take $(v-u)^+$ as test function in (2.3) and (2.2), then,

$$\int_\Omega \Big(A(x,v,Dv) - \bar{A}_M(x,u,Du) \Big)\, D(v-u)^+ \le 0$$

but on $\{x \in \Omega,\, v \ge u\}$ we have $\bar{A}_M(x,u,Du) = A(x,v,Du)$, then from (1.3), we obtain:

$$(v-u)^+ = 0$$

and so,

$$\text{(2.6)} \qquad u \ge v \ .$$

From (2.5) and (2.6), we can deduce that $\bar{A}_M(x,u,Du) = A(x,u,Du)$ and so u is in fact solution of (2.2). This proves Lemma 2.1.

We are now going to construct a sequence (u_n) in the following way:
we set

$$u_0 = 0 \ ;$$

suppose that the sequence is defined until u_{n-1} then u_n is a solution of:

$$\begin{cases} -\operatorname{div} A(x,u_n,Du_n) = \displaystyle\sum_{i=0}^{m} g_i(u_{n-1})\, f_{i,n}(x), \\ u_n \geq u_{n-1}, \\ u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \ . \end{cases} \tag{2.7}$$

We have to show that the sequence (u_n) is well defined:

For $n=0$,

$$\begin{cases} -\operatorname{div} A(x,0,0) = 0 \leq \displaystyle\sum_{i=0}^{m} f_{i,1}(x)\, g_i(0), \\ u_0 = 0 \leq 0 \quad \text{ on } \ \partial\Omega \ , \end{cases} \tag{2.8}$$

that is to say, u_0 is a subsolution of problem corresponding to u_1, and so from Lemma 2.1, u_1 exists. Suppose that the sequence is defined until u_{n-1}, then:

$$-\operatorname{div} A(x,u_{n-1},Du_{n-1}) = \sum_{i=0}^{m} g_i(u_{n-2})\, f_{i,n-1}(x) \quad \text{ in } \ \Omega$$

and

$$u_{n-1} \geq u_{n-2} \ ;$$

then, as for $0 \leq i \leq m$, g_i is nondecreasing,

$$-\operatorname{div} A(x,u_{n-1},Du_{n-1}) \leq \sum_{i=0}^{m} g_i(u_{n-1})\, f_{i,n}(x) \quad \text{ in } \ \Omega$$

and then u_n exists from Lemma 2.1. On another hand we construct a sequence (v_n), in the following way:
we set

$$v_0 = 0$$

and $v_n \in W_0^{1,p}(\tilde\Omega) \cap L^\infty(\tilde\Omega)$ is a solution of:

$$-\alpha\, \Delta_p v_n = \sum_{i=0}^{m} g_i(v_{n-1})\, \tilde f_{i,n}(x) \quad \text{ in } \ \tilde\Omega \ .$$

We are going to prove that the sequence (v_n) has the following property:

$$v_{n-1} \leq v_n \leq \psi \quad \forall\, n \geq 1 \ .$$

Recall that we suppose that ψ is a renormalized supersolution of problem (1.8). For $n=0$, we have $v_1 \geq v_0 = 0$. In the inequation satisfied by ψ, we take $w=(v_1-\psi)^+$ which is in $W_0^{1,p}(\tilde\Omega)\cap L^\infty(\tilde\Omega)$ and for h a function $h \in C^1(R)$ such that $h(s)=1$ if $s \leq \|v_1\|_\infty$, and $h(s)=0$ if $s \geq \|v_1\|_\infty+1$. Then $h(\psi)w=w$. In the equation satisfied by v_1 we take $(v_1-\psi)^+$ as test fuction. This leads to:

$$\alpha \int_\Omega \Big(|Dv_1|^{p-2}\, Dv_1 - |D\psi|^{p-2}\, D\psi\Big)\, D(v_1-\psi)^+ \leq$$
$$\leq \int_\Omega \Big(\sum_{i=0}^m g_i(v_0)\, \tilde f_{i,1}(x) - \sum_{i=0}^m g_i(\psi)\, \tilde f_i(x)\Big)\, (v_1-\psi)^+ \;\leq\; 0$$

and thus,

$$v_1 \leq \psi\ .$$

Suppose by induction that:

$$v_{n-2} \leq v_{n-1} \leq \psi\ .$$

Similarly, in the inequation satisfied by ψ, we take $w=(v_n-\psi)^+$ which is in $W_0^{1,p}(\tilde\Omega)\cap L^\infty(\tilde\Omega)$ and for h a function $h\in C^1(R)$ such that $h(s)=1$ if $s\leq \|v_n\|_\infty$, and $h(s)=0$ if $s\geq \|v_n\|_\infty+1$. In the equation satisfied by v_n we take $(v_n-\psi)^+$ as test function. As g_i is nondecreasing, we obtain:

$$\int_\Omega \Big(|Du_n|^{p-2}\, Dv_n - |D\psi|^{p-2}\, D\psi\Big)\, D(v_n-\psi)^+ \leq$$
$$\leq \int_\Omega \Big(\sum_{i=0}^m g_i(v_{n-1})\, \tilde f_{i,n}(x) - \sum_{i=0}^m g_i(\psi)\, \tilde f_i(x)\Big)\, (v_n-\psi)^+ \;\leq\; 0\ .$$

Now if we take $(v_{n-1}-v_n)^+$ as test function in the equations satisfied by v_{n-1} and v_n, after substraction, we obtain:

$$\int_\Omega \Big(|Dv_{n-1}|^{p-2}\, Dv_{n-1} - |Dv_n|^{p-2}\, Dv_n\Big)\, D(v_{n-1}-v_n)^+ \leq$$
$$\leq \int_\Omega \Big(\sum_{i=0}^m g_i(v_{n-2})\, \tilde f_{i,n-1}(x) - \sum_{i=0}^m g_i(v_{n-1})\, \tilde f_{i,n}(x)\Big)\, (v_{n-1}-v_n)^+\ ,$$

g_i is nondecreasing, and by induction $v_{n-2}\leq v_{n-1}$, thus:

$$\int_\Omega \Big(|Dv_{n-1}|^{p-2}\, Dv_{n-1} - |Dv_n|^{p-2}\, Dv_n\Big)\, D(v_{n-1}-v_n)^+ \;\leq\; 0$$

and thus,

$$v_{n-1} \leq v_n\ .$$

For all s in R, we note $s^- = -\inf(s,0)$. Let τ be a function of $W_0^{1,p}(\tilde\Omega)$ such that $0 \le \tau \le 1$, then, $(v_n - k\tau)^- \in W_0^{1,p}(\tilde\Omega) \cap L^\infty(\tilde\Omega)$ and $\|(v_n - k\tau)^-\|_\infty \le k$. We take $-(v_n - k\tau)^-$ as test function in the equation satisfied by v_n, and we note $C_{k,\tau}$ different constant which depend on k and τ,

$$\alpha \int_{\{v_n \le k\tau\}} |Dv_n|^p \, dx \le$$
$$\le k \int_{\{v_n \le k\tau\}} |Dv_n|^{p-1} \, |D\tau| \, dx - \int_{\{v_n \le k\tau\}} \sum_{i=0}^{m} g_i(v_{n-1}) \, \tilde f_{i,n}(x) \, (v_n - k\tau)^-$$

then,

$$\alpha \int_{\{v_n \le k\tau\}} |Dv_n|^p \, dx \ \le \ C_{k,\tau} \Big(\int_{\{v_n \le k\tau\}} |Dv_n|^p \, dx \Big)^{\frac{p-1}{p}} dx + C_{k,\tau}$$

and thus,

$$\int_{\{v_n \le k\tau\}} |Dv_n|^p \, dx \le C_{k,\tau}$$

a fortiori,

$$\int_{\{\tau \equiv 1\}} |DT_k v_n|^p \, dx \le C_{k,\tau} \ . \tag{2.9}$$

We now specify the choice of τ, we take $\tau = T_1\left((\psi - C_\psi - 1)^+\right)$, then $w \equiv 1$ on $\{\tau < 1\}$, and $w \equiv 0$ on $\{\psi \ge C_\psi + 2\}$. In the equation satisfied by v_n, we take $w \, v_n$ which is in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, as test function, and we obtain

$$\alpha \int_\Omega |Dv_n|^p \, w \, dx + \alpha \int_\Omega |Dv_n|^{p-2} \, Dv_n \, Dw \, v_n \, dx = \int_\Omega \sum_{i=0}^{m} g_i(v_{n-1}) \, \tilde f_{i,n}(x) \, w \, u_n \, dx$$

then,

$$\alpha \int_{\{w \equiv 1\}} |Dv_n|^p \, dx + \alpha \int_{\{w < 1\}} |Dv_n|^p \, w \, dx \le$$
$$\le C \int_\Omega |Dv_n|^{p-1} \, |Dw| \, dx + C$$
$$\le C \int_{\{w \equiv 1\}} |Dv_n|^{p-1} \, |Dw| \, dx + C \int_{\{w < 1\}} |Dv_n|^{p-1} \, |Dw| \, dx + C$$

but, $\{x \in \Omega, \, w(x) < 1\} \subset \{x \in \Omega, \, \tau(x) = 1\}$, then from (2.9),

$$\alpha \int_{\{w \equiv 1\}} |Dv_n|^p \, dx \ \le \ C \int_{\{w \equiv 1\}} |Dv_n|^{p-1} \, |Dw| \, dx + C$$

and thus,

$$\alpha \int_{\{w\equiv 1\}} |Dv_n|^p \, dx \le C$$

but, $\{x \in \Omega,\ \tau(x) < 1\} \subset \{x \in \Omega,\ w(x) < 1\}$, then,

$$\alpha \int_{\{w\equiv 1\}} |Dv_n|^p \, dx \le C \ ; \tag{2.10}$$

from (2.9) and (2.10) we deduce that:

$$T_K v_n \ \text{ is bounded in } \ W_0^{1,p}(\Omega) \ . \tag{2.11}$$

We are now going to show that $\tilde{u}_n \le v_n$ a.e. in $\tilde{\Omega}$.
For $n=0$, $\tilde{u}_0 = 0 = v_0$.
We set:

$$\varphi(s) = \begin{cases} 0 & \text{if } \ s \le t, \\ \dfrac{1}{h}\,(s-t) & \text{if } \ t < s \le t+h, \\ 1 & \text{if } \ s > t+h \ . \end{cases}$$

We can take $\varphi(u_n)$ as test function in the equation satisfied by u_n, that leads to:

$$\frac{1}{h}\int_{\{t<u_n\le t+h\}} A(x,u_n,Du_n)\,Du_n\,dx = \frac{1}{h}\int_{\{t<u_n\le t+h\}} \sum_{i=0}^m g_i(u_{n-1})\,f_{i,n}(x)\,(u_n - t)\,dx$$
$$+ \int_{\{t+h<u_n\}} \sum_{i=0}^m g_i(u_{n-1})\,f_{i,n}(x)\,dx \ .$$

From (1.4), and because $0 < \frac{u_n - t}{h} \le 1$ on $\{t < u_n \le h+t\}$, we get:

$$\frac{\alpha}{h}\int_{\{t<u_n\le t+h\}} |Du_n|^p \le \int_{\{t<u_n\le t+h\}} \sum_{i=0}^m g_i(u_{n-1})\,f_{i,n}(x)\,dx$$
$$+ \int_{\{t+h<u_n\}} \sum_{i=0}^m g_i(u_{n-1})\,f_{i,n}(x)\,dx \ ;$$

from Hölder,

$$\alpha\Big(\frac{1}{h}\int_{\{t<u_n\le t+h\}} |Du_n|\Big)^p \Big(\frac{1}{h}\int_{\{t<u_n\le t+h\}} dx\Big)^{-\frac{p}{p'}} \le$$
$$\le \int_{\{t<u_n\le t+h\}} \sum_{i=0}^m g_i(u_{n-1})\,f_{i,n}(x)\,dx + \int_{\{t+h<u_n\}} \sum_{i=0}^m g_i(u_{n-1})\,f_{i,n}(x)\,dx \ .$$

We note $\nu(t) = |u_n > t|$. Let h tend to zero.

$$\alpha\Big(-\frac{d}{dt}\int_{\{t<u_n\}} |Du_n|\Big)^p \Big(-\nu'(t)\Big)^{-\frac{p}{p'}} \leq \int_{\{t<u_n\}} \sum_{i=0}^m g_i(u_{n-1})\, f_{i,n}(x)\, dx\ .$$

From the definition of the perimeter of De Giorgi, and the isoperimetric inequality, we have:

$$-\frac{d}{dt}\int_{\{t<u_n\}} |Du_n| \,\geq\, N\,\omega_N^{1/N}\nu(t)^{1-1/N}$$

then,

$$\alpha\, N^p\,\omega_N^{p/N}\nu(t)^{p-p/N}\,(-\nu'(t))^{-\frac{p}{p'}} \,\leq\, \sum_{i=0}^m \int_{t<u_n} g_i(u_{n-1})\, f_{i,n}(x)\, dx$$

but, from the extension of Hardy–Littlewood theorem, which is proved in [6],

$$\sum_{i=0}^m \int_{t<u} g_i(u_{n-1})\, f_{i,n}(x)\, dx \,\leq\, \sum_{i=0}^m \int_0^{\nu(t)} (g_i(u_{n-1}))^\star(\sigma)\, f^\star_{i,n}(\sigma)\, d\sigma\ .$$

As g_i is nondecreasing, we obtain:

$$\sum_{i=0}^m \int_{t<u} g_i(u_{n-1})\, f_{i,n}(x)\, dx \,\leq\, \int_0^{\nu(t)} \sum_{i=0}^m g_i(u^\star_{n-1})(\sigma)\, f^\star_{i,n}(\sigma)\, d\sigma$$

thus,

$$1 \,\leq\, \frac{1}{\alpha}\, N^{-p}\,\omega_N^{-p/N}\nu(t)^{-p+p/N}\,(-\nu'(t))^{p/p'} \int_0^{\nu(t)} \sum_{i=0}^m g_i(u^\star_{n-1})(\sigma)\, f^\star_{i,n}(\sigma)\, d\sigma$$

and thus,

$$1 \,\leq\, \alpha^{-p'/p}\, N^{-p'}\omega_N^{-p'/N}\nu(t)^{-p'+p'/N}\,(-\nu'(t))\,\Big(\int_0^{\nu(t)} \sum_{i=0}^m g_i(u^\star_{n-1})(\sigma)\, f^\star_{i,n}(\sigma)\, d\sigma\Big)^{p'/p}$$

then, we integrate between 0 and $u^\star_n(s)-\epsilon$ with $\epsilon>0$. We know that:

$$\Big|u_n > u^\star_n(s)-\epsilon\Big| \,=\, \Big|u^\star_n > u^\star_n(s)-\epsilon\Big| \,\leq\, \Big|u^\star_n > u^\star_n(s)\Big| \,\leq\, s$$

then,

$$u^\star_n(s)-\epsilon \,\leq\, \alpha^{-p'/p}N^{-p'}C_N^{-p'/N}\int_s^{|\Omega} r^{-p'+p'/N}\Big(\int_0^r \sum_{i=0}^m g_i(u^\star_{n-1})(\sigma)\, f^\star_{i,n}(\sigma)\, d\sigma\Big)^{p'/p} dr\ .$$

As it is true for every $\epsilon > 0$, we obtain:

$$u_n^\star(s) \le \alpha^{-p'/p} N^{-p'} C_N^{-p'/N} \int_s^{|\Omega} r^{-p'+p'/N} \Big(\int_0^r \sum_{i=0}^m g_i(u_{n-1}^\star)(\sigma)\, f_{i,n}^\star(\sigma)\, d\sigma \Big)^{p'/p} dr \; .$$

We suppose by induction that,

$$u_{n-1}^\star(\sigma) \le v_{n-1}^\star(\sigma)$$

then,

$$\begin{aligned} u_n^\star(s) \; &\le \; \alpha^{-p'/p} N^{-p'} C_N^{-p'/N} \int_s^{|\Omega} r^{-p'+p'/N} \Big(\int_0^r \sum_{i=0}^m g_i(v_{n-1}^\star)(\sigma)\, f_{i,n}^\star(\sigma)\, d\sigma \Big)^{p'/p} dr \\ &= \; v_n^\star(s) \; . \end{aligned}$$

The last step consists in proving that (u_n) converges to a renormalized solution of (2.1). First we take $T_k u_n$ as test function in the equation satisfied by u_n,

$$\int_\Omega A(x, u_n, Du_n)\, DT_k u_n \, dx \; \le \; \int_\Omega \sum_{i=0}^m g_i(u_{n-1})\, f_{i,n}(x)\, T_k u_n \, dx \; ;$$

this implies from (1.4) that (we note C_k different constants independent of n, but which depend on k)

$$\begin{aligned} \alpha \int_\Omega |DT_K u_n|^p \, dx \le \int_{u_n \le k} \sum_{i=0}^m g_i(u_{n-1})\, f_{i,n}(x)\, u_n \, dx \\ + k \int_{u_n \ge k} \sum_{i=0}^m g_i(u_{n-1})\, f_{i,n}(x)\, dx \; . \end{aligned}$$

We know that if $u_n(x) \le k$ then $u_{n-1}(x) \le k$, then on $\{u_n \le k\}$, we have $g_i(u_{n-1}) \le C_k$ and $f_{i,n}(x) \le f_i(x)$. Moreover in the second term of the right handside of the previous inequality, we can use the extension of the Hardy–Littlewood theorem which is given in [6], and we obtain:

$$\alpha \int_\Omega |DT_K u_n|^p \, dx \; \le \; C_k \int_\Omega \sum_{i=0}^m f_i(x)\, dx + k \sum_{i=0}^m \int_{\{\tilde u_n \ge k\}} g_i(\tilde u_{n-1})\, \tilde f_{i,n}(x)\, dx \; .$$

We can add $\sum_{i=0}^m \int_{\{\tilde u_n < k\}} g_i(\tilde u_{n-1})\, \tilde f_{i,n}(x)\, \tilde u_n(x)\, dx$ which is nonnegative in the right handside, and so,

$$\begin{aligned}\alpha\int_\Omega |DT_K u_n|^p\,dx \;&\le\; C_k\int_\Omega\sum_{i=0}^m f_i(x)\,dx + k\sum_{i=0}^m\int_{\{\tilde u_n\ge k\}} g_i(\tilde u_{n-1})\,\tilde f_{i,n}(x)\,dx\\ &\quad+\sum_{i=0}^m\int_{\{\tilde u_n<k\}} g_i(\tilde u_{n-1})\,\tilde f_{i,n}(x)\,\tilde u_n(x)\,dx\\ &=\; C_k\int_\Omega\sum_{i=0}^m f_i(x)\,dx+\sum_{i=0}^m\int_{\tilde\Omega} g_i(\tilde u_{n-1})\,\tilde f_{i,n}(x)\,T_k\tilde u_n(x)\,dx\;.\end{aligned}$$

We have seen that, $\forall\, n\in N$, $\tilde u_n\le v_n\le\psi$ a.e. in $\tilde\Omega$, then,

$$\begin{aligned}\alpha\int_\Omega |DT_K u_n|^p\,dx \;&\le\; C_k\int_\Omega\sum_{i=0}^m f_i(x)\,dx+\sum_{i=0}^m\int_{\tilde\Omega} g_i(v_{n-1})\,\tilde f_{i,n}(x)\,T_k v_n\,dx\\ &=\; C_k\int_\Omega\sum_{i=0}^m f_i(x)\,dx+\int_{\tilde\Omega}|DT_k v_n|^p\,dx\\ &\le\; C_k\end{aligned}$$

because of (2.11), so we have proved that,

$$\|DT_K u_n\|_p\;\le\; C_k\;. \tag{2.12}$$

As the sequence (u_n) is nondecreasing, p.p. $x\in\Omega$, $u_n(x)$ tends to infinity or converges to a finite limit, we note $u(x)$. Let $A=\{x\in\Omega,\ u_n(x)\to+\infty\}$, and let $B_{n,k}=\{x\in\Omega,\ u_n(x)>k\}$, then $\forall\, k\ge 0$,

$$A\subset\bigcup_{n=0}^{+\infty} B_{n,k}$$

and

$$\Big|\bigcup_{n=0}^{+\infty} B_{n,k}\Big| = \lim_{n\to\infty}|B_{n,k}|\qquad\forall\, k\ge 0$$

because (u_n) is nondecreasing. But,

$$\Big|\{x\in\Omega,\ u_n(x)>k\}\Big|\;\le\;\Big|\{x\in\tilde\Omega,\ v_n(x)>k\}\Big|\;\le\;\Big|\{x\in\tilde\Omega,\ \psi(x)>k\}\Big|$$

then,

$$|A|\le\Big|\{x\in\tilde\Omega,\ \psi(x)>k\}\Big|,\qquad\forall\, k\in N$$

and consequently

$$|A|\le\lim_{k\to+\infty}\Big|\{x\in\tilde\Omega,\ \psi(x)>k\}\Big|=0$$

then,

$$u_n(x) \to u(x) \quad \text{a.e. } \; x \in \Omega \;.$$

We are now going to show that (u_n) converges to a renormalized solution of (2.1).

From (2.12), we can deduce that $DT_k u_n \to DT_k u$ in $L^p(\Omega)$ *weak*. We are now going to show that $DT_k u_n \to DT_k u$ in $L^p(\Omega)$ *strong*. We take $T_k u_n - T_k u$ as test function in the equation satisfied by u_n, then,

$$\int_\Omega A(x,u_n,Du_n)\, D(T_k u_n - T_k u)\, dx \;= \int_\Omega \sum_{i=0}^m g_i(u_{n-1})\, f_{i,n}(x)\, (T_k u_n - T_k u)\, dx$$

thus,

$$\begin{aligned}\int_\Omega \Big(A(x,u_n,Du_n) - A(x,u_n,DT_k u)\Big)\, D(T_k u_n - T_k u)\, dx \,+ \\ + \int_\Omega A(x,u_n,DT_k u)\, D(T_k u_n - T_k u)\, dx = \\ = \int_\Omega \sum_{i=0}^m g_i(u_{n-1})\, f_{i,n}(x)\, (T_k u_n - T_k u)\, dx \;.\end{aligned}$$

As $u_n(x) \le u(x)$, $T_k u_n - T_k u \equiv 0$ on $\{x \in \Omega,\, u_n(x) \ge k\}$. Then the previous inequality becomes:

$$\begin{aligned}\int_{\{u_n \le k\}} \Big(A(x,T_k u_n,DT_k u_n) - A(x,T_k u_n,DT_k u)\Big)\, D(T_k u_n - T_k u)\, dx \,+ \\ + \int_{\{u_n \le k\}} A(x,T_k u_n,DT_k u)\, D(T_k u_n - T_k u)\, dx = \\ = \int_{\{u_n \le k\}} \sum_{i=0}^m g_i(u_{n-1})\, f_{i,n}(x)\, (T_k u_n - T_k u)\, dx \;.\end{aligned} \tag{2.13}$$

Let n tend to $+\infty$, by Lebesgue theorem, we can see that:

$$\int_{\{u_n \le k\}} \sum_{i=0}^m g_i(u_{n-1})\, f_{i,n}(x)\, (T_k u_n - T_k u)\, dx \;\to\; 0$$

and

$$\begin{aligned}&A(x,T_k u_n,DT_k u)\, \xi_{\{u_n \le k\}} \;\to\; A(x,T_k u,DT_k u)\, \xi_{\{u \le k\}} \quad \text{in} \;\; L^{p'}(\Omega) \text{ strong} \\ &D(T_k u_n - T_k u) \to 0 \quad \text{in} \;\; L^p(\Omega) \text{ weak}\end{aligned}$$

then,

$$\int_{\{u_n \le k\}} A(x,T_k u_n,DT_k u)\, D(T_k u_n - T_u)\, dx \;\to\; 0 \;.$$

We can now use the following lemma which is proved in [2].

Lemma 2.2. *Suppose that A verifies (1.2), (1.3), (1.4), (1.5), if (z_n) is a sequence such that:*

- *z_n is bounded in $L^\infty(\Omega)$,*
- *$z_n \to z$ in $W_0^{1,p}(\Omega)$ weak and a.e. in Ω,*
- $\displaystyle\lim_{n\to 0}\int_\Omega \Big(A(x,z_n,Dz_n)-A(x,z_n,Dz)\Big)\,D(z_n-z)=0;$

then, $z_n \to z$ in $W_0^{1,p}(\Omega)$ strong.

We can apply this lemma to $T_K u_n$ and deduce that: $T_K u_n \to T_K u$ in $W_0^{1,p}(\Omega)$ strong. Let $w \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $h \in C^1(R)$ or piecewise affine, and with compact support, and let k such that $h\equiv 0$ on $]-\infty,-k[\,\cup\,]k,+\infty[$

$$\int_\Omega A(x,u_n,Du_n)\,h'(u_n)\,w\;dx+\int_\Omega A(x,u_n,Du_n)\,h(u_n)\,Dw\;dx = \\ =\int_\omega\sum_{i=0}^m g_i(u_{n-1})\,f_{i,n}(x)\,h(u_n)\,w\;dx$$

that is to say, from the choice of k,

$$\int_\Omega A(x,T_ku_n,DT_ku_n)\,h'(T_ku_n)\,w\;dx+\int_\Omega A(x,T_ku_n,DT_ku_n)\,h(T_ku_n)\,Dw\;dx = \\ =\int_\Omega\sum_{i=0}^m g_i(T_ku_{n-1})\,f_{i,n}(x)\,h(T_ku_n)\,w\;dx\;, \tag{2.14}$$

$$A(x,T_ku_n,DT_Ku_n)\to A(x,T_ku,DT_ku)\qquad\text{a.e. in }\;\Omega,\;\text{ moreover from (1.5)}\,,$$

$$\Big|A(x,T_ku_n,DT_Ku_n)\Big|^{p'}\;\le\;\beta(k)^{p'}\Big(|DT_ku_n|^{p-1}+b(x)\Big)^{p'}\;.$$

The right handside converges in $L^1(\Omega)$ strong, consequently $|A(x,T_ku_n,DT_ku_n)|^{p'}$ is equiintegrable, and then from Vitali's lemma $|A(x,T_ku_n,DT_ku_N)|^{p'}\to|A(x,T_ku,DT_ku)|^{p'}$. Finally, $A(x,T_ku_n,DT_ku_N)\to A(x,T_ku,DT_ku)$ in $L^{p'}(\Omega)$ strong and we can pass to the limit in (2.14), and we obtain

$$\int_\Omega A(x,T_ku,DT_ku)\,h'(T_ku)\,w\;dx+\int_\Omega A(x,T_ku,DT_ku)\,h(T_ku)\,Dw\;dx = \\ =\int_\Omega\sum_{i=0}^m g_i(T_ku)\,f_i(x)\,h(T_ku)\,w\;dx$$

that is to say,

$$\int_\Omega A(x,u,Du)\,h'(u)\,w\;dx + \int_\Omega A(x,u,Du)\,h(u)\,Dw\;dx = \\ = \int_\Omega \sum_{i=0}^m g_i(u)\,f_i(x)\,h(u)\,w\;dx~.$$

3 – A sub-supersolution theorem

Theorem 3.1. *We suppose that A satisfies (1.2), (1.3), (1.4), (1.5), and that F satisfies (1.6), (1.7), if there exists a nonnegative renormalized supersolution ψ of (1.1), then there exists a nonnegative renormalized solution u of (1.1), such that $u \le \psi$ a.e. in Ω.*

Proof: We can remark that $\varphi = 0$ is a subsolution of (1.1) (we could remark that the hypothesis $F(x,s) \ge 0$ could be replaced by: there exists a weak subsolution $\varphi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that $\varphi \le \psi$).
Let $n \ge 1$, we consider the problem:

$$\begin{cases} -\operatorname{div} A(x,u,Du) = F_n(x,u(x)) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega~, \end{cases} \tag{3.1}$$

where $F_n(x,s) = \dfrac{F(x,s)}{1+\frac{1}{n}F(x,s)}$.

Lemma 3.1. *Under the hypotheses of Theorem 3.1, we suppose that there exists a weak subsolution $v \in L^\infty(\Omega)$ of problem (3.1), such that $0 \le v \le \psi$, then there exists a solution $u \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ of problem (3.1) such that $v \le u \le \psi$.*

Proof of the Lemma: Let M such that $\|v\|_\infty \le M$. We set:

$$\bar A_M\Big(x,u(x),Du(x)\Big) = \begin{cases} A\Big(x,T_M\psi(x),Du(x)\Big) & \text{if } u(x) \ge T_M\psi(x), \\ A\Big(x,u(x),Du(x)\Big) & \text{if } v(x) \le u(x) \le T_M\psi(x), \\ A\Big(x,v(x),Du(x)\Big) & \text{if } u(x) \le v(x)~, \end{cases}$$

and

$$\bar F(x,u(x)) = \begin{cases} F(x,\psi(x)) & \text{if } u(x) \ge \psi(x), \\ F(x,u(x)) & \text{if } v(x) \le u(x) \le \psi(x), \\ F(x,v(x)) & \text{if } u(x) \le v(x)~, \end{cases}$$

$$\bar{F}_n(x,u(x)) = \frac{\bar{F}(x,u(x))}{1+\frac{1}{n}\,\bar{F}(x,u(x))} \ .$$

Then, $-\operatorname{div}\bar{A}_M(x,u(x),Du(x)) - \bar{F}_n(x,u(x))$ verifies the hypotheses of the theorem of [4, p. 180], and the problem:

$$\text{(3.2)} \qquad \begin{cases} -\operatorname{div}\bar{A}_M\Big(x,u(x),Du(x)\Big) = \bar{F}_n(x,u(x)), \\ u=0 \quad \text{ on } \ \partial\Omega \ , \end{cases}$$

has at least one solution u in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that:

$$\|u\|_\infty \,\le\, \frac{1}{-p'+\frac{p'}{N}+\frac{p'}{p}+1}\,|\Omega|^{-p'+\frac{p'}{N}+\frac{p'}{p}+1}\,\alpha^{-p/p'}N^{-p'}n^{\frac{p'}{p}} \,=\, D_n \ .$$

In the following we shall suppose that $M \ge D_n$. We are going to show that moreover, $u \le \psi$. We take $(u-\psi)^+$ as test function in (3.2), and in the inequation satisfied by ψ, we take $w = (u-\psi)^+ \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, and a function $h \in C^1(R)$ such that $h(s)=1$ if $s \le M$ and $h(s)=0$ if $s \ge M+1$. As in the previous section, that leads to:

$$\int_\Omega \Big(\bar{A}_M\Big(x,u(x),Du(x)\Big) - A\Big(x,\psi(x),D\psi(x)\Big)\Big)\,D(u-\psi)^+\,dx \,\le$$
$$\le\, \int_\Omega \Big(\bar{F}_n(x,u(x)) - F(x,\psi(x)\Big)\,(u-\psi)^+\,dx \,=\, 0$$

thus,

$$\int_\Omega \Big(\bar{A}_M\Big(x,u(x),Du(x)\Big) - A\Big(x,\psi(x),D\psi(x)\Big)\Big)\,D(u-\psi)^+\,dx \,\le\, 0$$

and we deduce that $(u-\psi)^+=0$ a.e. in Ω.

We take now $(v-u)^+$ as test function in (3.2) and in the inequation satisfied by v, and we can show like previously that $u \ge v$. Finally, we have shown that u is a solution of (3.1), and proved the lemma.

We construct a sequence (u_n), such that:

$$u_0 = 0 \ ,$$

$$u_n \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) \ ,$$

$$-\operatorname{div} A(x,u_n,Du_n) = F_n(x,u_n(x)) \ ,$$

$$u_{n-1} \le u_n \le \psi \ .$$

Using the previous lemma, we can show by induction that we can construct this sequence, if we remark that

$$-\operatorname{div} A(x, u_{n-1}, Du_{n-1}) = F_{n-1}(x, u_{n-1}(x)) \leq F_n(x, u_{n-1}(x))$$

that is to say, u_{n-1} is a subsolution of the equation satisfied by u_n.

To prove that $T_k u_n$ is bounded in $W_0^{1,p}(\Omega)$, we use the same method as in the previous section to prove (2.11). Let τ be a function of $W_0^{1,p}(\Omega)$ such that $0 \leq \tau \leq 1$, then, $(u_n - k\tau)^- \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\|(u_n - k\tau)^-\|_\infty \leq k$. We take $-(u_n - k\tau)^-$ as test function in the equation satisfied by u_n, and we note $C_{k,\tau}$ different constant which depend on k and τ,

$$\int_{\{u_n \leq k\tau\}} A(x, u_n, Du_n)\, D(u_n - k\tau)\, dx \;=\; -\int_\Omega F_n(x, u_n)\, (u_n - \tau k)^-\, dx$$

then, from (1.4) and (1.5),

$$\alpha \int_{\{u_n \leq k\tau\}} |Du_n|^p\, dx \;\leq\; C_{k,\tau} \int_{\{u_n \leq k\tau\}} \Big(|Du_n|^{p-1} + b(x)\Big)\, D\tau\, dx \;+\; C_{k,\tau}$$

then,

$$\alpha \int_{\{u_n \leq k\tau\}} |Du_n|^p\, dx \;\leq\; C_{k,\tau} \Big(\int_{\{u_n \leq k\tau\}} |Du_n|^p\, dx\Big)^{\frac{p-1}{p}}\, dx \;+\; C_{k,\tau}$$

and then,

$$\int_{\{u_n \leq k\tau\}} |Du_n|^p\, dx \;\leq\; C_{k,\tau}$$

a fortiori,

$$\int_{\{\tau \equiv 1\}} |DT_k u_n|^p\, dx \;\leq\; C_{k,\tau}\;. \tag{3.3}$$

We now specify the choice of τ, we take $\tau = T_1\left((\psi - C_\psi - 1)^+\right)$, then $w \equiv 1$ on $\{\tau < 1\}$ and $w \equiv 0$ on $\{\psi \geq C_\psi + 2\}$. In the equation satisfied by u_n, we take $w\, u_n$ which is in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function.

$$\int_\Omega A(x, u_n, Du_n)\, Du_n\, w\; dx \;+\; \int_\Omega A(x, u_n, Du_n)\, Dw\, u_n\; dx \;=\; \int_\Omega F_n(x, u_n)\, w\, u_n\; dx$$

then,

$$\begin{aligned}
\alpha \int_{\{w \equiv 1\}} |Du_n|^p\, dx \;+\; \alpha \int_{\{w<1\}} |Du_n|^p\, w\; dx \;\leq&\\
\leq C \int_\Omega \Big(|Du_n|^{p-1} + b(x)\Big)\, |Dw|\; dx \;+\; C&\\
\leq C \int_{\{w \equiv 1\}} |Du_n|^{p-1}\, |Dw|\, dx \;+\; C \int_{\{w<1\}} |Du_n|^{p-1}\, |Dw|\, dx \;+\; C&
\end{aligned}$$

but, $\{x\in\Omega,\ w(x)<1\}\subset\{x\in\Omega,\ \tau(x)=1\}$, then from (3.3),

$$\alpha\int_{\{w\equiv 1\}}|Du_n|^p\,dx\ \le\ C\int_{\{w\equiv 1\}}|Du_n|^{p-1}\,|Dw|\,dx\ +\ C$$

and thus,

$$\alpha\int_{\{w\equiv 1\}}|Du_n|^p\,dx\ \le\ C$$

but, $\{x\in\Omega,\tau(x)<1\}\subset\{x\in\Omega,\ w(x)<1\}$, then,

$$\alpha\int_{\{w\equiv 1\}}|Du_n|^p\,dx\ \le\ C \tag{3.4}$$

from (3.3) and (3.4) we deduce that $T_K u_n$ is bounded in $W_0^{1,p}(\Omega)$. On another hand, as (u_n) is nondecreasing and $u_n\le\psi$, u_n converges almost everywhere in Ω to a function u. This implies that $DT_K u_n\to DT_K u$ in $L^p(\Omega)$ weak. In the same way, with slight modifications, we can prove as in the previous section that $T_k u_n\to T_k u$ in $W_0^{1,p}(\Omega)$ strong, and that u is a renormalized solution of (1.1). This proves Theorem 3.1.

We can now prove Theorem 1.1: suppose that there exists a renormalized supersolution $\psi\ge 0$ for problem (1.8), then problem (2.1) has a renormalized solution $\bar u$ such that $|\bar u>t|\ \le\ |\psi>t|$, $\forall\,t\ \ge\ 0$. But, $\bar u$ is also a renormalized supersolution of (1.1), and then by Theorem 3.1, there exists a nonnegative renormalized solution u of problem (1.1), such that $u\le\bar u$ a.e. in Ω, and thus such that $|u>t|\le|\psi>t|$.

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