PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 3 – 1999

CHARACTERIZATION FOR RELATIONS ON SOME SUMMABILITY METHODS

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Abstract: In this paper we characterize a previous result proved by us connecting the summability methods $|\overline{N}, p_n|_k$ with either $|N, q_n|_k$ or $|\overline{N}, w_n|_k$ for given sequences $\{p_n\}$, $\{q_n\}$ and $\{w_n\}$ of positive real constants. Other results are also deduced.

1 – Introduction

Let $\sum a_n$ be an infinite series with partial sums s_n. Let σ_n^δ and η_n^δ denote the n-th Cesàro mean of order δ ($\delta > -1$) of the sequences $\{s_n\}$ and $\{n\,a_n\}$ respectively. The series $\sum a_n$ is said to be summable $|C,\delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}\,|\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty\ ,$$

or equivalently

$$\sum_{n=1}^{\infty} \frac{1}{n}\,|\eta_n^\delta|^k < \infty\ .$$

Let $\{p_n\}$ be a sequence of real or complex constants such that

$$P_n = p_0 + p_1 + \cdots + p_n \quad (p_{-1} = P_{-1} = 0)\ .$$

The series $\sum a_n$ is said to be summable $|N, p_n|$ if

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < \infty\ , \tag{1.1}$$

Received: October 19, 1997.

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v}\, s_v \qquad (T_{-1} = 0) \ .$$

We write $p = \{p_n\}$ and

$$M = \Big\{p\colon\ p_n > 0 \text{ and } p_{n+1}/p_n \le p_{n+2}/p_{n+1} \le 1,\ \ n = 0, 1, ...\Big\} \ .$$

It is known for $p \in M$ (1.1) holds iff (see [5])

$$\sum_{n=1}^{\infty} \frac{1}{n\, P_n} \left| \sum_{v=1}^{n} p_{n-v}\, v\, a_v \right| < \infty \ .$$

For $p \in M$, we say that $\sum a_n$ is summable $|N, p_n|_k$, $k \ge 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{1}{P_n} \sum_{v=1}^{n} p_{n-v}\, v\, a_v \right|^k < \infty \qquad \text{(Sulaiman [6])} \ .$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, where A_n^r is the coefficient of x^n in the power series expansion of $(1-x)^{-r-1}$ for $|x| < 1$, $|N, p_n|_k$ reduces to $|C, r|_k$ summability. The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \ge 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty \qquad \text{(Bor [1])} \ ,$$

where

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v\, s_v \ .$$

If we take $p_n = 1$, then $|\overline{N}, p_n|_k$ summability is equivalent to $|C, 1|_k$ summability. In general these two summabilities are not comparable.

Throughout this paper we set

$$Q_n = q_0 + q_1 + \cdots + q_n \ , \qquad q_{-1} = Q_{-1} = 0 \ ,$$

$$W_n = w_0 + w_1 + \cdots + w_n \ , \qquad w_{-1} = W_{-1} = 0 \ ,$$

$$\Delta f_n = f_n - f_{n+1} \ .$$

Let $\{p_n\}$ and $\{q_n\}$ be sequences of positive real constants such that $q \in M$. $\sum a_n$ is said to be summable $|N, p_n, q_n|_k$, $k \ge 1$, if

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n\, R_{n-1}^k} \left| \sum_{v=1}^{n} P_{v-1}\, q_{n-v}\, a_v \right|^k < \infty \qquad \text{(Sulaiman [7])} \ ,$$

where

$$R_n = p_0\, q_n + p_1\, q_{n-1} + \cdots + p_n\, q_0 \ .$$

Clearly $|N, p_n, 1|_k$ is the same as $|\overline{N}, p_n|_k$.

The following results are known.

Theorem A (Bor [1]). *Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$*

$$\text{(1.2)} \qquad \begin{aligned} &\textbf{(i)} \qquad n\, p_n = O(P_n) \ , \\ &\textbf{(ii)} \qquad P_n = O(n\, p_n) \ . \end{aligned}$$

If $\sum a_n$ is summable $|C, 1|_k$, then it is summable $|\overline{N}, P_n|_k$, $k \geq 1$.

Theorem B (Bor [2]). *Let $\{p_n\}$ be a sequence of positive real constants such that it satisfies (1.2). If $\sum a_n$ is summable $|\overline{N}, p_n|_k$ then it is also summable $|C, 1|_k$.*

Theorem C (Sulaiman [7]). *Let $\{p_n\}$, $\{q_n\}$ and $\{w_n\}$ be sequences of positive real constants such that $q \in M$ and $\{p_n / P_n\, R^k_{n-1}\}$ is nonincreasing for $q_n \neq c$. Let t_n denote the $(\overline{N}, w_n)$-mean of the series $\sum a_n$. Let $\{\varepsilon_n\}$ be a sequence of constants. If*

$$\text{(1.3)} \qquad \sum_{n=v+1}^{m} \frac{p_n\, q_{n-v-1}}{P_n\, R_{n-1}} = O(P_v^{-1}) \, , \qquad m \to \infty \ ,$$

$$\text{(1.4)} \qquad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\varepsilon_n|^k \, |\Delta t_{n-1}|^k < \infty \ ,$$

$$\text{(1.5)} \qquad \sum_{n=1}^{\infty} \frac{p_n}{P_n} \left(\frac{W_n}{w_n}\right)^{k} |\varepsilon_n|^k \, |\Delta t_{n-1}|^k < \infty \ ,$$

$$\text{(1.6)} \qquad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{W_n}{w_n}\right)^{k} |\Delta\varepsilon_n|^k \, |\Delta t_{n-1}|^k < \infty \ ,$$

and

$$\text{(1.7)} \qquad \sum_{n=1}^{\infty} \frac{p_n}{P_n} \left(\frac{P_{n-1}}{R_{n-1}}\right)^{k} \left(\frac{W_n}{w_n}\right)^{k} |\varepsilon_n|^k \, |\Delta t_{n-1}|^k < \infty \ ,$$

then the series $\sum a_n$ is summable $|N, p_n, q_n|_k$, $k \geq 1$.

It may be mentioned that Theorems A and B are special cases of Theorem C.

The object of this paper is to prove the following

Theorem D. *Let (p_n), (q_n) and (w_n) be sequences of positive real constants such that $q \in M$ and $(p_n/P_n\,R^k_{n-1})$ nonincreasing for $q_n \neq c$. Suppose that*

$$R_{n-1} = O(P_{n-1}) \ , \tag{1.8}$$

$$P_n\,w_n = O(p_n\,W_n) \ , \tag{1.9}$$

$$\Delta\left(\frac{w_n\,P_n\,R_{n-1}}{W_n\,p_n\,P_{n-1}}\right) = O\left(\frac{w_n}{W_n}\right) \ , \tag{1.10}$$

$$\Delta\left(\frac{W_n\,p_n\,P_{n-1}}{w_n\,P_n\,R_{n-1}}\,\varepsilon_n\right) = O\left(\frac{p_n\,P_{n-1}}{P_n\,R_{n-1}}\right) \ . \tag{1.11}$$

Then necessary and sufficient conditions that $\sum a_n\,\varepsilon_n$ be summable $|N, p_n, q_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, w_n|_k$, $k \geq 1$, are

$$\text{(i)} \qquad \varepsilon_n = O\left(\frac{w_n\,P_n\,R_{n-1}}{W_n\,p_n\,P_{n-1}}\right) \ ,$$

$$\text{(ii)} \qquad \Delta\varepsilon_n = O\left(\frac{w_n}{W_{n-1}}\right) \ .$$

2 – Lemmas

Lemma 1 (Sulaiman [7]). *Let $q \in M$. Then for $0 < \gamma \leq 1$,*

$$\sum_{n=v+1}^{\infty} \frac{q_{n-v-1}}{n^\gamma\,Q_{n-1}} = O(v^{-\gamma}) \ .$$

Lemma 2 (Bor [4]). *Let $k \geq 1$ and let $A = (a_{nv})$ be an infinite matrix. In order that $A \in (\ell^k; \ell^k)$ it is necessary that*

$$a_{nv} = O(1) \qquad (\text{for all } \ n, v) \ . \tag{2.1}$$

Lemma 3. *Suppose that $\varepsilon_n = O(f_n\,g_n)$, $f_n, g_n \geq 0$, $f_{n+1}\,g_{n+1} = O(f_n\,g_n)$, $\Delta(f_n\,g_n) = O(f_n)$ and $\Delta(\varepsilon_n/f_n\,g_n) = O(1/g_n)$. Then $\Delta\varepsilon_n = O(f_n)$.*

Proof: We have

$$\varepsilon_n = k_n\,f_n\,g_n, \qquad \text{where } \ k_n = \frac{\varepsilon_n}{f_n\,g_n} = O(1) \ ,$$

$$\Delta\varepsilon_n = k_n\,f_{n+1}\,\Delta g_n + k_n\,g_{n+1}\,\Delta f_n + f_{n+1}\,g_{n+1}\,\Delta k_n \ .$$

Since

$$f_n\,\Delta g_n + g_{n+1}\,\Delta f_n = O(f_n)\ ,$$

then

$$\begin{aligned}\Delta\varepsilon_n &= k_n\,f_n\,\Delta g_n + k_n\Big[O(f_n) - f_n\,\Delta g_n\Big] + f_{n+1}\,g_{n+1}\,\Delta k_n\\ &= k_n\,O(f_n) + O(f_n\,g_n\,|\Delta k_n|)\\ &= O(f_n) + O(f_n)\\ &= O(f_n)\ . \ \blacksquare\end{aligned}$$

3 – Proof of Theorem D

Write

$$T_n = \sum_{v=1}^{n} P_{v-1}\,q_{n-v}\,a_v\,\varepsilon_v\,, \qquad t_n = \frac{w_n}{W_n\,W_{n-1}}\,\sum_{v=1}^{n} W_{v-1}\,a_v\ ,$$

$$\begin{aligned}T_n &= \sum_{v=1}^{n} W_{v-1}\,a_v\left(\frac{P_{v-1}}{W_{v-1}}\,q_{n-v}\,\varepsilon_v\right)\\ &= \sum_{v=1}^{n-1}\left(\sum_{r=1}^{v} W_{r-1}\,a_r\right)\Delta_v\left(\frac{P_{v-1}}{W_{v-1}}\,q_{n-v}\,\varepsilon_v\right) + \left(\sum_{r=1}^{n} W_{r-1}\,a_r\right)\frac{P_{n-1}}{W_{n-1}}\,q_0\,\varepsilon_n\\ &= \sum_{v=1}^{n-1}\left\{P_{v-1}\,\Delta_v q_{n-v}\,\frac{W_v}{w_v}\,\varepsilon_v\,t_v + P_{v-1}\,q_{n-v-1}\,\varepsilon_v\,t_v\right.\\ &\qquad\left. + p_v\,q_{n-v-1}\,\frac{W_{v-1}}{w_v}\,\varepsilon_v\,t_v - P_v\,q_{n-v-1}\,\frac{W_{v-1}}{w_v}\,\Delta\varepsilon_v\,t_v\right\} + P_{n-1}\,q_0\,\frac{W_n}{w_n}\,\varepsilon_n\,t_n\\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} + T_{n,5}\ , \qquad \text{say}\ .\end{aligned} \tag{3.1}$$

In order to prove sufficiency, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{p_n}{P_n\,R_{n-1}^k}\,|T_{n,r}|^k < \infty\,, \qquad r = 1,\,2,\,3,\,4,\,5\ .$$

Applying Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |T_{n,1}|^k &= \sum_{n=1}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \left| \sum_{v=1}^{n-1} P_{v-1}\, \Delta_v q_{n-v}\, \frac{W_v}{w_v}\, \varepsilon_v\, t_v \right|^k \\
&\le \sum_{n=1}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \sum_{v=1}^{n-1} P_{v-1}^k\, |\Delta_v q_{n-v}| \left(\frac{W_v}{w_v}\right)^k |\varepsilon_v|^k\, |t_v|^k \\
&\quad \cdot \left\{ \sum_{v=1}^{n-1} |\Delta_v q_{n-v}| \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} P_{v-1}^k \left(\frac{W_v}{w_v}\right)^k |\varepsilon_v|^k\, |t_v|^k \sum_{n=v}^{m+1} \frac{p_n\, |\Delta_v q_{n-v}|}{P_n R_{n-1}^k} \\
&= O(1) \sum_{v=1}^{m} \frac{p_v}{P_v} \left(\frac{P_{v-1}}{R_{v-1}}\right)^k \left(\frac{W_v}{w_v}\right)^k |\varepsilon_v|^k\, |t_v|^k ,
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |T_{n,4}|^k &= \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \left| \sum_{v=1}^{n-1} P_v\, q_{n-v-1}\, \frac{W_{v-1}}{w_v}\, \Delta\varepsilon_v\, t_v \right|^k \\
&\le \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v\, q_{n-v-1} \left(\frac{W_{v-1}}{w_v}\right)^k |\Delta\varepsilon_v|^k\, |t_v|^k \\
&\quad \cdot \left\{ \sum_{v=1}^{n-1} \frac{p_v\, q_{n-v-1}}{R_{n-1}} \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v \left(\frac{W_{v-1}}{w_v}\right)^k |\Delta\varepsilon_v|^k\, |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n\, q_{n-v-1}}{P_n R_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} \left(\frac{W_{v-1}}{w_v}\right)^k |\Delta\varepsilon_v|^k\, |t_v|^k .
\end{aligned}$$

Similarly we can show that

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |T_{n,2}|^k = O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} |\varepsilon_v|^k\, |t_v|^k ,$$

$$\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} |T_{n,3}|^k = O(1) \sum_{v=1}^{m} \frac{p_v}{P_v} \left(\frac{W_v}{w_v}\right)^k |\varepsilon_v|^k\, |t_v|^k ,$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n R_{n-1}^k} |T_{n,5}|^k = O(1) \sum_{n=1}^{m} \frac{p_n}{P_n} \left(\frac{P_{n-1}}{R_{n-1}}\right)^k \left(\frac{W_n}{w_n}\right)^k |\varepsilon_n|^k\, |t_n|^k .$$

The sufficiency follows.

Necessity of (i). Using the result of Bor in [4], the transformation from $((P_n/p_n)^{1-1/k}\, t_n)$ into $([(p_n/P_n)^{1/k}/R_{n-1}]\, T_n)$ maps ℓ^k into ℓ^k and hence the diagonal elements of this transformation are bounded (by Lemma 2) and so (i) is necessary.

Necessity of (ii). This follows from Lemma 3 and the necessity of (i) by taking $f_n = w_n/W_n,\ g_n = P_n R_{n-1}/p_n P_{n-1}$.

4 – Applications

Corollary 1. *Let $\{p_n\}$ and $\{w_n\}$ be sequences of positive real constants such that (1.9) is satisfied.*

Then the necessary and sufficient conditions such that $\sum a_n$ be summable $|\overline{N}, p_n|_k$ whenever it is summable $|\overline{N}, w_n|_k$, $k \ge 1$, is

$$p_n\, W_n = O(P_n\, w_n) \ . \tag{4.1}$$

The proof follows from Theorem D by putting $\varepsilon_n = 1$, $q_n = 1$.

Corollary 2 (Bor and Thorpe [3])**.** *Let $\{p_n\}$ and $\{w_n\}$ be sequences of positive real constants such that (1.9) and (4.1) are satisfied.*

Then the series $\sum a_n$ is summable $|\overline{N}, p_n|_k$ iff it is summable $|\overline{N}, w_n|_k$, $k \ge 1$.

The proof follows from Corollary 1.

Corollary 3. *Let (p_n), (w_n) be sequences of positive real constants such that (1.9) is satisfied and*

$$\Delta\!\left(\frac{w_n\, P_n}{W_n\, p_n}\right) = O\!\left(\frac{w_n}{W_n}\right) ,$$

$$\Delta\!\left(\frac{W_n\, p_n}{w_n\, P_n}\, \varepsilon_n\right) = O\!\left(\frac{p_n}{P_n}\right) .$$

Then necessary and sufficient conditions that $\sum a_n\, \varepsilon_n$ be summable $|\overline{N}, p_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, w_n|_k$, $k \ge 1$, are

$$\varepsilon_n = O\!\left(\frac{w_n\, P_n}{W_n\, p_n}\right), \qquad \Delta\varepsilon_n = O\!\left(\frac{w_n}{W_n}\right) .$$

The proof follows from Theorem D by putting $q_n = \varepsilon_n = 1$.

Remark. It may be mentioned that Theorems A and B could be obtained from Corollary 2.

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