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LINES ON DEL PEZZO SURFACES WITH $K_S^2 = 1$ IN CHARACTERISTIC 2 IN THE SMOOTH CASE

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Abstract: In the case when the branch divisor of the antibicanonical map is smooth, we prove the existence in characteristic 2 of 240 (-1)-curves on a smooth projective surface with $q = 0$, $K_S^2 = 1$, $|-K_S|$ ample and containing an irreducible reduced curve, concluding in this case the proof of Castelnuovo's criterion of rationality.

1 – Introduction

In this paper we prove the following theorem:

Theorem 1.1. *Let S be a smooth, projective surface over an algebraically closed field $\mathbb{K}$ of characteristic 2. Assume that:*

(i) $q(S) = 0$;

(ii) $-K_S$ *is ample;*

(iii) $K_S^2 = 1$.

Then the anti bicanonical map $\phi_2 = \phi_{|-2K_S|}$ is a $2:1$ morphism whose image is a quadric cone $\mathcal{Q} \subset \mathbb{P}^3$. Suppose moreover that the branch divisor $A \subset \mathcal{Q}$ of ϕ_2 is smooth. Then S contains 240 (-1)-curves.

More precisely, there are 120 distinct planes H in $\mathbb{P}^3$ not passing through the vertex V of the cone $\mathcal{Q}$ such that $\phi_2^*(H) = \Gamma_1 + \Gamma_2$ where Γ_1, Γ_2 are two (-1)-curves such that $\Gamma_1 \cdot \Gamma_2 = 3$. Every (-1)-curve arises in this way.

Call a pair $\{\Gamma_1, \Gamma_2\}$ of (-1)-curves of type $(1,1,1)$, $(2,1)$, (3) if $\Gamma_1 \cap \Gamma_2$ contains 3 points, respectively 2, respectively 1 point (cf. 2.3) and denote $n_{(1,1,1)}$, $n_{(2,1)}$, $n_{(3)}$ the number of such pairs. Then the possible values of $n_{(1,1,1)}$, $n_{(2,1)}$, $n_{(3)}$ are shown in (72) and table 1.

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Since the surface S can't be minimal, Theorem 1.1 gives together with the results proved in [10], [11] a proof in positive characteristic of Castelnuovo's rationality criterion $q = P_2 = 0$ for a smooth algebraic surface in all cases but $K_S^2 = 1$, $\mathrm{Char}(\mathbb{K}) = 2$ and the branch divisor of ϕ_2 is not smooth (cf. 2.5 (iv)). Indeed, (see [5]) a minimal surface S for which $q = P_2 = 0$ is either rational or else

(1) $\mathrm{Pic}(S) = \mathbb{Z}[-K_S]$,

(2) $|-K_S|$ contains an irreducible reduced curve,

(3) $K_S^2 > 0$.

We are left to exclude the second possibility, which is done in [10] in the case $K_S^2 \geq 2$ and in [11] and Theorem 1.1 in the case $K_S^2 = 1$ when $\mathrm{Char}\,\mathbb{K} \neq 2$ or the branch divisor A of ϕ_2 is smooth.

Our proof uses elementary methods and is based on the fact that the (-1)-curves in S occur as the pull-back $\phi_2^*(H)$ having at least 3 singular points of planes $H \subset \mathbb{P}^3$ (see 2.2).

There exists a Segre–Hirzebruch $\mathbb{F}_{10}$ surface $\mathbb{F}$ over $A \cong \mathbb{P}^1$ and a morphism $\psi\colon \mathbb{F} \to \mathbb{P}^{3\vee}$ with the property that the planes through $Q \in A$ such that their pull-back is singular above Q are parametrized up to a purely inseparable extension of degree 2 by the image $\psi(\mathbb{F}_Q)$ of the fiber of $\mathbb{F}$ over Q (see (7)). Then the pairs of (-1)-curves correspond to the triple points of $\psi(\mathbb{F})$, hence to triples of nodes of the double curve Λ of ψ.

To prove the irreducibility of Λ and count the number of nodes, we determine the contribution to its arithmetic genus of the other singularities, and this requires heavy computer calculations. On the other side, we find all the (-1)-curves on the given surface and not just prove the existence of one and get information on their type and configuration.

To perform the computations in section 6 we used CoCoA, a Gröbner-basis based symbolic system (by A. Capani, G. Niesi, L. Robbiano, Dept. of Mathematics, University of Genova) running on a unix machine.

The rationality criterion is proved or sketched in [1], [2], [3], [4], [6], [8], [9].

Notations and conventions

If C is a curve, $p_a(C) = h^1(C, \mathcal{O}_C)$ is its arithmetic genus. If S is a smooth surface, $q(S) = h^1(S, \mathcal{O}_S)$, K_S is a canonical divisor and $P_2(S) = h^0(S, 2K_S)$. A (-1)-curve in S is an irreducible curve Γ s.t. $\Gamma^2 = \Gamma K_S = -1$.

Char $(\mathbb{K})$ is its characteristic of the field $\mathbb{K}$. $\mathbb{F}_d$ is the Segre–Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$. $\mathbb{P}^{n\vee}$ is the projective space of the hyperplanes of $\mathbb{P}^n$. $T\mathcal{Q}_Q$ is the tangent space of the hypersurface $\mathcal{Q}$ at $Q \in \mathcal{Q}$.

$\mathbb{K}[[t]]$ is the ring of formal power series, (t) is its maximal ideal and $\mathbb{K}[[t]]^* = \{\xi \in \mathbb{K}[[t]] \colon \xi(0) \neq 0\}$; $o_n(t) \in (t^n)$. By a formal neighborhood of a point P on a curve C we mean the finite set of local parametrizations of C by power series centered at P; each parametrization corresponds to a branch of C through P. If f is a polynomial, $f_X = \frac{\partial f}{\partial X}$; if $\xi(t) \in \mathbb{K}[[t]]$ then $\xi'(t) = \frac{d\xi}{dt}$.

$M(n, m, \mathcal{S})$ is the vector space of $m \times n$ matrices with entries in a ring $\mathcal{S}$; if $M \in M(n, m, \mathbb{K})$, $S_R(M)$ is the subspace of $\mathbb{K}^m$ generated by its rows. If $\mathrm{Char}\mathbb{K} = 2$ and $N = (n_{i,j}) \in M(n, m, \mathbb{K})$, then $N^{[2]} = (n_{i,j}^2)$, $N^{[\frac{1}{2}]} = (\sqrt{n_{i,j}})$.

$W^\perp$ is the orthogonal of the subspace W of $\mathbb{K}^n$ with respect to the standard bilinear symmetric form ${}^tX \cdot Y$. $\#A$ is the number of elements of a finite set A. $\mathrm{Im}\, f$ is the image of the function f.

2 – The anticanonical model of S

Let S be a smooth, projective surface defined over an algebraically closed field $\mathbb{K}$ of characteristic 2 satisfying the hypothesis (i), (ii), (iii) of Theorem 1.1. Remark that (i), (ii), (iii) imply that every divisor in the linear system $|-K_S|$ is irreducible and reduced, and $|-K_S|$ has projective dimension equal to 1, as shown in [11]. The following facts up to 2.3 are based on [7] and proved in [11].

If $\mathcal{S} = \mathbb{K}[X_0, X_1, W, Z]$ is graded by $\deg X_0 = \deg X_1 = 1$, $\deg W = 2$, $\deg Z = 3$, and $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}_n$, $\mathcal{R}_n = H^0(S, -nK_S)$, is the anticanonical ring of S, there exists a surjective graded $\mathbb{K}$-algebra homomorphism $\mathcal{S} \to \mathcal{R}$ mapping X_0, X_1, W, Z to $x_0, x_1 \in \mathcal{R}_1$, $w \in \mathcal{R}_2$, $z \in \mathcal{R}_3$. An isomorphism $\mathcal{S}/(\sigma) \cong \mathcal{R}$ is induced, where $\sigma = Z^2 + Z\, a(X_0, X_1, W) + b(X_0, X_1, W)$, and σ, a, b are homogeneous of degree 6, 3, 6 respectively.

$$
\begin{tikzcd}
\mathrm{S} \arrow[r, "\eta\ \cong"] \arrow[dr, "\phi_2"] & \Sigma \arrow[r, hook] \arrow[dr, "\pi"] & \mathbb{P}(1,1,2,3) \arrow[d, "\Pi"] \\
& \mathcal{Q} \arrow[d] & \mathbb{P}(1,1,2) \arrow[l, "\cong"'] \arrow[dl, "j"] \\
& \mathbb{P}^3 &
\end{tikzcd}
$$

If $\Sigma = \operatorname{Proj}\mathcal{R} \subset \mathbb{P}(1,1,2,3) = \operatorname{Proj}\mathcal{S}$ is the anticanonical model of S, then $\eta\colon S \to \Sigma$, $\eta = (x_0, x_1, w, z)$, is an isomorphism, as $-K_S$ is ample.

The map $j:\ \mathbb{P}(1,1,2) \to \mathbb{P}^3$, $j(x_0,x_1,w) = (x_0^2, x_0x_1, x_1^2, w)$, induces an isomorphism between $\mathbb{P}(1,1,2)$ and the quadric cone $\mathcal{Q} = \{T_0T_2 - T_1^2 = 0\}$ in $\mathbb{P}^3$, and the antibicanonical map $\phi_2: S \to \mathbb{P}^3$, $\phi_2 = (x_0^2, x_0x_1, x_1^2, w)$, factors through $\mathcal{Q}$. The projection $\Pi\colon \mathbb{P}(1,1,2,3) \to \mathbb{P}(1,1,2)$ sending (x_0,x_1,w,z) to (x_0,x_1,w) induces $\pi:\ \Sigma \to \mathbb{P}(1,1,2)$, which corresponds to the antibicanonical map after the identifications $S \cong \Sigma$ and $\mathbb{P}(1,1,2) \cong \mathcal{Q}$, which we shall assume from now on. It follows that ϕ_2 is $2:1$ onto $\mathcal{Q}$.

In conclusion, Σ is the $2:1$ covering of $\mathbb{P}(1,1,2)$ defined by

$$\sigma = Z^2 + Z\,a(X_0,X_1,W) + b(X_0,X_1,W) = 0\ . \tag{1}$$

Remark 2.1. *Let $V = (0,0,0,1)$ be the vertex of the quadric cone $\mathcal{Q}$. Then*

- **(i)** *if $E \in |-K_S|$ then $\phi_2(E)$ is a line in $\mathcal{Q}$ passing through the vertex V;*
- **(ii)** *if $\Gamma \subset S$ is a (-1)-curve, then $\phi_2|_\Gamma:\ \Gamma \to \phi_2(\Gamma)$ is $1:1$ and $\phi_2(\Gamma)$ is a smooth conic in $\mathcal{Q}$, the intersection of $\mathcal{Q}$ with a plane H in $\mathbb{P}^3$ s.t. $V \notin H$;*
- **(iii)** $p_a(\mathcal{A}^*) = p_a(A^*) = 4$. ∎

Key-lemma 2.2. *Let H be a plane in $\mathbb{P}^3$, $H \not\ni V$; then the divisor $\phi_2^*(H)$ has a (-1)-curve as component $\iff$ it has (at least) 3 (maybe infinitely near) singular points. If this happens, $\phi_2^*(H) = \Gamma_1 + \Gamma_2$ where Γ_i are (-1)-curves for $i = 1,2$ and $\Gamma_1 \cdot \Gamma_2 = 3$. Every (-1)-curve in S arises in this way.* ∎

Remark 2.3. *If Γ_1, Γ_2 is as in 2.2 then $\Gamma_1 + \Gamma_2$ is smooth outside the ramification and singular in $\Gamma_1 \cap \Gamma_2$, hence $1 \le \#(\Gamma_1 \cap \Gamma_2) \le 3$ and $\Gamma_1 \cap \Gamma_2 = A^* \cap \Gamma_i$. We call Γ_1, Γ_2 a pair of (-1)-curves of type $(1,1,1)$, respectively $(2,1)$, respectively (3) if $\#(\Gamma_1 \cap \Gamma_2)$ is 3, 2, 1.* ∎

Let $a = \alpha_3 + \alpha_1 W$, $b = \beta_6 + \beta_4 W + \beta_2 W^2 + \beta_0 W^3$, where $\alpha_i, \beta_i \in \mathbb{K}[X_0, X_1]_i$.

Remark 2.4. *The non-singularity of Σ implies:*

- **(i)** *$\beta_0 \neq 0$;*
- **(ii)** $a \neq 0$.

Proof: The non-singularity of Σ in $\mathbb{P}(1,1,2,3)$ is equivalent to

$$\Big\{\sigma = \sigma_{X_0} = \sigma_{X_1} = \sigma_W = \sigma_Z = 0\Big\} = \emptyset\ .$$

(i) If $\beta_0 = 0$ then Σ is singular in $\pi^{-1}(0,0,1)$.

(ii) Otherwise, $\sigma = z^2 + b$.

We want to show that $\emptyset \neq \{b_{X_0} = b_{X_1} = b_W = 0\}$. The last set is equal to $\{\beta_4^2\,\beta_{2X_i}^2 + \beta_4\,\beta_{4X_i}^2 + \beta_{6X_i}^2 = 0,\ W^2 = \beta_4\}_{i=1,2}$. The first equation has 10 solutions in $\mathbb{P}^1$; if for $i = 0$ we have a solution $(a_0, a_1) \in \mathbb{P}^1$ with $a_1 \neq 0$, then Euler's Theorem on homogeneous functions, gives the relation $X_0\,b_{X_0} + X_1\,b_{X_1} = 0$ and we are done. The same if for $i=1$ we have a solution (a_0, a_1) with $a_0 \neq 0$. We are left with the case $\beta_4^2\,\beta_{2X_0}^2 + \beta_4\,\beta_{4X_0}^2 + \beta_{6X_0}^2 = X_1^{10}$ and $\beta_4^2\,\beta_{2X_1}^2 + \beta_4\,\beta_{4X_1}^2 + \beta_{6X_1}^2 = X_0^{10}$. Multiplying the first by X_0^2, the second by X_1^2, summing up, using Euler's relation $X_0\,\beta_{dX_0} + X_1\,\beta_{dX_1} = 0$ for d even and the fact that $\operatorname{Char}\mathbb{K} = 2$, we get $0 = X_0^2\,X_1^{10} + X_1^2\,X_0^{10}$ as polynomials, excluded. ∎

It follows that π has $\mathcal{A} = \{a = 0\}$ as branch divisor and $\mathcal{A}^* = \pi^{-1}(\mathcal{A})$ as ramification divisor; then $A = j(\mathcal{A})$ and $A^* = \eta^{-1}(\mathcal{A}^*)$ are the branch divisor and the ramification divisor of ϕ_2. The vertex $V = (0,0,0,1)$ of the quadric cone $\mathcal{Q}$ is an isolated branch point, since S is smooth.

Remark 2.5. *We have:*

(i) $p_a(A^*) = p_a(A) = 4$*;*

(ii) $V \in A \subset \mathcal{Q}$ *and* $\deg A = 3$*,* V *being the vertex of the cone* $\mathcal{Q}$*;*

(iii) A *is smooth* $\iff \{\alpha_3 = \alpha_1 = 0\} = \emptyset$ *in* $\mathbb{P}^1$*;*

(iv) *the following cases may occur:*

- **(1)** *if A is smooth, then it is a twisted cubic curve in $\mathbb{P}^3$;*
  if A is not smooth then it decomposes into:
- **(2)** *a smooth conic and a line;*
- **(3)** *three distinct lines;*
- **(4)** *a double line and a line;*
- **(5)** *a triple line.*

Proof:

(i) holds because $\phi_2|_{\mathcal{A}^*} : \mathcal{A}^* \to \mathcal{A}$ is purely inseparable and by 2.1 (iii).

(ii) The fact that the vertex V of the cone $\mathcal{Q}$ lies in A follows from (iv); moreover $\mathcal{A}^* \in |-2K_\Sigma|$ so $\deg A \cdot \deg \phi_2 = \mathcal{A}^* \cdot (-2K_\Sigma) = 6$.

(iii) $\boxed{\Rightarrow}$ If $\{\alpha_3 = \alpha_1 = 0\} \neq \emptyset$, we may choose projective coordinates s.t. $a = X_0W + X_0\,q$, $q \in \mathbb{K}[X_0, X_1]_2$. Then $(a_{X_0}, a_{X_1}, a_W) = (0,0,0)$ in $(X_0, X_1, W) = (0, 1, q(0,1))$.

$\boxed{\Leftarrow}$ If α_1 and α_2 have no common roots, we may assume projective coordinates s.t. $\alpha_1 = X_0$, $\alpha_3 = m\,X_1^3 + X_0\,q$, $m \in \mathbb{K} - \{0\}$, q as before. Then if $(a_{X_0}, a_{X_1}, a_W) = (0,0,0)$, we get $X_0 = 0$, $X_1 = 0$, $q = 0$, $W = 0$, excluded.

(iv) $(0,0,1) \in \mathcal{A}$ implies $V = j((0,0,1)) \in A$. Suppose A smooth; we may assume $\alpha_1 = X_0$ and after $W \mapsto n^{-1}W + q$, q as before, $X_1 \mapsto nX_1$ for suitable $n \in \mathbb{K} - \{0\}$, we get $a = X_0W + X_1^3$. Then

$$\alpha\colon \mathbb{P}^1 \to \mathbb{P}^3 \qquad \alpha(x_0, x_1) = (x_0^3,\, x_0^2x_1,\, x_0x_1^2,\, x_1^3) \tag{2}$$

is an isomorphism and $A = \alpha(\mathbb{P}^1)$ is a twisted cubic curve.

If A is not smooth, we get in the same way for a the normal forms WX_0, $X_1(X_1+X_0)\,(X_1+pX_0)$, $p \in \mathbb{K} - \{0,1\}$, $X_1^2(X_1+X_0)$, X_1^3, which correspond to cases (2), (3), (4), (5). ∎

From now on we make the assumption of Theorem 1.1 that $A \cong \mathcal{A}$ is smooth. We choose coordinates (x_0, x_1, w) on $\mathbb{P}(1,1,2)$ so that $a = X_0W + X_1^3$ and $A = \alpha(\mathbb{P}^1)$ is the twisted cubic curve defined by (2).

We shall identify A to $\mathbb{P}^1$ by α and choose on A the canonical coordinates $X = (x_0, x_1)$ of $\mathbb{P}^1$, so that $V = (0,1)$.

3 – The surface $\mathbb{F}$ and the map $\psi\colon \mathbb{F} \to \mathbb{P}^{3\vee}$

To apply 2.2, we look for hyperplanes H in $\mathbb{P}^3$ for which $H \not\ni V$ and $\phi_2^*(H)$ has at least 3 singular points.

Let $H = \{h_0T_0 + h_1T_1 + h_2T_2 + h_3T_3 = 0\}$, $h_3 \neq 0$. Consider $H \in \mathbb{P}^{3\vee}$ and choose on $\mathbb{P}^{3\vee}$ the dual coordinates; then $H = (h_0, h_1, h_2, h_3)$ and

$$\phi_2^*(H):\quad \begin{cases} Z^2 + aZ + b = 0\ , \\ h_0\,X_0^2 + h_1\,X_0\,X_1 + h_2\,X_2^2 + h_3\,W = 0\ . \end{cases}$$

For $Q' = (x_0, x_1, w) \in \mathcal{A} - \{(0,0,1)\}$, $Q = j(Q') = (x_0^2, x_0x_1, x_1^2, w) \in A - \{V\}$, consider

$$F_Q = \left\{ H \in \mathbb{P}^{3\vee} \mid H \ni Q,\ \exists\, Q^* \in \phi_2^{-1}(Q) \text{ s.t. } \phi_2^*(H) \text{ is singular in } Q^* \right\}.$$

The singularity of $\phi_2^*(H)$ at Q^* can be expressed by

$$\operatorname{rank}\begin{pmatrix} a_{X_0}Z + b_{X_0} & a_{X_1}Z + b_{X_1} & a_WZ + b_W & a \\ h_1X_1 & h_1X_0 & h_3 & 0 \end{pmatrix}(Q') \ \le\ 1\ . \tag{3}$$

As $h_3 \neq 0$, if $Q \notin A$ then $F_Q = \emptyset$, while if $Q \in A - \{V\}$

$$F_Q \neq \emptyset \iff (h_3\, f_{X_0} + h_1\, X_1\, f_W)(Q) = (h_3\, f_{X_1} + h_1\, X_0\, f_W)(Q) = 0\ .$$

Let

$$M = \begin{pmatrix} X_0^2 & X_0\, X_1 & X_1^2 & W \\ 0 & X_1\, \sigma_W & 0 & \sigma_{X_0} \\ 0 & X_0\, \sigma_W & 0 & \sigma_{X_1} \end{pmatrix} \in M(3, 4, \mathcal{S})\ . \tag{4}$$

Euler's Theorem applied to σ gives $X_0\, \sigma_{X_0} + X_1\, \sigma_{X_1} = 0$. The smoothness of $\mathcal{A}$ implies $\operatorname{rank} M(Q') = 2$ if $Q' \in \mathcal{A} - \{(0,0,1)\}$, $\operatorname{rank} M((0,0,1)) = 1$. Let $S_R(M(Q'))$ be the subspace of $\mathbb{K}^4$ generated by the rows of $M(Q')$, and denote by $\perp$ the orthogonality in $\mathbb{K}^4$ with respect to $\langle X, Y\rangle = {}^tX \cdot Y$.

Remark 3.1. *For all $Q \in A - \{V\}$, $L(Q) = \mathbb{P}(S_R(M(Q'))$ is a line in $\mathbb{P}^3$ and $Q \in L(Q) \subset T\mathcal{Q}_Q$, where $T\mathcal{Q}_Q$ is the tangent plane to $\mathcal{Q}$ in Q. Moreover*

$$F_Q = \mathbb{P}\Big(S_R(M(Q'))^{\perp}\Big) = \Big\{H \in \mathbb{P}^{3\vee} \mid H \supset L(Q)\Big\} \subset \mathbb{P}^{3\vee} \tag{5}$$

is a line in $\mathbb{P}^{3\vee}$ which represents the net of the planes in $\mathbb{P}^3$ containing $L(Q)$. ∎

To determine F_Q, remark that $x_0 \neq 0$ for $Q' \in \mathcal{A} - \{(0,0,1)\}$, so we can multiply by X_0 the equations to modify degrees. Looking at the rows of M, we define the following matrix

$$N = \begin{pmatrix} X_1^2\, X_0^4 & 0 & X_0^6 & 0 \\ 0 & X_1\, \sigma_{X_0} & X_0\, \sigma_{X_0} + W\sigma_W & X_1^2\, \sigma_W \\ X_1\, \sigma_{X_1} + W\sigma_W & X_0\, \sigma_{X_1} & 0 & X_0^2\, \sigma_W \end{pmatrix}\ .$$

Remark 3.2. *$N \in M(3, 4, \mathcal{S}_6)$ and the following properties hold:*

(i) $\operatorname{rank} N(Q') = 2\ \ \forall\, Q' \in \mathcal{A}$;

(ii) $S_R(N(Q')) = S_R(M(Q'))^{\perp}\ \ \forall\, Q' \in \mathcal{A} - \{(0,0,1)\}$.

Proof: Using Euler's relation we get $S_R(N) \subset S_R(M)^{\perp}$; moreover we have $\operatorname{rank} N(Q') \geq 2$ if $Q' \in \mathcal{A} - \{(0,0,1)\}$, $\operatorname{rank} M((0,0,1)) = 2$. ∎

Using the relation $X_0\, W = X_1^3$ on $\mathcal{A}$ and remarking that W has exponent at most 3, we can eliminate W multiplying by X_0^3. On $\mathcal{A}$ we have $Z = \sqrt{b}$, so taking the square we eliminate Z. Thus let $\tilde{N} = X_0^6\, N^{[2]}$, where $N^{[2]}$ is the matrix obtained taking the square of each entry of N. Then

$$\tilde{N} = \begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} R_2 & 0 & R_1 & 0 \\ 0 & F_1 & R_0 & G_1 \\ R_0 & F_2 & 0 & G_2 \end{pmatrix} , \tag{6}$$

$$R_0 = X_0^8\,\sigma_{X_0}^2 + X_0^6\,W^2\,\sigma_W^2 = \beta_0^2\,X_1^{18} + X_0\,S\,, \qquad S \in \mathbb{K}[X_0, X_1]_{17}\ ,$$

$$R_1 = X_0^{18}\,, \qquad R_2 = X_0^{14}\,X_1^4\ ,$$

$$F_1 = X_0^6\,X_1^2\,\sigma_{X_0}^2 = X_0^4\,X_1^4\,\sigma_{X_1}^2\,, \qquad F_2 = X_0^8\,\sigma_{X_1}^2\ ,$$

$$G_1 = X_0^6\,X_1^4\,\sigma_W^2\,, \qquad G_2 = X_0^{10}\,\sigma_W^2\ .$$

Remark 3.3. *The following properties are consequences of 3.2.*

(i) $\tilde{N} \in M(3, 4, \mathbb{K}[X_0, X_1]_{18})$*;*

(ii) $R_0\,\nu_0 + R_1\,\nu_1 + R_2\,\nu_2 = 0$ *and for all* $X = (x_0, x_1) \in \mathbb{P}^1 \cong A$ *we have:*

- **(1)** $\operatorname{rank} \tilde{N}(X) = 2$, $\operatorname{rank}(R_0, R_1, R_2)(X) = 1$*;*
- **(2)** $x_0 \neq 0 \implies \nu_0(X)$, $\nu_2(X)$ *are independent;*
- **(3)** $X = V = (0, 1) \implies \nu_1(V)$, $\nu_2(V)$ *are independent.*

(iii) $S_R(\tilde{N}(Q)) = S_R(M^{[2]}(Q'))^{\perp}$, $\forall\, Q = j(Q') \in A - \{V\}$. ∎

The Kernel of the surjective linear map $\mathbb{K}^3 \to S_R(\tilde{N})$, $Y \mapsto y_0\nu_0 + y_1\nu_1 + y_2\nu_2$, where $Y = (Y_0, Y_1, Y_2)$, is spanned by the relations (R_0, R_1, R_2).

Let $\mathcal{V} = \mathbb{P}^1_X \times \mathbb{K}^3_Y \xrightarrow{\pi} \mathbb{P}^1$ be the trivial vector bundle with fiber $\mathbb{K}^3$ and let $\mathcal{K} = \{(X, Y) \in \mathcal{V} \mid Y \in \langle (R_0, R_1, R_2)(X) \rangle\}$ be the sub bundle generated by the relations. Define

$$\mathbb{F} = \mathbb{P}(\mathcal{V}/\mathcal{K}) \xrightarrow{p} \mathbb{P}^1 \cong A \cong \mathcal{A}$$

to be the associated $\mathbb{P}^1$-bundle and denote by $\mathbb{F}_X = p^{-1}(X)$ its fiber. Let

$$\psi\colon\ \mathbb{F} \to \mathbb{P}^{3\vee} \qquad \psi(X, Y) = y_0\,\nu_0(X) + y_1\,\nu_1(X) + y_2\,\nu_2(X)\ .$$

Let $(\_)^{[2]} : \mathbb{P}^n \to \mathbb{P}^n$ be the purely inseparable morphism $(x_i) \mapsto (x_i^2)$ and $(\_)^{[\frac{1}{2}]}$ be the inverse bijection.

Remark 3.4. *From 3.3 it follows*

(i) ψ *is a morphism and for all* $X = Q = j(Q') \in A$ *the restriction map* $\psi|_{\mathbb{F}_Q} \colon \mathbb{F}_Q \to \mathbb{P}^{3\vee}$ *is linear, so that* $\psi(\mathbb{F}_Q) = \mathbb{P}\,S_R(\tilde{N}(Q))$ *is a line in* $\mathbb{P}^{3\vee}$*. Moreover, if* $Q \neq V$ *then*

$$\psi(\mathbb{F}_Q)^{[\frac{1}{2}]} = \mathbb{P}\Big(S_R(M(Q'))^{\perp}\Big) = F_Q\ . \tag{7}$$

(ii) *For $H \in F_Q$, we have $H \supset L(Q) \ni Q$ and $\deg A = 3$, so ψ is finite, $\#\psi^{-1}(H) \leq 3$ for all $H \in \psi(\mathbb{F})$, and $\psi(\mathbb{F})$ is a surface.* ∎

We have determined F_Q, the planes H for which $\phi_2^*(H)$ is singular over Q; by 2.2 the (-1)-curves on S correspond to $F_{Q_1} \cap F_{Q_2} \cap F_{Q_3}$, hence to the triple points of the surface $\psi(\mathbb{F})$. The proof of Theorem 1.1 reduces therefore to show that the surface $\psi(\mathbb{F}) - \{h_3 = 0\}$ contains 120 triple points.

We need normal forms for F_i, G_i up to coordinate change in $\mathbb{P}(1,1,2,3)$. The projective transformations $PGL(2,\mathbb{K})$ of $\mathbb{P}^1_{(x_0,x_1)}$ send any 3 distinct points to any 3 distinct points; since we are in case (1) of 2.5 (iv) as in that proof we may fix 1 of the 3 points and get $a = X_0\, W + X_1^3$.

After $X_0 \mapsto \beta_0^{1/3} X_0$ and $W \mapsto W/\beta_0^{1/3}$, we may assume $\beta_0 = 1$. Denote by $\square$ the square of a polynomial. Then $Z \mapsto Z + \gamma_1(X_0,X_1)(W) + \gamma_1(X_0,X_1)$, $\gamma_i \in \mathbb{K}[X_0,X_1]_i$, gives $b \mapsto b + a\,\gamma_1 W + a\,\gamma_3 + \square = W^3 + (\beta_2 + X_0\,\gamma_1)\,W^2 + (\beta_4 + X_0\,\gamma_3 + X_1^3\,\gamma_1)\,W + \beta_6 + X_1^3\,\gamma_3 + \square$. Choose γ_1 so that $(\beta_2 + X_0\,\gamma_1)\,W^2 = c\,X_1^2\,W^2 = \square$ and γ_3 so that $\beta_4 + X_0\,\gamma_3 + X_1^3\,\gamma_1 = c_3 X_1^4$.

Hence $a = X_0 W + X_1^3$ and $b = W^3 + c_3 X_1^4\, W + X_0 X_1 H^2 + L^2$, where $H = c_0 X_0^2 + c_1 X_0 X_1 + c_2 X_1^2$ and $L = d_0 X_0^3 + d_1 X_0^2 X_1 + d_2 X_0 X_1^2 + d_3 X_1^3 + d_4 W X_1$, with $c_i, d_j \in \mathbb{K}$, $(c_0, d_0) \neq (0,0)$ because Σ is smooth above $(1,0,0)$.

Setting $t = X_1/X_0$ it follows

$$
\begin{aligned}
&F_1 = t^4 F_2\ ,\\
&G_1 = t^4 G_2\ ,\\
(8)\qquad &R_0 = X_1^{18} + c_3^2\, X_0^4 X_1^{14} + X_0^8 X_1^2 H^4\ ,\\
&F_2 = X_0^5 X_1^{13} + c_3\, X_0^7 X_1^{11} + X_0^9 X_1^5 H^2 + X_0^8 X_1^4 L^2 + X_0^{10} H^4\ ,\\
&G_2 = X_0^6 X_1^{12} + X_0^9 X_1^9 + c_3^2\, X_0^{10} X_1^8 + c_3\, X_0^{11} X_1^7 + X_0^{13} X_1 H^2 + X_0^{12} L^2\ .
\end{aligned}
$$

The $\mathbb{P}^1$-bundle $\mathbb{F}$ is a Segre–Hirzebruch surface $\mathbb{F}_d = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$. We want to find generators of $\operatorname{Pic}\mathbb{F}$ and determine d.

For $i = 1,2$ and $j = 0,1,2$ let $r_j = R_j(1,t)$, $f_i = F_i(1,t)$, $g_i = G_i(1,t)$; $t_V = 1/t = X_0/X_1$, $r_{V,j} = R_j(t_V,1)$, $f_{V,i} = F_i(t_V,1)$, $g_{V,i} = G_i(t_V,1)$. Then

$$
\begin{aligned}
&r_0 = (t^9 + c_3\, t^7 + t\, h^2)^2 \;=\; t^2\, g^4\,, \qquad r_1 = 1\,, \qquad r_2 = t^4\ ,\\
(9)\qquad &f_2 = t^{13} + c_3\, t^{11} + t^5\, h^2 + t^4\, l^2 + h^4\ ,\\
&g_2 = t^{12} + t^9 + c_3^2\, t^8 + c_3\, t^7 + t\, h^2 + l^2\ ,
\end{aligned}
$$

where $h = c_0 + c_1 t + c_2 t^2$, $l = d_0 + d_1 t + d_2 t^2 + d_3 t^3 + d_4 t^4$. It follows

$$(10)\qquad \begin{aligned} g_2' &= \frac{d}{dt} g_2(t) = g^2 , \\ g &= c_0 + c_1\, t + c_2\, t^2 + \sqrt{c_3}\, t^3 + t^4 , \\ t^4 g_2 + f_2 &= g^4 . \end{aligned}$$

Proposition 3.5. *For $Q \in A - \{V\}$, the line $L(Q)$ defined in 3.1 verifies*

(i) *$L(Q)$ is the tangent to A in $Q \iff g(Q) = 0 \iff g_2'(Q) = 0$;*

(ii) *$L(Q) \ni V \iff g_2(Q) = 0$.*

From the geometry of the twisted cubic A and (10) it follows $\{g_2 = g_2' = 0\} = \{f_2 = g_2 = 0\} = \emptyset$, i.e. g_2 has 12 distinct roots in $\mathbb{K}$.

Proof:

(i) The line $L(Q) = \mathbb{P}(S_R(M(Q')))$ is tangent to $A = \alpha(\mathbb{P}^1)$ in $Q \iff \alpha'(t) = (0, 1, 0, t^2) \in S_R(M)$. But $x_0 \neq 0$, so the condition is equivalent to the fact that the 2 vectors $(1, X_1 \sigma_W, X_0\, \sigma_W)$ and $(t^2, \sigma_{X_0}, \sigma_{X_1})$ are dependent, which is equivalent — by Euler's relation and by $(X_0\, \sigma_W, \sigma_{X_1}) \neq (0, 0)$ (smoothness of A) — to $\sigma_{X_1} + t^2 X_0\, \sigma_W = 0$ and to $X_0^8(\sigma_{X_1}^2 + t^4 X_0^2\, \sigma_W^2) = 0$ and finally to $t^4 g_2 + f_2 = g^4 = 0$.

(ii) $V = (0, 0, 0, 1) \in (S_R(M(Q')) \iff$ the first 3 columns of M have rank 1 $\iff f_W = 0 \iff g_2(Q) = 0$. ∎

Let $U = \{(x_0, x_1) \in \mathbb{P}^1 \mid x_0 \neq 0\}$, $U_V = \{(x_0, x_1) \in \mathbb{P}^1 \mid R_0(x_0, x_1) \neq 0\}$.

By 3.3 (ii) it follows that $\{U, U_V\}$ is an open cover of $\mathbb{P}^1$, $U_V \ni V$, and that $\nu_0(X)$, $\nu_2(X)$ are independent for $X \in U$, $\nu_1(X)$, $\nu_2(X)$ are independent for $X \in U_V$. Local affine coordinates on $\mathcal{V}/\mathcal{K}$ are

on $\pi^{-1}(U)$ $\quad t = x_1/x_0$ and y_0, y_2, $\quad$ where $y_i = y_i|_U$;

on $\pi^{-1}(U_V)$ $\quad t_V = x_0/x_1$ and $y_{V,1}$, $y_{V,2}$, $\quad$ where $y_{V,i} = y_i|_{U_V}$.

Let $\mathbb{F} = \tilde{U} \cup \tilde{U}_V$, where $\tilde{U} = p^{-1}(U) = \tilde{U}_0 \cup \tilde{U}_2$, $\tilde{U}_V = p^{-1}(U_V) = \tilde{U}_{V,1} \cup \tilde{U}_{V,2}$

$$\tilde{U}_0 = \left\{(X, Y) \in \tilde{U} \mid Y_0 \neq 0\right\} ,$$

$$\tilde{U}_2 = \left\{(X, Y) \in \tilde{U} \mid Y_2 \neq 0\right\} ,$$

$$\tilde{U}_{V,1} = \Big\{(X,Y) \in \tilde{U}_V \mid Y_{V,1} \neq 0\Big\} \ ,$$
$$\tilde{U}_{V,2} = \Big\{(X,Y) \in \tilde{U}_V \mid Y_{V,2} \neq 0\Big\} \ .$$

Affine coordinates are

$$\begin{array}{lll} \text{on } \tilde{U}_0 & (t, u_0) & \text{where } u_0 = y_2/y_0 \ , \\ \text{on } \tilde{U}_2 & (t, u_2) & \text{where } u_2 = y_0/y_2 \ , \\ \text{on } \tilde{U}_{V,1} & (t_V, v_1) & \text{where } v_1 = y_{V,2}/y_{V,1} \ , \\ \text{on } \tilde{U}_{V,2} & (t_V, v_2) & \text{where } v_2 = y_{V,1}/y_{V,2} \ . \end{array}$$

Let $\psi_{\tilde{U}}$, $\psi_{\tilde{U}_V}$, ψ_i, $\psi_{V,i}$, be the restrictions of ψ to $\tilde{U}$, $\tilde{U}_V$, $\tilde{U}_i$, $\tilde{U}_{V,i}$.

By (6), we have $\nu_0 = (t^4, 0, 1, 0)$, $\nu_2 = (r_0, f_2, 0, g_2)$, $\nu_{V,1} = (0,\ f_{V,1},\ r_{V,0},\ g_{V,1})$, $\nu_{V,2} = (r_{V,0},\ f_{V,2},\ 0,\ g_{V,2})$. It follows

$$\begin{aligned} &\psi_{\tilde{U}} = Y_0\,\nu_0 + Y_2\,\nu_2\,, \qquad \psi_{\tilde{U}_V} = Y_{V,1}\,\nu_{V,1} + Y_{V,2}\,\nu_{V,2} \ , \\ &\psi_0 = \nu_0 + u_0\,\nu_2 = (t^4 + u_0 r_0,\ u_0 f_2,\ 1,\ u_0 g_2) \ , \\ &\psi_2 = u_2\,\nu_0 + \nu_2 = (t^4 u_2 + r_0,\ f_2,\ u_2,\ g_2) \ , \\ &\psi_{V,1} = \nu_{V,1} + v_1\,\nu_{V,2} = (v_1\,r_{V,0},\ f_{V,1} + v_1 f_{V,2},\ r_{V,0},\ g_{V,1} + v_1 g_{V,2}) \ , \\ &\psi_{V,2} = v_2\,\nu_{V,1} + \nu_{V,2} = (r_{V,0},\ v_2 f_{V,1} + f_{V,2},\ v_2 r_{V,0},\ v_2 g_{V,1} + g_{V,2}) \ . \end{aligned} \tag{11}$$

We define on $\mathbb{F}$ the divisors D, E, $\mathbb{F}_X = p^{-1}(X)$ for $X \in \mathbb{P}^1$.

Remarking that $R_0 \neq 0$ in U_V

$$\begin{aligned} D = \psi^*\{h_2 = 0\} &= \{Y_0 X_0^{18} + Y_1 R_0 = 0\} \\ &= \Big\{(\tilde{U}, Y_0 X_0^{18}),\ (\tilde{U}_V, Y_{V,1} R_0 = 0)\Big\} \\ &= \Big\{(\tilde{U}_0, 1),\ (\tilde{U}_2, u_2),\ (\tilde{U}_{V,1}, 1),\ (\tilde{U}_{V,2}, v_2)\Big\} \ , \end{aligned}$$

$$\begin{aligned} E &= \Big\{(\tilde{U}, Y_2),\ (\tilde{U}_V, X_1^4 Y_{V,1} + X_0^4 Y_{V,2})\Big\} \\ &= \Big\{(\tilde{U}_0, u_0),\ (\tilde{U}_2, 1),\ (\tilde{U}_{V,1}, 1 + t_V^4 v_1),\ (\tilde{U}_{V,2}, v_2 + t_V^4)\Big\} \ . \end{aligned}$$

Proposition 3.6. *D, E, $\mathbb{F}_X$ are irreducible, smooth, rational divisors on $\mathbb{F}$. If we denote their classes in* $\operatorname{Pic}\mathbb{F}$ *respectively by d, e, f and by $h^\vee$ the class of a hyperplane $H^\vee$ in $\mathbb{P}^{3\vee}$, then*

$$f^2 = 0\,, \qquad d \cdot f = e \cdot f = 1\,, \qquad d \cdot e = 4 \ ,$$
$$\psi^*(h^\vee) = d = e + 14\,f\,, \qquad d^2 = 18\,, \qquad e^2 = -10 \ .$$

Hence $\mathbb{F}$ is a Segre–Hirzebruch surface $\mathbb{F}_{10}$. Moreover, for all $(a,b) \in \mathbb{P}^1$

$$\psi^*\{a\,h_1 + b\,h_3 = 0\} \;=\; E + \{a\,f_2(t) + b\,g_2(t) = 0\}|_{\tilde{U}} + r\,\mathbb{F}_V \tag{12}$$

where $r = 1 \Leftrightarrow a \neq 0$ and $r = 2 \Leftrightarrow a = 0$.

In particular, $\psi^*\{h_3 = 0\} = E + \{g_2(t) = 0\}|_{\tilde{U}} + 2\,\mathbb{F}_V$, according to 3.5 and the fact that $\{h_3 = 0\} = \{H^\vee \in \mathbb{P}^{3\vee} \mid H^\vee \ni V\}$.

Proof: E is a divisor since $\frac{1}{y_2}(x_1^4\,y_{V,1} + x_0^4\,y_{V,2}) = t^4 \neq 0$ in $\tilde{U} \cap \tilde{U}_V$.

The other assertions follow from the local definition of D, E and the fact that $\psi^*(h^\vee) = e + 14\,f$, which is a consequence of (12).

The proof of 12 goes as follows: $\psi^*\{ah_1 + bh_3 = 0\} = \{a(Y_1F_1 + Y_2F_2) + b(Y_1G_1 + Y_2G_2) = 0\} = \{(\tilde{U}, Y_2(aF_2 + b\,G_2)), (\tilde{U}_V, (Y_{V,1} + t_V^4 Y_2)\,(aF_1 + b\,G_1))\}$, ((6), (8)). The value of r follows from $aF_1 + b\,G_1 = aX_0X_1^{17} + bX_0^2X_1^{16} + X_0^3(\ldots)$ and $\mathbb{F}_V = \{X_0 = 0\}$. ∎

Corollary 3.7. *Let L_0 be the line $\{h_1 = h_3 = 0\} \subset \mathbb{P}^{3\vee}$.*

- **(i)** *$\psi(E)$ and $\psi(\mathbb{F}_X)$, for all $X \in \mathbb{P}^1$, are lines in $\mathbb{P}^{3\vee}$;*
- **(ii)** *$\psi(E) = \psi(\mathbb{F}_V) = L_0$ and $\psi(\mathbb{F}_X) \neq L_0$, for all $X \in U$;*
- **(iii)** *$\psi|_E\colon E \to L_0$ is a purely inseparable morphism of degree 4;*
- **(iv)** *$\deg\psi(\mathbb{F}) = 18$ and $\psi\colon \mathbb{F} \to \psi(\mathbb{F})$ is a birational morphism.*

Proof:

(iii) $E|_{\tilde{U}_0} = \{Y_2 = 0\}$ and $\psi(E \cap \tilde{U}_0) = Y_0\,\nu_0 = Y_0(t^4, 0, 1, 0)$.

The second assertion of **(ii)** follows from the injectivity of $\psi|_E$.

(iv) $\deg\psi(\mathbb{F})$ divides $d^2 = 18$ and $\deg\psi(\mathbb{F}) \geq 12$, because by 3.5 (ii), 3.6 and (i), (ii) above $\psi(\mathbb{F}) \cap \{h_3 = 0\}$ contains the distinct lines $\psi(\mathbb{F}_{c_i})$, $i = 1 \ldots 12$, where c_i are the distinct roots of g_2. ∎

Remark 3.8. *The critical set $\{\operatorname{rank} d\psi < 2\}$ of ψ is $E \cup \bigcup_{\{\rho \in U | g(\rho) = 0\}} \mathbb{F}_\rho$. From 3.4 (ii) it follows that the singularities of $\psi(\mathbb{F})$ are*

$$\operatorname{Sing}(\psi(\mathbb{F})) \;=\; L_0 \cup \bigcup_{g(\rho)=0} \psi(\mathbb{F}_\rho) \cup \{\textit{double points}\} \cup \{\textit{triple points}\}\;. \tag{13}$$

Proof: By 3.4 (i), the critical set is $\{(x,y) \in \mathbb{F} \mid \frac{\partial\psi}{\partial x}(x,y) = 0\}$. In $\tilde{U}$ by (11) we have $\partial\psi_{\tilde{U}}/\partial t = Y_2(0, t^4 g_2', 0, g_2')$, while in $\mathbb{F}_V = \{t_V = 0\} \subset \tilde{U}_V$, $\partial\psi_{\tilde{U}_V}/\partial t_V = (Y_{V,1} + t_V^4\,Y_{V,2})\,(0, f_{V,1}', 0, g_{V,1}')$ and $f_{V,1}' = 1$ for $t_V = 0$. ∎

4 – The map ξ and the double curve $\Lambda \subset \mathbb{F}$

The map ψ can be better understood through the following definition. For every $t \in U \subset \mathbb{P}^1$, the lines $\psi(\mathbb{F}_t)$ and $L_0 = \psi(E)$ in $\mathbb{P}^{3\vee}$ are distinct by 3.7 and intersect in $\psi(E \cap \mathbb{F}_t)$, hence span a plane $\xi(t) = \langle \psi(\mathbb{F}_t), L_0 \rangle \in \Phi$, where $\Phi = \{H^\vee \subset \mathbb{P}^{3\vee} \mid H^\vee \supset L_0\} = \{ah_1 + bh_3 = 0 \mid (a,b) \in \mathbb{P}^1\}$ is the pencil of planes containing L_0. A morphism $\xi\colon \mathbb{P}^1 \to \Phi$ is defined, and we have

$$\begin{array}{ccccccc} & S & & \mathbb{F} & \xrightarrow{\psi} & \mathbb{P}^{3\vee} & \xrightarrow{(\_)^{[\frac{1}{2}]}} \mathbb{P}^{3\vee} \\ & \downarrow \phi_2 & & \downarrow p & & & \\ \mathbb{P}^3 \supset & \mathcal{Q} \supset A & \xleftarrow{\alpha\,\cong} & \mathbb{P}^1 & \xrightarrow{\xi} & \Phi \cong \mathbb{P}^1 & \end{array}$$

Remark 4.1. *Choose on the pencil Φ the projective coordinates for which $\{ah_1 + bh_3 = 0\} = (a,b)$, and let $\infty = (0,1) \in \Phi$. Then*

- **(i)** *$\xi^*(\infty) = c_1 + \ldots + c_{12} + V$, where c_i are the distinct roots of g_2. Hence ξ is separable, $\xi(V) = \infty$, $\deg \xi = 13$ and ∞ is not a branch point.*
- **(ii)** *In the affine coordinate b/a on $\Phi - \{\infty\}$, for $t \in U$, $t_V \in U_V$ we have*

$$\xi(t) = \frac{f_2}{g_2} = t^4 + \frac{g^4}{g_2}\,, \qquad \xi(t_V) = \frac{f_{V,1}}{g_{V,1}}\,, \qquad \xi'(t) = \frac{g^6}{g_2^2}\,. \tag{14}$$

- **(iii)** *The critical $\{d\xi = 0\}$ set of ξ is $\mathcal{R} = \{\rho \in U \mid g(\rho) = 0\} = \{\rho_1 \ldots \rho_d\}$, where ρ_i are the distinct roots of g and $1 \le d \le 4$ ((10)). The ramification of ξ at ρ_i is $6\,n_i$, where n_i is the multiplicity of the root ρ_i. The branch locus of ξ is $\xi(\mathcal{R}\} = \{\rho_1^4 \ldots \rho_d^4\} \subset \Phi - \{\infty\}$; it follows that $\xi|_{\mathcal{R}}$ is injective.*
- **(iv)** *Up to the identification $\mathbb{P}^1 \cong A$, the singular set of $A^* = \phi_2^{-1}(A)$ is $\phi_2^{-1}(\mathcal{R})$ and the contribution of $\phi_2^{-1}(\rho_i)$ to $p_a(A^*)$ is n_i; A^* is rational.*

Proof: By (12) $\psi^*\{ah_1 + bh_3 = 0\} = E + \mathbb{F}_V + F_{(a,b)}$, where $F_{(a,b)} = \{af_2(t) + b\,g_2(t) = 0\}|_{\tilde{U}} + (n-1)\,\mathbb{F}_V = \sum_i r_i\,\mathbb{F}_{X_i}$, $\sum_i r_i = 13$. The relation $t^4 g_2 + f_2 = g^4$ in (10) and 3.5 (ii) imply that f_2, g_2 have no common roots, so $\{F_{(a,b)}\}_{(a,b)\in\mathbb{P}^1}$ is a pencil without fixed component and $F_{(a,b)} = p^*\,\xi^*(a,b)$. The case $a{=}0$ proves **(i)**, while looking at $F_{(a,b)}|_{\tilde{U}}$, $F_{(a,b)}|_{\tilde{U}_V}$ we get **(ii)**; (10) gives ξ'.

(iii) As ∞ is not a branch point, $\mathcal{R} \subset U \cap \{g_2 \ne 0\}$, so $\mathcal{R} = \{\xi' = g^6/g_2^2 = 0\}$. The other assertions follow from $\deg g = 4$, and the relation (10).

(iv) A local computation shows that $\mathcal{A}^*$ is singular in $\pi^{-1}(Q) \iff L(Q)$ is tangent to A in Q; then we use 3.5 (i). If $\mathcal{A}_2$ is a copy of $\mathcal{A}$ and $\phi'\colon \mathcal{A}_2 \to \mathcal{A}$ is the $\mathbb{K}$-linear Frobenius map, $\pi^{-1} \circ \phi'^2$ is a normalization of $\mathcal{A}^*$, hence the assertion on n_i. A^* is rational since $p_a(A^*) = 4 = \deg g = \sum n_i$ (2.5 (i)). ∎

To understand $\psi(\mathbb{F})$, let $H^\vee = (a,b) \in \Phi$, a plane in $\mathbb{P}^{3\vee}$ containing L_0.

If $H^\vee \notin \xi(\mathcal{R})$, i.e. $(a,b) \neq (1,\rho_i^4)$, then $\psi^*(H^\vee) = E + \mathbb{F}_V + F_{(a,b)}$, where $F_{(a,b)} = \mathbb{F}_{e_1} + \ldots + \mathbb{F}_{e_{13}}$, $\{e_1,\ldots,e_{13}\} = \xi^{-1}(a,b)$. The scheme intersection between $H^\vee$ and $\psi(\mathbb{F})$ is $H^\vee \cdot \psi(\mathbb{F}) = \psi_* \psi^*(H^\vee) = 5\,L_0 + \psi(\mathbb{F}_{e_1}) + \ldots + \psi(\mathbb{F}_{e_{13}})$ because of 3.7 (ii), 3.4 (i). The 13 lines $\psi(\mathbb{F}_{e_i})$ are all distinct and intersect L_0 in different points, since $\psi|_E$ is injective. One of the $\psi(\mathbb{F}_{e_i})$ is $L_0 \iff H^\vee = \infty \iff a = 0$.

If $H^\vee = (1,\rho^4) \in \xi(\mathcal{R})$, $H^\vee \cdot \psi(\mathbb{F}) = 5\,L_0 + m\,\psi(\mathbb{F}_\rho) + \psi(\mathbb{F}_{e_1}) + \ldots + \psi(\mathbb{F}_{e_{13-m}})$.

From 3.8, we see that $\psi(\mathbb{F})$ is singular along L_0 and the d lines $\psi(\mathbb{F}_{(1,\rho)})$. The other singularities in $H^\vee$ arise from the intersections of the lines $\psi(\mathbb{F}_{e_i})$; there are $\binom{13-m}{2}$ double points and triple points. Varying $H^\vee \in \Phi$, the double points describe by transversality a curve Δ in $\psi(\mathbb{F})$, which is the closure of

$$\bigcup_{\substack{e \neq e^* \in \mathbb{P}^1 \\ \xi(e)=\xi(e^*)=H^\vee \in \Phi}} \psi(\mathbb{F}_e) \cap \psi(\mathbb{F}_{e^*})$$

and the triple points are its nodes. Let $\Lambda = \psi^*(\Delta)$ be the so-called double curve of ψ. The restriction $\psi\colon \Lambda \to \Delta$ is a $2:1$ morphism.

To count the nodes of Λ, we need local parametrizations. If $(a,b) \in \Phi$ and $e \in \xi^{-1}(a,b)$, let Λ_e be a formal neighborhood of $\Lambda \cap \mathbb{F}_e$, by which we mean a set of parametrizations of Λ at its points. Let $\Lambda_{(a,b)} = \bigcup_{e \in \xi^{-1}(a,b)} \Lambda_e$; we study $\Lambda_{(a,b)}$ in the cases $(a,b) \in \Phi - (\xi(\mathcal{R}) \cup \{\infty\})$, $(a,b) \in \xi(\mathcal{R})$, $(a,b) = \infty$.

Case I. Suppose $(a,b) \in \Phi - (\xi(\mathcal{R}) \cup \{\infty\})$; then $\xi^{-1}(a,b) = \{e_1 \ldots e_{13}\} \subset U \cap \{g_2 \neq 0,\ g \neq 0\}$. Let $e, e^* \in \xi^{-1}(a,b)$, $e \neq e^*$; then $\psi(\mathbb{F}_e)$ and $\psi(\mathbb{F}_{e^*})$ lie in the plane $(a,b) \in \Phi$ and must intersect outside L_0, so $\psi^{-1}(\psi(\mathbb{F}_e) \cap \psi(\mathbb{F}_{e^*})) = \{P, P^*\}$, where $P \in \mathbb{F}_e - E$, $P^* \in \mathbb{F}_{e^*} - E$, hence $P, P^* \in \tilde{U}_2$.

Therefore we look for power series solutions of the equation

$$\left\{ \begin{array}{c} \psi_2(t,u_2) = \psi_2(t^*,u_2^*) \\ t \neq t^* \end{array} \right. \tag{15}$$

with (t,u_2) centered at P, (t^*,u_2^*) centered at P^*.

As $g(e)$, $g(e^*)$, $g_2(e)$, $g_2(e^*)$ are all $\neq 0$, by (11) we have

$$\psi_2(t,u_2) = \left(\frac{u_2\,t^4 + t^2\,g(t)^4}{g_2(t)}\,,\ \xi(t)\,,\ \frac{u_2}{g_2(t)} \right).$$

It follows that (15) is equivalent to the 2 conditions

$$\begin{cases} \xi(t) = \xi(t^*) \\ t \neq t^* \end{cases} \tag{16}$$

$$L(t,t^*)\begin{pmatrix} u_2 \\ u_2^* \end{pmatrix} = B(t,t^*) \tag{17}$$

where

$$L = \begin{pmatrix} g_2(t^*) & g_2(t) \\ \dfrac{t^4}{g_2(t)} & \dfrac{t^{*4}}{g_2(t^*)} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \dfrac{t^2\, g^4(t)}{g_2(t)} + \dfrac{t^{*2}\, g^4(t^*)}{g_2(t^*)} \end{pmatrix} . \tag{18}$$

If $\det L = (t+t^*)^4$ is invertible, every solution of (16) gives a unique solution of (17). We are assuming that $(a,b) \notin \xi(\mathcal{R})$ is not a branch point, so there exists a power series $t^*(t) \in \mathbb{K}[[t-e]]$ such that

$$\xi(t^*(t)) = \xi(t) , \qquad t^*(e) = e^* , \qquad \frac{dt^*}{dt}(e) \neq 0 . \tag{19}$$

As $t + t^*(t) \in \mathbb{K}[[t-e]]^*$, (17) has solutions $u_2(t), u_2^*(t) \in \mathbb{K}[[t-e]]$. Let

$$\lambda(t) = (t, u_2(t)) , \qquad \lambda^*(t) = (t^*(t), u_2^*(t)) . \tag{20}$$

Then λ, λ^* are parametrizations of Λ at $P = (e, u_2(e))$, $P^* = (e^*, u_2^*(e))$ and if $\psi(P) = \psi(P^*)$ is not a triple point there are no other branches through these points, hence Λ is smooth, horizontal in neighborhoods of P, P^*. Using (10), we can compute the explicit solution of (17)

$$u_2(t) = \frac{g^4(t)}{(t+t^*)^2} + t^{*2}\, g_2(t) , \qquad u_2'(t) = t^{*2}\, g^2(t) . \tag{21}$$

Remark that the situation is symmetric in t, t^*, i.e. $t^{**} = t$.

It remains to see what happens near the points of Λ which are inverse images of triple points of Δ. Let $P \in \mathbb{F}_e$, $P^* \in \mathbb{F}_{e^*}$, $P^\diamond \in \mathbb{F}_{e^\diamond}$, $e, e^*, e^\diamond \in \xi^{-1}(a,b)$, e, e^*,$e^\diamond$ distinct s.t. $\psi(P) = \psi(P^*) = \psi(P^\diamond)$ is a triple point in the plane (a,b), intersection of the distinct lines $\psi(\mathbb{F}_e)$, $\psi(\mathbb{F}_{e^*})$, $\psi(\mathbb{F}_{e^\diamond})$. Then $P, P^*, P^\diamond \notin E$, hence $P, P^*, P^\diamond \in \tilde{U}_0$, and as before we get 2 parametrizations for Λ near P: $\lambda_{P,P^*}(t) = (t, u_{2P,P^*}(t))$ relative to (P, P^*) and $\lambda_{P,P^\diamond}(t) = (t, u_{2P,P^\diamond}(t))$ relative to $(P, P^\diamond)$.

By (21), $u'_{2P,P^*}(e) + u'_{2P,P^\diamond}(e) = (e^* + e^\diamond)^2\, g^2(e) \neq 0$ because $e^* \neq e^\diamond$ and $g(e) \neq 0$, hence there are 2 smooth branches of Λ through P with different tangents. In conclusion, Λ has a node in P and, by symmetry, in P^*, $P^\diamond$. If $H = \psi(P)^{[1/2]} = \psi(P^*)^{[1/2]} = \psi(P^\diamond)^{[1/2]}$, then by 2.2 $\phi_2^*(H) = \Gamma_1 + \Gamma_2$ and Γ_1, Γ_2 is a pair of (-1)-curves of **type (1,1,1)** (see 2.3), i.e. having 3 distinct intersection above $\alpha(p(e))$, $\alpha(p(e^*))$, $\alpha(p(e^\diamond))$ (figures 1 and 2).

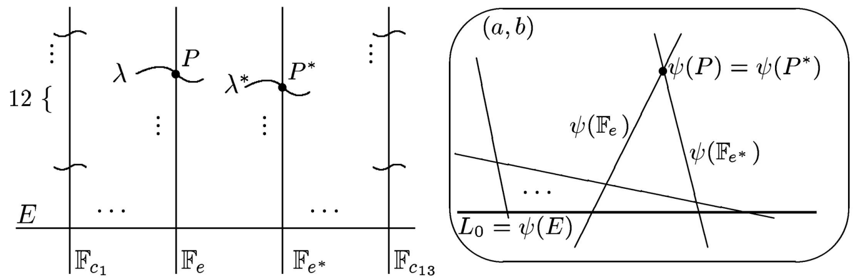

Fig. 1: $\Lambda_{(a,b)}$ for $(a,b) \in \Phi - (\xi(\mathcal{R}) \cup \{\infty\})$.

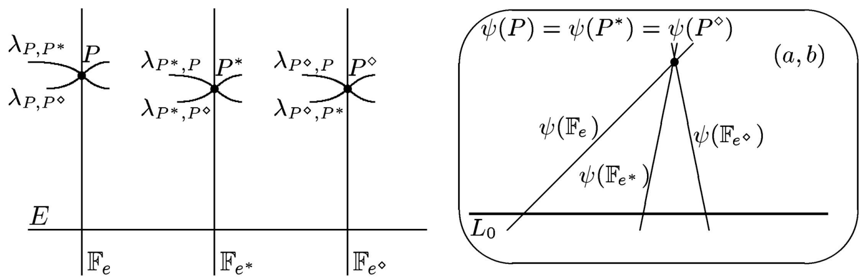

Fig. 2: Nodes of $\Lambda_{(a,b)}$ for $(a,b) \in \Phi - (\xi(\mathcal{R}) \cup \{\infty\})$.

Remark 4.2. $\Lambda_{\Phi-(\xi(\mathcal{R})\cup\{\infty\})} = \bigcup_{(a,b)\in\Phi-(\xi(\mathcal{R})\cup\{\infty\})} \Lambda_{(a,b)}$ *has only nodes as singularities, is contained in* $\tilde{U}_2$ *and the number of nodes in* $\Lambda_{\Phi-(\xi(\mathcal{R})\cup\{\infty\})}$ *equals 3 times the number of triple points of* $\psi(\mathbb{F})$ *in* $\bigcup_{H^\vee\in\Phi-(\xi(\mathcal{R})\cup\{\infty\})} H^\vee$.

Each triple point corresponds to 1 pair of (-1)*-curves of type (1,1,1).*

If λ *denotes the class of* Λ *in* $\operatorname{Pic}\mathbb{F}$*, then* $\lambda \cdot f = 12$.

Proof: If $e \in \xi^{-1}(\Phi - (\mathcal{R} \cup \{\infty\}))$, $\xi^{-1}(a,b) = \{e, e_1, \ldots, e_{12}\}$, Λ is parametrized at its intersection with $\mathbb{F}_e$ by $\lambda_{e,e_i}(t)$ as above. So $\mathbb{F}_e$ intersects $\Lambda_{\Phi-(\xi(\mathcal{R})\cup\{\infty\})}$ transversally in 12 points and $\lambda \cdot f = 12$. ∎

Case II. Suppose $(a,b)=\xi(\rho)$, $\rho\in\mathcal{R}$. Then $\xi^*((a,b)) = m\rho+e_1+\ldots+e_{13-m}$, and $e_j \notin \mathcal{R}$ by 4.1 (iii), $m \geq 2$. Let $e \in \{e_1, \ldots, e_{13-m}\}$. Since e is not a ramification point and $\xi(\rho) = \xi(e)$, there exists $t^*(t) \in \mathbb{K}[[t-\rho]]$ such that $\xi(t^*(t)) = \xi(t)$, $t^*(\rho) = e$. Let n be the multiplicity of the root ρ of g (4.1). Then the vanishing order of $t^{*'}$ at ρ is $6n$ and equals the vanishing order of ξ' at ρ. The corresponding solutions of (15) are the parametrizations

$$\lambda_{\rho,e} = (t, u_2(t))\,, \qquad \lambda^*_{\rho,e} = (t^*(t), u_2^*(t)) \tag{22}$$

of Λ, where u_2 is defined by 21 and

$$u_2^*(t) = \frac{g^4(t^*(t))}{(t+t^*)^2} + t^2\, g_2(t^*(t))\,, \qquad u_2^{*'}(t) = t^2\, g^2(t^*(t))\, t^{*'}(t)\ . \tag{23}$$

It follows that $\lambda'(\rho) = (1, u_2'(\rho)) \neq (0,0)$, i.e. that branch of Λ is smooth at $(\rho, u_2(\rho))$, while $\lambda^{*'}(\rho) = (0,0)$, i.e. Λ is singular at $(e, u_2^*(\rho))$.

Since $(a,b)\cdot\psi(\mathbb{F}) = 5\,L_0 + m\,\psi(\mathbb{F}_\rho) + \psi(\mathbb{F}_{e_1}) + \ldots + \psi(\mathbb{F}_{e_{13-m}})$, $\mathbb{F}_e$ intersects Λ in $13-m$ points; $12-m$ of them are not necessarily distinct, correspond to $\psi(\mathbb{F}_e)\cap\psi(\mathbb{F}_{e_i})$, $e_i \in \{e_1, \ldots, e_{13-m}\} - \{e\}$, and lie in the smooth, transversal to $\mathbb{F}_e$ parametrizations λ_{e,e_i} of Λ_e. The last is singular, different from the previous ones by 3.5 and 2.5, corresponds to $\psi(\mathbb{F}_e)\cap\psi(\mathbb{F}_\rho)$ and lies in $\lambda^*_{\rho,e}$. Thus $\mathbb{F}_e\cdot\Lambda = 12 - m + \mathbb{F}_e\cdot \mathrm{Im}\,(\lambda^*) = 12$ because $\mathbb{F}_e\cdot\mathrm{Im}\,(\lambda^*) = m$, the vanishing order at e of $\xi(t)$. As $\lambda\cdot f = 12$, we have determined all Λ_e, consisting of the $12-m$ smooth curves λ_{e,e_i} and the singular curve $\lambda^*_{\rho,e}$

$$\Lambda_e \;=\; \{\lambda_{e,e_i}\}_{e_i\in\{e_1,\ldots,e_{13-m}\}-\{e\}} \cup \{\lambda^*_{\rho,e}\} \tag{24}$$

λ_{e,e_i} may intersect λ_{e,e_j} in $\mathbb{F}_e$ forming a node iff $\psi(\mathbb{F}_e)$, $\psi(\mathbb{F}_{e_i})$, $\psi(\mathbb{F}_{e_j})$ intersect in a triple point, as in case I; $\lambda^*_{\rho,e}$ does not intersect λ_{e,e_i}.

To study Λ_ρ, remark that $\psi(\mathbb{F}_\rho)$ intersects $\psi(\mathbb{F}_{e_i})$ in $13-m$ distinct points: if $H^{[2]} \in \psi(\mathbb{F}_\rho)\cap\psi(\mathbb{F}_{e_i})\cap\psi(\mathbb{F}_{e_j})$, then $H\cdot A \geq 4$ by 3.5, excluded by 2.5. These intersections lie in the $13-m$ parametrizations λ_{ρ,e_i}, transversal to $\mathbb{F}_\rho$. Since $\mathbb{F}_\rho\cdot\Lambda = 12$, further parametrizations of Λ_ρ giving $m-1$ intersections with $\mathbb{F}_\rho$ must be found, arising from the self intersections of $m\,\psi(\mathbb{F}_\rho)$.

We need m and the power series of ξ at ρ. Remark that to find the normal forms we have fixed 1 of 3 points in $\mathbb{P}^1$. We may thus assume (see 4.1)

$$\rho = 0\,,\quad e = 1\,,\quad c_0 = 0\,,\quad g_2(0) = d_0 \neq 0\,,\quad \xi(\rho) = 0\,,\quad f_2(1) = 0\ . \tag{25}$$

By (9), (10), (14) it follows that

$$\begin{aligned} g_2 \;&=\; d_0^2 + d_1^2\,t^2 + c_1^2\,t^3 + d_2^2\,t^4 + c_2^2\,t^5 + d_3^2\,t^6 + c_3\,t^7 \\ &\quad + (c_3+d_4)^2\,t^8 + t^9 + t^{12}\;, \\ g \;&=\; c_1\,t + c_2\,t^2 + \sqrt{c_3}\,t^3 + t^4\;, \\ \xi(t) \;&=\; t^4 + g^4/g_2 \;=\; a_4\,t^4 + a_6\,t^6 + a_7\,t^7 + \sum_{i\geq 8} a_i\,t^i\;, \end{aligned} \tag{26}$$

hence $m \in \{4,6,7\}$ if $n=1$, as

$$a_4 = 1 + \frac{c_1^4}{d_0^2}\,, \qquad a_6 = \frac{c_1^4\,d_1^2}{d_0^4}\,, \qquad a_7 = \frac{c_1^6}{d_0^4}\;. \tag{27}$$

Case II splits as follows, n being the multiplicity of the root ρ of g.

II 1_4. $n=1$, $m=4$ hence $c_1 \neq 0$, $d_0 + c_1^2 \neq 0$, $a_4 \neq 0$. We look for solutions of (16) with t close to t^*. Set $t^* = t\,(1+\zeta)$; then (16) becomes

$$\begin{cases} \xi(t) = \xi\Big(t\,(1+\zeta)\Big) \\ \zeta \neq 0\;. \end{cases} \tag{28}$$

Substituting and dividing by $t^4\,\zeta$, we get an affine curve $\mathcal{C}$ in the plane (t,ζ)

$$a_4\,\zeta^3 + a_6\,t^2\,\zeta + a_7\,t^3 + a_7\,t^3\,\zeta + o_5(t,\zeta) \;=\; 0\,, \qquad o_5(t,\zeta) \in (t,\zeta)^5\;. \tag{29}$$

$\mathcal{C}$ has in $(0,0)$ a triple point with tangents $\zeta = \epsilon_j\,t$, $j=1,2,3$, ϵ_j roots of

$$a_4 X^3 + a_6 X + a_7 \;\in\; \mathbb{K}[X]\;. \tag{30}$$

Since $a_7 \neq 0$ and there is no X^2, it follows that $\epsilon_j \neq 0$ and that the ϵ_j are distinct; so $(0,0)$ is an ordinary triple point with branches $t \mapsto (t, \zeta_j(t))$, $\zeta_j(0) = 0$, $\zeta_j'(0) = \epsilon_j$. We get 3 solutions of (16), $t_j^*(t) = t\,(1+\zeta_j(t)) \in \mathbb{K}[[t]]$. The corresponding parametrizations of Λ_ρ are given by (17), (18), (20), (21)

$$\tilde{\lambda}_{\rho,j}(t) = (t, \tilde{u}_{2j}(t))\,, \qquad \tilde{u}_{2j}(t) = \frac{g^4(t)}{t^2\,\zeta_j(t)^2} + t^2\,\Big(1+\zeta_j(t)\Big)^2\,g_2(t)\;. \tag{31}$$

Hence $\tilde{\lambda}_{\rho,j}(t)$ are transversal to $\mathbb{F}_\rho$ and $\tilde{P}_{\rho,j} = \tilde{\lambda}_{\rho,j}(0) = (0, c_1^4/\epsilon_j^2)$, $\tilde{\lambda}'_{\rho,j}(0) = (1,0)$. As $\mathbb{F}_\rho \cdot \mathrm{Im}(\tilde{\lambda}_{\rho,j}) = 1$, we have found the 3 missing intersections, and

$$\Lambda_\rho \;=\; \{\lambda_{\rho,e_i}\}_{i=1\ldots 9} \cup \{\tilde{\lambda}_{\rho,j}\}_{j=1,2,3}\;. \tag{32}$$

All λ_{ρ,e_i}, $\tilde{\lambda}_{\rho,j}$ are smooth, transversal to $\mathbb{F}_\rho$ and their intersections with $\mathbb{F}_\rho$ are respectively $P_{\rho,i} = \lambda_{\rho,e_i}(0)$, which are distinct by 3.5, 2.5, and $\tilde{P}_{\rho,j}$, which are distinct since ϵ_j are distinct. It may be $P_{\rho,i} = \tilde{P}_{\rho,j} = P$. Consider in this case the plane $H = \psi(P)^{[\frac{1}{2}]}$. We already know that $\phi_2^*(H)$ has 1 singular point above $\alpha(p(\rho))$ and 1 above $\alpha(p(e_i))$; it may have 2 above $\alpha(p(\rho))$.

Remark 4.3. *Let $P\in\mathbb{F}_Q$, $Q\in U$ and $H=\psi(P)^{[1/2]}$ a plane containing $L(Q)$. By 3.4, $\phi_2^*(H)$ has at least 1 singular point above $\alpha(Q)$.*

Let $I = I(H, A; Q)$ be the intersection multiplicity of H, A at $\alpha(Q)$. Then

(i) *$I \geq 1$ and $I \geq 2 \iff P \in E \cup \bigcup_{\rho\in\mathcal{R}} \mathbb{F}_\rho = \{d\psi=0\} \cap \tilde{U}$;*

(ii) *$I = 3 \iff P \in D \cap \mathbb{F}_\rho,\ \ \rho \in \mathcal{R} \iff P = (\rho, u_2) \in \tilde{U}_2$ with $u_2 = 0$.*

Let $P = (0, u_2) \in \mathbb{F}_\rho \cap \tilde{U}_2$.

(iii) *$\phi_2^*(H)$ has 2 infinitely near singular points above $\alpha(Q)$ $\iff$ $\iff$ u_2 is a root of*

$$X^3 + d_1^4 X^2 + d_0^4(d_0^4 + c_1^8) \in \mathbb{K}[X] \ ; \tag{33}$$

(iv) *$\phi_2^*(H)$ has 3 infinitely near singular points above $\alpha(Q) \iff$ we have*

$$\begin{cases} u_2 = 0 \\ d_0 + c_1^2 = 0 \\ d_1 + d_2^2 + c_2^4 = 0 \ ; \end{cases} \tag{34}$$

(v) *the 3 distinct points $\tilde{P}_{\rho,j}$ are exactly the points of $\mathbb{F}_\rho\cap\tilde{U}$ s.t. $\phi_2^*(H)$ has 2 infinitely near singular points above $\alpha(Q)$;*

(vi) *assuming (25), $\tilde{P}_{\rho,j} = P_{\rho,i} = P \iff \epsilon_j = c_1^2/d_0 \iff$ if $H = \psi(P)^{[\frac{1}{2}]}$ then $\phi_2^*(H) = \Gamma_1 + \Gamma_2$ with Γ_i having intersection 2 above $\alpha(p(\rho))$ and 1 above $\alpha(p(e_i))$, i.e. Γ_1, Γ_2 is a pair of (-1)-curves of* **type (2,1)** *(2.3).*

Proof:

(i) follows from 3.5, 3.8.

(ii) If $I = 3$, we must be in case (i) and $P \notin E$, since otherwise $H \ni V$ by 3.6 and $V \in A$, $V \neq \alpha(Q)$, $\deg A = 3$. So $P \in \mathbb{F}_\rho \cap \tilde{U}_2$, we may assume $\rho = 0$ and in the coordinates (t, u_2) of $\tilde{U}_2$ $P = (0, a)$. Then $L(Q)$ is tangent to A at $\alpha(Q)$, and the pencil of planes containing $L(Q)$ is isomorphic to A sending a plane to its third intersection with A. So there is exactly 1 plane H s.t. $I = 3$, i.e. $H \cap (A - \{\alpha(Q)\}) = \emptyset$. And if $P \in D$, i.e. $a = 0$, then $\psi(P) = (h_0, h_1, 0, h_3)$, so $\alpha(t) = (1, t, t^2, t^3) \notin H$ for $t \neq 0$.

(iii) Let $P = (0, u_2) \in \mathbb{F}_\rho \cap \tilde{U}_2$; then $H = \psi_2(P)^{[1/2]} = (0, 0, \sqrt{u_2}, d_0)$. In the affine chart of $\mathbb{P}(1,1,2,3)$ containing $\pi^{-1}(\alpha(Q))$ with coordinates $z = Z/X_0^3$, $w = W/X_0^2$, $t = X_1/X_0$, $\phi_2^*(H)$ is defined by

$$(35) \qquad \begin{cases} z^2 + (w + t^3)\, z + b = 0 \ , \\ \sqrt{u_2}\, t^2 + d_0 w = 0 \ . \end{cases}$$

Substituting $w = d_0^{-1}\sqrt{u_2}\, t^2$ in the first equation, after $z + d_0 \mapsto z$ we get

$$z^2 + A\, t^2 z + t^3 z + B\, t^2 + C\, t^3 + D\, t^4 + E\, t^5 + F\, t^6 \ = \ 0$$

where $A = \sqrt{u_2}/d_0$, $B = d_1^2 + \sqrt{u_2}$, $C = d_0 + c_1^2$, $D = d_2^2$, $E = c_2^2$. Blowing up at $(t, z) = (0, 0)$, which is singular as expected, the strict transform lies in $z = t\, v$; after $v \mapsto v + \sqrt{B}$, we get $v^2 + A\, t\, v + (A\sqrt{B} + C)\, t + t^2\, v + (\sqrt{B} + D)\, t^2 + Et^3 + Ft^4 = 0$. So we have 2 infinitely near singular points at $(t, z) = (0, 0) \iff A\sqrt{B} + C = 0$, from which (iii) follows.

(iv) Suppose $A\sqrt{B} + C = 0$; blowing up at $(t, v) = (0, 0)$, the strict transform lies in $v = t\, s$ and is $s^2 + A\, s + \sqrt{B} + D + (E + s)\, t + F\, t^2 = 0$. Let α_i be the roots of $X^2 + AX + \sqrt{B} + D$; after $s + \alpha_i \mapsto s$ we get $s^2 + A\, s + (E + \alpha_i)\, t + t\, s + F\, t^2 = 0$. So we have 3 infinitely near singular points at $(t, z) = (0, 0) \iff A = E + \alpha_i = 0$, which is equivalent to 34.

(v) Just check that c_1^4/ϵ_j^2, $j = 1, 2, 3$, are roots of 33.

(vi) $\tilde{P}_{\rho,j} = (0, c_1^4/\epsilon_j^2)$ and by (25), (20), (21) $P_{\rho,i} = (0, d_0^2)$. All the assertions follow from (iii) and (iv). ∎

Proposition 4.4. *In case* $\mathbf{1_4}$ *(*$n = 1$, $m = 4$*), if* $(a, b) = \xi(\rho)$, $\rho \in \mathcal{R}$, *then*

$$(36) \qquad \xi^*((a, b)) = 4\,\rho + e_1 + \ldots + e_9 \,, \qquad \Lambda_{(a,b)} = \Lambda_\rho \cup \Lambda_{e_1} \cup \ldots \cup \Lambda_{e_9} \ .$$

Λ_ρ *is defined in (22), (31), (32) and* $\Lambda_{e_i} = \{\lambda_{e_i,e_j}\}_{j \neq i} \cup \{\lambda^*_{\rho,e_i}\}$ *(24);* Λ_ρ *intersects* $\mathbb{F}_\rho$ *in* $P_{\rho,1}, \ldots, P_{\rho,9}$*, distinct, and* $\tilde{P}_{\rho,1}$, $\tilde{P}_{\rho,2}$, $\tilde{P}_{\rho,3}$*, distinct.*

There are pairs of (-1)*-curves of type (2,1)* $\iff P_{\rho,i} = \tilde{P}_{\rho,j} \iff \epsilon_j = c_1^2/d_0$ *(4.3 (vi)). If their number is* $n_{\rho,(2,1)}$*, then* $0 \leq n_{\rho,(2,1)} \leq 3$.

The singularities in $\Lambda_{(a,b)}$ *are, except nodes, the* $n_{\rho,(2,1)}$ *singularities* $P_{\rho,i} = \tilde{P}_{\rho,j} \in \mathrm{Im}\,(\lambda_{\rho,e_i}) \cap \mathrm{Im}\,(\tilde{\lambda}_{\rho,j})$ *and the 9 cusps* $P^*_{\rho,i} \in \mathrm{Im}(\lambda^*_{\rho,e_i})$ *(figure 3).* ∎

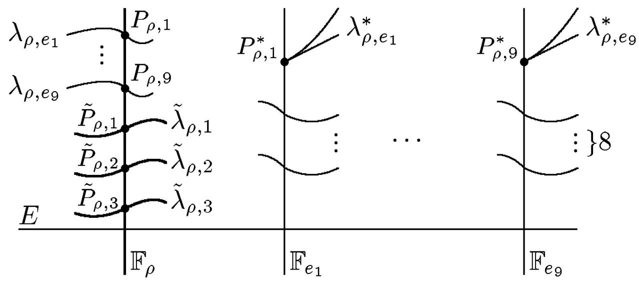

Fig. 3 : $\Lambda_{(a,b)}$ of type $1_4, 2, 3, 4$.

II 1_6. $n = 1$, $m = 6$ hence $c_1 \neq 0$, $d_1 \neq 0$, $d_0 = c_1^2$, $a_4 = 0$, $a_6 \neq 0$, $a_7 \neq 0$.

As in II 1_4, we solve the first of (28). Substituting and dividing by $t^6\,\zeta$, we get the equation of an affine curve $\mathcal{C}'$ in the plane (t, ζ):

$$\begin{aligned} a_6\,\zeta\,(1+\zeta+\zeta^2)^2 + a_7\,t\left[1+\zeta\,(1+\zeta+\zeta^2)^2 + \zeta^2\,(1+\zeta+\zeta^2)^2\right] + \\ + \; a_8\,t^2\,\zeta^7 + t^3(\ldots) \;=\; 0\;. \end{aligned} \tag{37}$$

Let η_1, η_2 be the distinct roots of $1+X+X^2$; $\mathcal{C}'$ intersects $\{t=0\}$ in the 3 distinct points $B_0 = (0,0)$, $B_1 = (0,\eta_1)$, $B_2 = (0,\eta_2)$. The above equation becomes $a_6\,\zeta\,(\zeta+\eta_1)^2\,(\zeta+\eta_2)^2 + a_7\,t + o_3 = 0$ and defines the following parametrizations at B_j : $t \mapsto (t, \zeta(t))$, $\zeta(t) = \frac{a_7}{a_6}\,t + o_2(t) \in \mathbb{K}[[t]]$, $o_2(t) \in (t)^2$ for $j = 0$ and $\zeta \mapsto (t_j(\zeta), \zeta)$, $t_j(\zeta) = \eta_j\,\frac{a_6}{a_7}\,(\zeta+\eta_j)^2 + o_3\,(\zeta+\eta_j) \in \mathbb{K}[[\zeta+\eta_j]]$ for $j = 1, 2$. We get therefore 3 solutions of (16), $(t, t^*(t))$ and $(t_j(\zeta), t_j^*(\zeta))$, $j = 1, 2$, centered at 0 the first, at η_j the second: $t^*(t) = t(1+\zeta(t))$, $t_j^*(\zeta) = t_j(\zeta)\,(1+\zeta)$.

The corresponding parametrizations of Λ_ρ as in (31) are

$$\tilde{\lambda}_\rho(t) = (t, \tilde{u}_2(t))\,, \qquad \tilde{u}_2(t) = \frac{g^4(t)}{t^2\,\zeta^2(t)} + t^2\left(1+\zeta(t)\right)^2 g_2(t)\;, \tag{38}$$

where $\tilde{u}_2$ is defined by (21), and for $j = 1, 2$

$$\tilde{\lambda}_{\rho,j}(\zeta) = \left(t_j(\zeta), \tilde{u}_{2,j}(\zeta)\right), \quad \tilde{u}_{2,j}(\zeta) = \frac{g^4(t_j(\zeta))}{t_j^2(\zeta)\,\zeta^2} + t_j^2(\zeta)\,(1+\zeta)^2\,g_2(t_j(\zeta))\;. \tag{39}$$

Thus $\tilde{\lambda}_\rho(t) = (t, d_1^4 + o_1(t))$, $\tilde{\lambda}_{\rho,j}(\zeta) = (\eta_j\,\frac{d_1^2}{c_1^2}\,(\zeta+\eta_j)^2 + o_3(\zeta+\eta_j),\, o_5(\zeta+\eta_j))$. As $\mathbb{F}_\rho\cdot \mathrm{Im}(\tilde{\lambda}_\rho) = 1$, $\mathbb{F}_\rho\cdot \mathrm{Im}(\tilde{\lambda}_{\rho,j}) = 2$ we found all Λ_ρ

$$\Lambda_\rho \;=\; \{\lambda_{\rho,e_i}\}_{i=1\ldots 7} \cup \{\tilde{\lambda}_\rho\} \cup \{\tilde{\lambda}_{\rho,j}\}_{j=1,2}\;. \tag{40}$$

Let $\tilde{P}_\rho = \tilde{\lambda}_\rho(0) = (0, d_1^4)$, $\tilde{P}_{\rho,*} = \tilde{\lambda}_{\rho,j}(\eta_j) = (0,0)$, $P_{\rho,i} = \lambda_{\rho,e_i}(0) = (0, d_0^2)$ (by (25), $e_i = 1$). From 4.3 it follows

Proposition 4.5. *In case* $\mathbf{1_6}$, *($n=1$, $m=6$), if $(a,b) = \xi(\rho)$, $\rho \in \mathcal{R}$, then*

$$\xi^*((a,b)) = 6\,\rho + e_1 + \ldots + e_7\,, \qquad \Lambda_{(a,b)} = \Lambda_\rho \cup \Lambda_{e_1} \cup \ldots \cup \Lambda_{e_7}\ . \tag{41}$$

Λ_ρ is defined in (22), (38), (39), (40) and Λ_{e_i} is as in 4.4. Λ_ρ intersects $\mathbb{F}_\rho$ in the distinct points $P_{\rho,1}, \ldots, P_{\rho,7}$, $\tilde{P}_\rho$, and in $\tilde{P}_{\rho,*}$, $\tilde{P}_{\rho,*} \neq \tilde{P}_\rho$.

$H = \psi(\tilde{P}_{\rho,*})^{[1/2]}$ has intersection $I = 3$ with A at $\alpha(p(\rho))$ and $\phi_2^*(H)$ has 3 infinitely near singular points above $\alpha(p(\rho)) \iff d_1 + d_2^2 + c_2^4 = 0$.

In this case, by 2.1 $\phi_2^*(H)$ is a pair of (-1)-curves having intersection multiplicity 3 above $\alpha(p(\rho))$, i.e. a pair of **type (3)** *(2.3). If we denote by $n_{\rho,(3)}$ the number of such pairs, we have therefore $0 \le n_{\rho,(3)} \le 1$.*

There are pairs of (-1)-curves of type (2,1) $\iff P_{\rho,i} = \tilde{P}_\rho \iff d_0 = d_1^2$; their number is $0 \le n_{\rho,(2,1)} \le 1$. The singularities in $\Lambda_{(a,b)}$ are, except nodes, the $n_{\rho,(2,1)}$ singularities $P_{\rho,i} = \tilde{P}_\rho \in \mathrm{Im}(\lambda_{\rho,e_i}) \cap \mathrm{Im}(\tilde{\lambda}_\rho)$; $\tilde{P}_{\rho,*} \in \mathrm{Im}(\tilde{\lambda}_{\rho,1}) \cap \mathrm{Im}(\tilde{\lambda}_{\rho,2})$; the 7 cusps $P^*_{\rho,i} \in \mathrm{Im}(\lambda^*_{\rho,e_i})$ (see figure 4). ∎

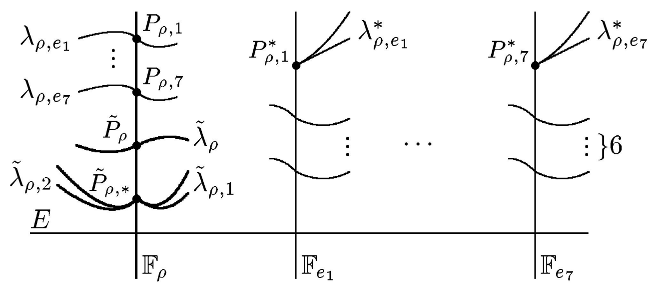

Fig. 4: $\Lambda_{(a,b)}$ of type 1_6.

II $\mathbf{1_7}$. $n=1$, $m=7$ hence $c_1 \neq 0$, $d_1 = 0$, $d_0 = c_1^2$, $a_4 = a_6 = 0$, $a_7 \neq 0$.

Substituting and dividing by $t^7\zeta$, we get a curve $\mathcal{C}''$: $a_7(1+\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5+\zeta^6) + t(\ldots) = 0$ in the plane (t,ζ).

The polynomial $1+X+X^2+X^3+X^4+X^5+X^6 \in \mathbb{K}[X]$ has 6 distinct roots $\mu_j \neq 0$, $j=1\ldots 6$, in $\mathbb{K}$. Thus $\mathcal{C}''$ intersects $\{t=0\}$ in the 6 points $(0,\mu_j)$, and at $(0,\mu_j)$ we have $\zeta_j(t) = \mu_j + o_1(t) \in \mathbb{K}[[t]]$ and hence $t_j^*(t) = (1+\mu_j)\,t + o_2(t)$. The corresponding parametrizations of Λ_ρ as in (31) are

$$\tilde{\lambda}_{\rho,j}(t) = (t, \tilde{u}_{2,j}(t))\,, \qquad \tilde{u}_{2,j}(t) = \frac{g^4(t)}{t^2\,\zeta_j^2(t)} + t^2\,(1+\zeta_j(t))^2\,g_2(t)\ . \tag{42}$$

Then $\tilde\lambda_{\rho,j}(t) = (t,\, [\frac{c_1^4}{\mu_j^2} + (1+\mu_j^2)\, d_0^2]\, t^2 + o_4(t))$ and $\mathbb{F}_\rho\cdot \mathrm{Im}(\tilde\lambda_{\rho,j}) = 1$. As before

$$\Lambda_\rho \;=\; \{\lambda_{\rho,e_i}\}_{i=1\ldots 6} \cup \{\tilde\lambda_{\rho,j}\}_{j=1\ldots 6}\ . \tag{43}$$

Let $\tilde P_\rho = \tilde\lambda_{\rho,j}(0) = (0,0)$, $P_{\rho,i} = \lambda_{\rho,e_i}(0) = (0, d_0^2)$ (by (25), $e_i = 1$). Then

Proposition 4.6. *In case* $\mathbf{1_7}$ *(*$n=1$, $m=7$*), if* $(a,b) = \xi(\rho)$, $\rho\in\mathcal{R}$, *then*

$$\xi^*((a,b)) = 7\rho + e_1 + \ldots + e_6\,, \qquad \Lambda_{(a,b)} = \Lambda_\rho \cup \Lambda_{e_1} \cup \ldots \cup \Lambda_{e_6}\ . \tag{44}$$

Λ_ρ is defined in (22), (42), (43) and Λ_{e_i} is as in 4.4. Λ_ρ intersects $\mathbb{F}_\rho$ in $P_{\rho,1},\ldots,$ $P_{\rho,6}$, $\tilde P_\rho$. The plane $H = \psi(\tilde P_\rho)^{[1/2]}$ has intersection $I = 3$ with A at $\alpha(p(\rho))$ and $\phi_2^*(H)$ has 3 infinitely near singular points above $\alpha(p(\rho)) \Longleftrightarrow d_2^2 + c_2^4 = 0$. In this case, $\phi_2^*(H) = \Gamma_1 + \Gamma_2$ is a pair of (-1)-curves of type (3); thus $0 \le n_{\rho,(3)} \le 1$. There are no pairs of (-1)-curves of type (2,1), since $P_{\rho,i} \ne \tilde P_\rho$. The singularities in $\Lambda_{(a,b)}$ are, except nodes, $\tilde P_\rho \in \bigcap_{j=1\ldots 6} \mathrm{Im}\,(\tilde\lambda_{\rho,j})$ and the 6 cusps $P^*_{\rho,i} \in \mathrm{Im}(\lambda^*_{\rho,e_i})$ (figure 5). ∎

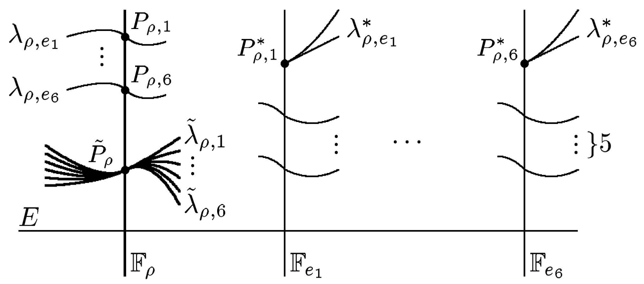

Fig. 5 : $\Lambda_{(a,b)}$ of type 1_7.

II 2. $n=2$ hence $c_1 = 0$, $c_2 \ne 0$, $m=4$, $a_4 = 1$, $a_6 = a_7 = 0$, $a_8 = c_2^4/d_0^2 \ne 0$, $a_9 = a_{11} = 0$, $a_{13} = c_2^6/d_0^4 \ne 0$. Set $t^* = t\,(1+t^3\zeta)$; substituting in (16) and dividing by $t^{16}\,\zeta$, we get the curve $\zeta^3 + a_{10}\,\zeta + a_{13} + t(\ldots) = 0$. The polynomial

$$X^3 + a_{10}X + a_{13} \;\in\; \mathbb{K}[X] \tag{45}$$

has 3 distinct roots ω_j, $\omega_j \ne 0$, $j = 1,2,3$ in $\mathbb{K}$. The curve intersects $\{t=0\}$ in the 3 points $(0,\omega_j)$, where we have $\zeta_j(t) = \omega_j + o_1(t)$ and hence $t_j^*(t) = t + \omega_j\, t^4 + o_5(t)$. The corresponding parametrizations of Λ_ρ are

$$\tilde\lambda_{\rho,j}(t) = (t, \tilde u_{2,j}(t))\,, \qquad \tilde u_{2,j}(t) = \frac{g^4(t)}{t^8\,\zeta_j^2(t)} + t^2\left(1 + t^3\,\zeta_j(t)\right)^2 g_2(t)\ . \tag{46}$$

It follows $\tilde{\lambda}_{\rho,j}(t) = (t, \frac{c_2^4}{\omega_j^2} + o_2(t))$. Since $\mathbb{F}_\rho \cdot \mathrm{Im}(\tilde{\lambda}_{\rho,j}) = 1$, Λ_ρ is as in (32). Let $\tilde{P}_{\rho,j} = \tilde{\lambda}_{\rho,j}(0) = (0, \frac{c_2^4}{\omega_j^2})$, $P_{\rho,i} = \lambda_{\rho,e_i}(0) = (0, d_0^2)$ ($e_i = 1$ by (25)). Then

Proposition 4.7. *In case* **2** *($n = 2$, $m = 4$), if $(a,b) = \xi(\rho)$, $\rho \in \mathcal{R}$, then $\xi^*((a,b))$ and $\Lambda_{(a,b)}$ are as in (36), Λ_ρ is defined in (22), (46), (32) and Λ_{e_i} is as in 4.4. Λ_ρ intersects $\mathbb{F}_\rho$ in the points $P_{\rho,1}, \dots, P_{\rho,9}$, $\tilde{P}_{\rho,j}$, $j = 1,2,3$.*

If $H = \psi(\tilde{P}_{\rho,j})^{[1/2]}$, then $\phi_2^*(H)$ has 2 infinitely near singular points above $\alpha(p(\rho))$, since c_2^4/ω_j^2 are the roots of the polynomial Q_ρ defined in (33).

There are of (-1)-curves of type $(2,1) \iff P_{\rho,i} = \tilde{P}_{\rho,j} \iff d_0 = c_2^2/\omega_j$; hence $0 \le n_{\rho,(2,1)} \le 3$. There are no (-1)-curves of type (3). The singularities in $\Lambda_{(a,b)}$ are, except nodes, the $n_{\rho,(2,1)}$ points $P_{\rho,i} = \tilde{P}_{\rho,j} \in \mathrm{Im}(\lambda_{\rho,e_i}) \cap \mathrm{Im}(\tilde{\lambda}_{\rho,j})$; the 9 cusps $P^*_{\rho,i} \in \mathrm{Im}(\lambda^*_{\rho,e_i})$ (figure 3). ∎

II 3. $n = 3$ hence $c_1 = c_2 = 0$, $c_3 \neq 0$, $m = 4$, $\xi(t) = t^4 + a_{12}\, t^{12} + a_{14}\, t^{14} + a_{16}\, t^{16} + a_{18}\, t^{18} + a_{19}\, t^{19} + a_{20}(t)$, $a_{19} = c_3^3/d_0^4 \neq 0$. If $t^* = t\,(1 + t^5\,\zeta)$, from (16) dividing by $t^{24}\,\zeta$, we get the curve $\zeta^3 + a_{14}\,\zeta + a_{19} + t(\dots) = 0$. Then

$$X^3 + a_{14}X + a_{19} \in \mathbb{K}[X] \tag{47}$$

has 3 distinct roots $\omega'_j \in \mathbb{K}$, $j = 1,2,3$, $\omega'_j \neq 0$. The curve intersects $\{t = 0\}$ in the 3 points $(0, \omega'_j)$, where we have $\zeta_j(t) = \omega'_j + o_1(t)$ and hence $t^*_j(t) = t + \omega_j\, t^6 + o_7(t)$. The corresponding parametrizations of Λ_ρ are

$$\tilde{\lambda}_{\rho,j}(t) = (t, \tilde{u}_{2,j}(t)), \qquad \tilde{u}_{2,j}(t) = \frac{g^4(t)}{t^{12}\,\zeta_j^2(t)} + t^2\left(1 + t^5\,\zeta_j(t)\right)^2 g_2(t)\ . \tag{48}$$

It follows $\tilde{\lambda}_{\rho,j}(t) = (t, \frac{c_3^2}{\omega_j'^2} + o_2(t))$, $\mathbb{F}_\rho \cdot \mathrm{Im}(\tilde{\lambda}_{\rho,j}) = 1$ and Λ_ρ is as in (32). Let $\tilde{P}_{\rho,j} = \tilde{\lambda}_{\rho,j}(0) = (0, \frac{c_3^2}{\omega_j'^2})$, and $P_{\rho,i} = \lambda_{\rho,e_i}(0) = (0, d_0^2)$ ($e_i = 1$). Then

Proposition 4.8. *In case* **3***, ($n = 3$, $m = 4$), if $(a,b) = \xi(\rho)$, $\rho \in \mathcal{R}$, then $\xi^*((a,b))$ and $\Lambda_{(a,b)}$ are as in (36), Λ_ρ is defined in (22), (48), (32) and Λ_{e_i} is as in 4.4. Λ_ρ intersects $\mathbb{F}_\rho$ in $P_{\rho,1}, \dots, P_{\rho,9}$, $\tilde{P}_{\rho,j}$, $j = 1,2,3$.*

If $H = \psi(\tilde{P}_{\rho,j})^{[1/2]}$, then $\phi_2^*(H)$ has 2 infinitely near singular points above $\alpha(p(\rho))$, as $c_3^2/\omega_j'^2$ are the roots of the polynomial Q_ρ (see (33)).

There are (-1)-curves of type $(2,1) \iff P_{\rho,i} = \tilde{P}_{\rho,j} \iff d_0 = c_3/\omega'_j$; hence $0 \le n_{\rho,(2,1)} \le 3$. There are no (-1)-curves of type (3). The singularities in $\Lambda_{(a,b)}$ are, except nodes, the $n_{\rho,(2,1)}$ points $P_{\rho,i} = \tilde{P}_{\rho,j} \in \mathrm{Im}\,(\lambda_{\rho,e_i}) \cap \mathrm{Im}\,(\tilde{\lambda}_{\rho,j})$; the 9 cusps $P^*_{\rho,i} \in \mathrm{Im}(\lambda^*_{\rho,e_i})$ (figure 3). ∎

II 4. $n=4$ hence $c_1=c_2=c_3=0$, $m=4$, $\xi(t)=t^4+a_{16}\,t^{16}+a_{18}\,t^{18}+a_{20}\,t^{20}+a_{22}\,t^{22}+a_{24}\,t^{24}+a_{25}\,t^{25}+o_{26}(t)$, $a_{25}=1/d_0^4\neq 0$. If $t^*=t(1+t^7\zeta)$, from (16) dividing by $t^{32}\,\zeta$ we get the curve $\zeta^3+a_{18}\,\zeta+a_{25}+t(\dots)=0$. Then

$$X^3+a_{18}X+a_{25}\,\in\,\mathbb{K}[X] \tag{49}$$

has 3 distinct roots ω_j'', $\omega_j''\neq 0$, $j=1,2,3$ in $\mathbb{K}$. The curve intersects $\{t=0\}$ in the 3 points $(0,\omega_j'')$, where we have $\zeta_j(t)=\omega_j''+o_1(t)$ and hence $t_j^*(t)=t+\omega_j''\,t^8+o_9(t)$. The corresponding parametrizations of Λ_ρ are

$$\tilde\lambda_{\rho,j}(t)=(t,\tilde u_{2,j}(t))\,,\qquad \tilde u_{2,j}(t)=\frac{g^4(t)}{t^{16}\,\zeta_j^2(t)}+t^2\left(1+t^7\,\zeta_j(t)\right)^2 g_2(t)\ . \tag{50}$$

It follows $\tilde\lambda_{\rho,j}(t)=(t,\frac{1}{\omega_j''^2}+o_2(t))$. Since $\mathbb{F}_\rho\cdot\mathrm{Im}(\tilde\lambda_{\rho,j})=1$, Λ_ρ is as in (32). Let $\tilde P_{\rho,j}=\tilde\lambda_{\rho,j}(0)=(0,\frac{1}{\omega_j''^2})$, and $P_{\rho,i}=\lambda_{\rho,e_i}(0)=(0,d_0^2)$ $(e_i=1)$; then

Proposition 4.9. *In case* **4**, *$(n=4,\ m=4)$, if $(a,b)=\xi(\rho)$, $\rho\in\mathcal{R}$, then $\xi^*((a,b))$ and $\Lambda_{(a,b)}$ are as in (36), Λ_ρ is defined in (22), (50), (32) and Λ_{e_i} is as in 4.4. Λ_ρ intersects $\mathbb{F}_\rho$ in $P_{\rho,1},\,\dots,\,P_{\rho,9}$, $\tilde P_{\rho,j}$, $j=1,2,3$.*

If $H=\psi(\tilde P_{\rho,j})^{[1/2]}$, then $\phi_2^*(H)$ has 2 infinitely near singular points above $\alpha(p(\rho))$, since $1/\omega_j''^2$ are the roots of Q_ρ ((33)).

There are (-1)-curves of type $(2,1)$ $\iff$ $P_{\rho,i}=\tilde P_{\rho,j}$ $\iff$ $d_0=1/\omega_j''$; hence $0\le n_{\rho,(2,1)}\le 3$. There are no (-1)-curves of type (3). The singularities in $\Lambda_{(a,b)}$ are, except nodes, the $n_{\rho,(2,1)}$ points $P_{\rho,i}=\tilde P_{\rho,j}\in\mathrm{Im}\,(\lambda_{\rho,e_i})\cap\mathrm{Im}\,(\tilde\lambda_{\rho,j})$; the 9 cusps $P^*_{\rho,i}\in\mathrm{Im}(\lambda^*_{\rho,e_i})$ (figure 3). ∎

Case III. Suppose $(a,b)=\infty=(0,1)$; by 4.1, $\xi^*(\infty)=e_1+\ldots+e_{12}+V$, $g_2(e_i)=0$, $g_2'(e_i)\neq 0$. We know $\psi^*(\infty)=E+2\,\mathbb{F}_V+\mathbb{F}_{e_1}+\ldots+\mathbb{F}_{e_{12}}$, $\psi_*(E)=4\,L_0$, $\psi_*(\mathbb{F}_V)=L_0$ and hence $\infty\cdot\psi(\mathbb{F})=6\,L_0+\psi(\mathbb{F}_{e_1})+\ldots+\psi(\mathbb{F}_{e_{12}})$, $\psi|_E$ and $\psi|_{F_V}$ injective. It follows that in the plane ∞ the lines $\psi(\mathbb{F}_{e_i})$, $i=1,\dots,12$, intersect the line L_0 in 12 distinct points $\overline{A}_i$; let $\psi^{-1}(\overline{A}_i)=\{A_i,A_i^*\}$, $A_i=E\cap\mathbb{F}_{e_i}$, $A_i^*\in\mathbb{F}_V$, $A_i^*\neq\mathbb{F}_V\cap E$, A_i^* distinct (E is a section of $\mathbb{F}$).

For $1\le i<j\le 12$ the lines $\psi(\mathbb{F}_{e_i})$ and $\psi(\mathbb{F}_{e_j})$ in ∞ are different from L_0, hence $\psi(\mathbb{F}_{e_i})\cap\psi(\mathbb{F}_{e_j})=\{\overline{B}_{i,j}\}$, $\overline{B}_{i,j}\notin L_0$. Let $\psi^{-1}(\overline{B}_{i,j})=\{B_{i,j},B^*_{i,j}\}$, $B_{i,j}\in\mathbb{F}_{e_i}-E$, $B^*_{i,j}\in\mathbb{F}_{e_j}-E$.

For i, j, k distinct, $\psi(\mathbb{F}_{e_i})\cap\psi(\mathbb{F}_{e_j})\cap\psi(\mathbb{F}_{e_k})=\emptyset$, otherwise the corresponding plane would meet the cubic A in V, $\alpha(p(e_i))$, $\alpha(p(e_j))$, $\alpha(p(e_k))$.

In conclusion $\Lambda_\infty\cap\mathbb{F}_{e_j}=\{A_j\}\cup\{B_{i,j}\}_{i<j}\cup\{B^*_{j,k}\}_{j<k}$, $\Lambda_\infty\cap\mathbb{F}_V=\{A_i^*\}_{i=1\ldots12}$, $\Lambda_\infty\cap E=\{A_i\}_{i=1\ldots12}$. In particular, $\#(\Lambda_\infty\cap\mathbb{F}_{e_j})=\#(\Lambda_\infty\cap\mathbb{F}_V)=12$.

Since $\lambda \cdot f = 12$ ((4.2)), it follows that Λ_∞ is smooth, transversal to $\mathbb{F}_{e_j}$, $\mathbb{F}_V$.

Moreover, $\lambda \cdot e = 12$. To prove this, note that $\Lambda \cap E = \Lambda_\infty \cap E = \{A_i\}_{i=1\ldots 12}$, since in the preceding cases I, II we had $\Lambda_{(a,b)} \subset \tilde{U}_2$, hence $\Lambda_{(a,b)} \cap E = \emptyset$. It suffices therefore to show that the intersections A_i are transversal. The parametrization of Λ near A_i comes from $\psi_0(A_i) = \psi_{V1}(A_i^*)$. By (11) we have $\psi_0(t, u_0) = (t^4 + u_0 r_0$, $u_0 f_2$, 1, $u_0 g_2)$ and $\psi_{V,1}(t_V, v_1) = (v_1 r_{V,0}$, $f_{V,1}(1 + t_V^4 v_1)$, $r_{V,0}$, $g_{V,1}(1 + t_V^4 v_1))$. We obtain a 2×4 matrix $[P^1, P^2, P^3, P^4]$ whose rank must be 1. Looking at the 2×2 minors $[P^1, P^3]$ and $[P^2, P^3]$, we get

$$(51) \qquad L'(t, t_V) \begin{pmatrix} u_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} r_{V,0}\, t^4 \\ f_{V,1} \end{pmatrix}, \qquad L'(t, t_V) = \begin{bmatrix} r_{V,0}\, r_0 & r_{V,0} \\ r_{V,0}\, f_2 & t_V^4\, f_{V,1} \end{bmatrix}.$$

Looking at $[P^2, P^4]$, from 14 it follows $\xi(t) = \xi(t_V)$. But ∞ is not a branch point for ξ, so the last equation has solution $t = t(t_V) \in \mathbb{K}[[t_V]]$, $t(0) = c_i$, $t'(0) \neq 0$. Solving (51), we get $u_0 = f_{V1} \frac{1 + t^4 t_V^4}{r_{V,0} f_2 + t_V^4 r_0 f_{V,1}}$. From (8), (10) we see $r_{V,0} = 1 + o_4(t_V)$, $f_{V1} = t_V + o_3(t_V)$ and $f_2(e_i) \neq 0$, since $g_2(e_i) = 0$. Hence $u_0 = t_V + o_2(t_V)$, the required transversality in A_i. We proved:

Proposition 4.10. *Λ_∞ is smooth and $\lambda \cdot e = 12$. In particular, there are no nodes in Λ_∞ (figure 6).* ∎

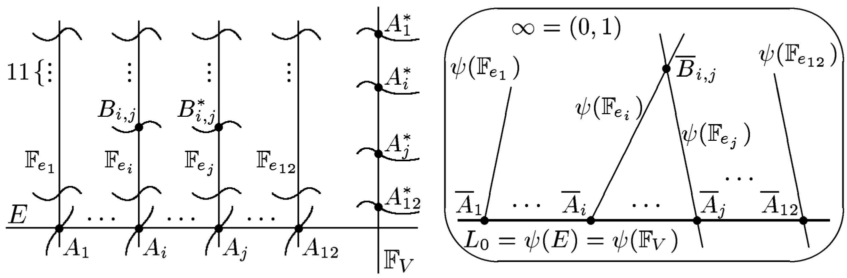

Fig. 6 : Λ_∞ $(i < j)$.

5 – The proof of Theorem 1.1

In this section we use the local analysis of Λ carried out in cases I, II, III in section 4 to prove 1.1, leaving the calculations to the next section.

Proposition 5.1. *The arithmetic genus of Λ is $p_a(\Lambda) = 781$.*

Moreover, the following possibilities occur with respect to the decomposition into irreducible components $\Lambda = \sum \Lambda_i$ *of* Λ*. The exponent denotes the arithmetic genus of the component and the latin lower case letters denote the corresponding classes in* $\operatorname{Pic}\mathbb{F}$*.*

(d1) $\Lambda = \Lambda^{781}$ *is irreducible and* $l = 12\,e + 132f$*;*

(d2) $\Lambda = \Lambda_1^{531} + \Lambda_2^{11}$ *and* $l_1 = 10\,e + 110f,\ l_2 = 2\,e + 22f$*;*

(d3) $\Lambda = \Lambda_1^{329} + \Lambda_2^{69}$ *and* $l_1 = 8\,e + 88f,\ l_2 = 4\,e + 44f$*;*

(d4) $\Lambda = \Lambda_1^{329} + \Lambda_2^{11} + \Lambda_3^{11}$ *and* $l_1 = 8\,e + 88f,\ l_1 = l_2 = 2\,e + 22f$*;*

(d5) $\Lambda = \Lambda_1^{175} + \Lambda_2^{175}$ *and* $l_1 = l_2 = 6\,e + 66f$*;*

(d6) $\Lambda = \Lambda_1^{175} + \Lambda_2^{69} + \Lambda_3^{11}$ *and* $l_1 = 6\,e + 66f,\ l_2 = 4\,e + 44f,\ l_3 = 2\,e + 22f$*;*

(d7) $\Lambda = \Lambda_1^{175} + \Lambda_2^{11} + \Lambda_3^{11} + \Lambda_4^{11}$ *and* $l_1 = 6\,e + 66f,\ l_2 = l_3 = l_4 = 2\,e + 22f$*;*

(d8) $\Lambda = \Lambda_1^{69} + \Lambda_2^{69} + \Lambda_3^{69}$ *and* $l_1 = l_2 = l_3 = 4\,e + 44f$*;*

(d9) $\Lambda = \Lambda_1^{69} + \Lambda_2^{69} + \Lambda_3^{11} + \Lambda_4^{11}$ *and* $l_1 = l_2 = 4\,e + 44f,\ l_3 = l_4 = 2\,e + 22f$*.*

Proof: If we call e, f, l, k the classes of E, $\mathbb{F}_X$, Λ, $K_{\mathbb{F}}$ in $\operatorname{Pic}\mathbb{F}$, from 3.6, 4.2, 4.10 it follows $l \cdot e = l \cdot f = 12$, hence $l = 12\,e + 132\,f$. Moreover $-2 = e^2 + e \cdot k = f^2 + fk$ and $k = -12f - 2\,e$. It follows $p_a(l) = 1 + \frac{1}{2}(l^2 + l \cdot k) = 781$.

The map $\psi\colon \Lambda \to \Delta$ is generically $2:1$. Call $\tilde{\Lambda}_j$ the irreducible components of Λ for which $\tilde{\Lambda}_j \xrightarrow{2:1} \psi(\tilde{\Lambda}_j) = \tilde{\Delta}_j$ and Λ'_k, Λ''_k the components for which $\Lambda'_k \xrightarrow{1:1} \psi(\Lambda'_k) = \Delta_k$ and $\Lambda''_k \xrightarrow{1:1} \psi(\Lambda''_k) = \Delta_k$.

Given $(a,b) \in \Phi - (\xi(\mathcal{R}) \cup \{\infty\})$, we define in the following way the symmetric 13×13 matrix $\tilde{I}_j = (\tilde{i}_{rs})$ with entries in $\{0,1\}$, having 0's on the main diagonal. Let $\xi^*(a,b) = e_1 + \ldots + e_{13}$; for $1 \le r, s \le 13$, $r \ne s$, there is a unique $P_{rs} \in \mathbb{F}_{c_r}$ s.t. $\psi(P_{rs}) \in \psi(\mathbb{F}_{c_r}) \cap \psi(\mathbb{F}_{c_s})$. Then $\tilde{i}_{rs} = 1 \Leftrightarrow P_{rs} \in \tilde{\Lambda}_j$.

Moreover we define the skew-symmetric 13×13 matrix $I_k = (i_{rs})$ with entries in $\{0,1,-1\}$ by: $i_{rs} = 1 \Leftrightarrow P_{rs} \in \Lambda'_k$. Then (1)–(6) hold:

(1) $\Lambda'_k \cdot \mathbb{F}_X = \Lambda''_k \cdot \mathbb{F}_X$, because $\Lambda'_k \cdot \mathbb{F}_X$ equals the number of 1's in each row of I_k, $\Lambda''_k \cdot \mathbb{F}_X$ equals the number of -1's in each row and $\sum_{rs} i_{rs} = 0$.

(2) $\tilde{\Lambda}_j \cdot \mathbb{F}_X$ is even, because this number equals the number of entries $= 1$ in each row of $\tilde{I}_j$, the number of rows is odd and $\sum_{rs} \tilde{i}_{rs}$ is even.

If $(a,b) \in \xi(\mathcal{R})$, i.e. $\xi^*(a,b) = m\,\rho + e_1 + \ldots + e_{13-m}$ we may define the matrices $\tilde{I}_j$ and I_k as well, imposing the further condition that the first m rows, and hence the first m columns are equal. Call $\tilde{I}'_j, I'_k \in M_m(\mathbb{K})$, respectively $\tilde{I}''_j, I''_k \in M_{13-m}(\mathbb{K})$, the submatrices of $\tilde{I}_j$ and I_k formed by the first m rows and columns, the last $13-m$ rows and columns. Let

$$\tilde{I}_j = \begin{bmatrix} \tilde{I}'_j & {}^t\tilde{B}_j \\ \tilde{B}_j & \tilde{I}''_j \end{bmatrix}, \qquad I_k = \begin{bmatrix} I'_k & -{}^tB_k \\ B_k & I''_k \end{bmatrix}.$$

(3) $I'_k = 0$, because this matrix is both symmetric and skew-symmetric.

Let $\{\tilde{\Lambda}_j\}_{1\le j\le t}$ be the components $\tilde{\Lambda}_j$ meeting at least one singularity P^*_{ρ,e_i}, which means $\tilde{B}_j \neq 0$ — see figures 3, 4, 5. Then

(4) If $\tilde{l}_j$ is the class of $\tilde{\Lambda}_j$, then $\tilde{l}_j \cdot f \geq m$ for $1 \leq j \leq t$, because the singularities P^*_{ρ,e_i} have intersection m with $\mathbb{F}_X$.

(5) If l_k is the class of Λ'_k, then $\tilde{l}_j \cdot f = \tilde{l}_j \cdot e$ and $l_k \cdot f = l_k \cdot e$. This follows from (1) and the local analysis in case III. By (1), l_k is also the class of Λ''_k.

(6) $l_k \cdot f \geq m$, because by (3) all Λ'_k, Λ''_k must contain at least one P^*_{ρ,e_i}.

— If case II 1_7 occurs, there exists $(a,b) \in \xi(\mathcal{R})$ for which $m=7$. Since $l \cdot f = 12$, (1) and (6) imply that there are no Λ'_k, Λ''_k. Moreover $t=1$ by (4) and $\tilde{l}_1 \cdot e \in \{8, 10, 12\}$ by (2). By (5) $l_1 \in \{8\,e+88f,\ 10\,e+110f,\ 12\,e+132f\}$, and we may have (d1), (d2), (d3), (d4).

— If case II 1_6 occurs, there exists $(a,b) \in \xi(\mathcal{R})$ for which $m=6$. If there are components Λ'_k, Λ''_k, by (6) there are 2 of them of class $6\,e+66f$, so in each row of I_1 we must have six 1's and six -1's, excluded by (3). If there are only components $\tilde{\Lambda}_j$, by (4) $1 \le t \le 2$. If $t=1$ then we may have (d1), (d2), (d3), (d4), (d6), (d7). If $t=2$ then necessarily we would have (d5), but then the 7×6 matrix $\tilde{B}_1$ would have 1 row (suppose the first) formed by 0's and all the 6 other rows formed by 1's, so $\tilde{I}''_1$ would have the first row and column (apart from the main diagonal) formed by 1's, and the last rows of $\tilde{I}_1$ would have at least seven 1's, excluded.

— If for all $(a,b) \in \xi(\mathcal{R})$ only II 1_4, II 2, II 3, II 4 occur, always $m=4$ and as before we have the following cases. If there are components Λ'_k, Λ''_k then by (6) there are 2 of them of class $4\,e+44f$, and we may have (d8) or (d9). If there are only components $\tilde{\Lambda}_j$, then $1 \leq t \leq 3$ and we may have (d1), (d2) for $t=1$; (d5), (d6) for $t=2$; (d8) for $t=3$. ∎

If $\Lambda_\star$ is an irreducible component of Λ, let $\nu_\star\colon N_\star \to \Lambda_\star$ be its normalization, $g_\star = p_a(N_\star)$, $l_\star$ the class of $\Lambda_\star$ in $\operatorname{Pic}\mathbb{F}$, $d_\star = l_\star \cdot f$. The Riemann–Hurwitz

Theorem applied to $p \circ \nu_\star : N_\star \xrightarrow{d_\star : 1} \mathbb{P}^1$ gives $g_\star = -d_\star + 1 + \frac{1}{2} \deg R'_\star$, where $R'_\star = \sum_{Q \in N_\star} r'_Q \cdot Q$ is the ramification divisor in $N_\star$.

Call respectively δ^*_{ρ,e_i}, $\tilde{\delta}_{\rho,*}$, $\tilde{\delta}_\rho$ contribution to the arithmetic genus of $\Lambda_\star$ of the singularities $P^*_{\rho,i} \in \lambda^*_{\rho,e_i}$ ((24)), $\tilde{P}_{\rho,*} \in \tilde{\lambda}_{\rho,j}$ ((40)), $\tilde{P}_\rho \in \tilde{\lambda}_{\rho,j}$ ((43)), which are contained in $\Lambda_\star$. Remark that $\tilde{\delta}_{\rho,*}$ depends on the number $\tilde{b}_{\rho,*}$, $1 \le \tilde{b}_{\rho,*} \le 2$, of parametrizations $\tilde{\lambda}_{\rho,j}$ centered at $\tilde{P}_{\rho,*}$ which belong to $\Lambda_\star$.

Corollary 5.2. Λ *is irreducible if following conditions are satisfied.*

(52) $\delta^*_{\rho,e_i} \ge 9$ *in cases* 1_4, 1_6, 1_7 ,

(53) $\tilde{\delta}_{\rho,*} = 3$ *in case* 1_6, *if* $\tilde{b}_{\rho,*} = 1$,

(54) $\tilde{\delta}_{\rho,*} \ge 18$ *in case* 1_6, *if* $\tilde{b}_{\rho,*} = 2$,

(55) $\delta^*_{\rho,e_i} \ge 18$ *in case* 2 ,

(56) $\delta^*_{\rho,e_i} \ge 27$ *in case* 3 ,

(57) $\delta^*_{\rho,e_i} \ge 36$ *in case* 4 ,

Proof: We must exclude cases (d2)–(d9) in 5.1.

— **(d2)**, **(d3)**, **(d4).** Consider Λ_2^{11} in (d2) or (d4). If $(a,b) \in \xi(\mathcal{R})$, as $l_2 \cdot f = 2$, Λ_2^{11} does not contain any singularity P^*_{ρ,e_i} ($\tilde{I}''_2 = 0$). If case II 1_6 does not occur, it follows from the local analysis of Λ that $p \circ \nu_2$ has no ramification and is $2:1$, which contradicts Hurwitz's formula. Thus case II 1_6 must occur twice or four times, since $r'_{\nu^{-1}(P_{\rho,*})} = 1$, as we see from $t_j(\zeta)$ in (39). It can't occur four times, because of (53) and $p_a(\Lambda_2^{11}) = 11$. If it occurs twice, then $g_2 = p_a(N_2) = 0$ and Λ_2^{11} must have $11 - 2 \cdot 3 = 5$ nodes by (53), excluded because the number of nodes is divisible by 3. The same for Λ_2^{69} in (d3): if II 1_6 does not occur, we have 4 situations II 1_7, no ramification, excluded by Hurwitz. So II 1_6 must occur 4 times, $\deg R' = 8$, Λ_2^{69} must contain both $\tilde{\lambda}_{\rho,1}$, $\tilde{\lambda}_{\rho,2}$ at $\tilde{P}_{\rho,*}$ i.e. $\tilde{n}_{\rho,*} = 2$. But the 4 singularities $\tilde{P}_{\rho,*}$ give to $p_a(\Lambda_2^{69})$ by (54) a contribution $\ge 4 \cdot 18 = 72$, impossible.

— **(d5).** Necessarily $m = 4$, $t = 2$ and only II 1_4, II 2, II 3, II 4 may occur. Call $n_{1,1}$ the number of nodes of Λ_1^{175}, $n_{2,2}$ the number of nodes of Λ_2^{175}; From (52), (55), (56), (57) we have $g_1 + g_2 + n_{1,1} + n_{2,2} + 324 \le 350$. Let $n_{1,2} = (6\,e + 66f)\,(6\,e + 66f) = 432$ be the number of nodes of Λ generated by the intersections of Λ_1^{175} and Λ_2^{175}. We have 4 kinds of triple points of $\Delta = \Delta_1 + \Delta_2$, $\Delta_1 = \psi(\Lambda_1^{175})$, $\Delta_2 = \psi(\Lambda_2^{175})$: $(1,1,2)$ with 2 branches of Δ_1 and 1 of Δ_2, $(1,2,2)$ with 1 branch of Δ_1 and 2 of Δ_2, $(1,1,1)$ with 3 branches of Δ_1, $(2,2,2)$ with 3 branches of Δ_1; let a, b, c, d be respectively the number of such points. Then we must have $n_{1,1} + n_{1,2} = a + 3\,c + b + 3\,d \le 26$ and $n_{1,2} = 2\,a + 2\,b = 432$, impossible.

— **(d6).** If we have 4 situations $\mathrm{II}\,1_6$, as $t=1$ all the 28 points $P^*_{\rho,i}$, $\rho\in\mathcal{R}$, $i=1,...,7$ must belong to Λ_1^{175}; from (52) we see $\sum_{\rho,i}\delta^*_{\rho,e_i}\geq 63\cdot 4=252>175$, impossible. If we have at most 3 situations $\mathrm{II}\,1_6$, as $\mathrm{II}\,1_7$ can't occur, from (52)–(57) we see $\sum_{\rho,i}\delta^*_{\rho,e_i}\geq 63\cdot 3+81=270>175+69+11$, impossible.

— **(d7)**, **(d8)**, **(d9).** As $\mathrm{II}\,1_7$ can't occur, (52)–(57) imply $\sum_{\rho,i}\delta^*_{\rho,e_i}\geq 252$, but in these cases $\sum_j p_a(\Lambda_j)\leq 208$, impossible. ∎

In the next section we shall check (52)–(57), proving the irreducibility of Λ, which we assume from now. Let $\nu:\ N\to\Lambda$ be its normalization, $g=p_a(N)$. Consider $p\circ\nu$: $N\xrightarrow{12:1}\mathbb{P}^1$ and let $R'=\sum_{Q\in N}r'_Q\,Q$ the ramification divisor. Hurwitz's formula gives $g=-11+\frac{1}{2}\deg R'$. Let $R=\sum_{\rho\in\mathcal{R}}r_\rho\,\rho$ be the ramification divisor of ξ, so that $\mathcal{R}=\operatorname{Supp}R$ (4.1 (iii)). Let $\mathcal{R}'=\operatorname{Supp}R'$. It follows from the local analysis that $\mathcal{R}'\subset\bigcup_{\rho\in\mathcal{R}}\nu^{-1}(\Lambda_{\xi(\rho)})$. For $\rho\in\mathcal{R}$, let $r'_\rho=\sum_{Q\in\nu^{-1}(\Lambda_{\xi(\rho)})}r'_Q$, so that $\deg R'=\sum_{\rho\in\mathcal{R}}r'_\rho$.

Remark 5.3. *For $\rho\in\mathcal{R}$ the following hold*

$$r'_\rho=\begin{cases}9\,r_\rho & \textit{in cases } 1_4,\ 2,\ 3,\ 4\\ 7\,r_\rho+r'_{Q_1}+r'_{Q_2} & \textit{in case } 1_6\\ 6\,r_\rho & \textit{in case } 1_7\end{cases}$$

where $\{Q_1,Q_2\}=\nu^{-1}(\tilde{P}_{\rho,*})$. *Moreover* $r_\rho=\begin{cases}6 & \textit{in cases } 1_4,\ 1_6,\ 1_7\\ 12 & \textit{in case } 2\\ 18 & \textit{in case } 3\\ 24 & \textit{in case } 4\ .\end{cases}$

Proof: If $\lambda(t)=(a(t),b(t))$ parametrizes one branch of Λ at P, then $p\circ\nu$ is defined at the corresponding point of $\nu^{-1}(P)$ by $a(t)$. If $\Lambda_{\xi(\rho)}$ is of type 1_4, 2, 3, 4 then it follows from the local analysis and (22), (31), (46), (48), (50) that $\nu^{-1}(P^*_{\rho,i})$ consists of one point $Q^*_{\rho,i}$ and the differential of $a(t)$ vanishes only at $Q^*_{\rho,i}$, hence $R'|_{\Lambda_{\xi(\rho)}}=r'_{\rho,1}\,Q^*_{\rho,9}+\ldots+r'_{\rho,9}\,Q^*_{\rho,9}$. By (22) at $Q^*_{\rho,i}$, if we denote by $\xi|_c$ the power series defined by ξ at c we have $a(t)=t^*(t)=(\xi|_{e_i})^{-1}\circ(\xi|_\rho)$ and $\xi|_{e_i}$ is an isomorphism, so that $r'_{\rho,i}=r_\rho$ and hence $r'_\rho=9\,r_\rho$. The cases 1_6 and 1_7 are analogous. The second assertion follows from 4.1 (iii). ∎

We prove 1.1. The exact sequence $0\to\mathcal{O}_\Lambda\to\nu_*\,\mathcal{O}_N\to\mathcal{D}\to 0$ of ν gives $p_a(\Lambda)=h^1(\mathcal{O}_\Lambda)=h^0(\mathcal{D})+g$, hence $781=h^0(\mathcal{D})-11+\frac{1}{2}\deg R'$, so that

$$792\ =\ h^0(\mathcal{D})+\frac{1}{2}\sum_{\rho\in\mathcal{R}}r'_\rho\ . \tag{58}$$

In 5.3 we computed r'_ρ. The support of the sheaf $\mathcal{D}$ is the singular set $\operatorname{Sing}\Lambda$ of Λ, hence $h^0(\mathcal{D}) = \sum_{P\in \operatorname{Sing}\Lambda} \delta_P$, where $\delta_P = \dim_{\mathbb{K}}\mathcal{D}_P$.

As in 4.3, 4.5 let $n_{(1,1,1)}$, $n_{(2,1)}$, $n_{(3)}$ be the number of pairs of (-1)-curves of type (1,1,1), (2,1), (3). We proved the following facts about $\operatorname{Sing}\Lambda$:

(1) Λ_∞ contains no singular points, by 4.10;

(2) $\bigcup_{(a,b)\in\Phi-(\xi(\mathcal{R})\cup\{\infty\})} \Lambda_{(a,b)}$ has only nodes as singularities by 4.2. Each pair of (-1)-curves of type $(1,1,1)$ corresponds to 3 nodes and 1 triple point of Δ;

(3) If $(a,b)\in\xi(\mathcal{R})$, $\Lambda_{(a,b)}$ may contain nodes (4.4–4.9). The number of these nodes and those in $\Lambda_{\Phi-(\xi(\mathcal{R})\cup\{\infty\})}$ (4.2) is $3\,n_{(1,1,1)}$.

The other singularities in $\Lambda_{(a,b)}$ are:

(4) $P^*_{\rho,i}$; their number is 9 in cases 1_4, 2, 3, 4, is 7 in case 1_6, is 6 in case 1_7;

(5) the singularity $\tilde{P}_{\rho,*}$ in case 1_6. There are $n_{\rho,(3)}$, $0\le n_{\rho,(3)}\le 1$, pairs of (-1)-curves of type (3) associated to this singularity and

$$n_{\rho,(3)} = 1 \iff d_1 + d_2^2 + c_2^4 = 0 \;; \tag{59}$$

(6) the singularity $\tilde{P}_\rho$ in case 1_7. There are $n_{\rho,(3)}$, $0\le n_{\rho,(3)}\le 1$, pairs of (-1)-curves of type (3) associated to this singularity and

$$n_{\rho,(3)} = 1 \iff d_2^2 + c_2^4 = 0 \;; \tag{60}$$

(7) the $n_{\rho,(2,1)}$, $0\le n_{\rho,(2,1)}\le 3$, singularities $P_{\rho,i}=\tilde{P}_{\rho,j}$ in cases 1_4, 2, 3, 4:

$$P_{\rho,i}=\tilde{P}_{\rho,j} \iff \begin{cases} d_0 = c_1^2/\epsilon_j & \text{in case } 1_4 \quad \text{— see 4.3 (vi)} \\ d_0 = c_2^2/\omega_j & \text{in case } 2 \\ d_0 = c_3^2/\omega_j & \text{in case } 3 \\ d_0 = 1/\omega_j & \text{in case } 4 \;; \end{cases} \tag{61}$$

(8) the $n_{\rho,(2,1)}$, $0\le n_{\rho,(2,1)}\le 1$, singularity $P_{\rho,i}=\tilde{P}_\rho$ in case 1_6

$$P_{\rho,i}=\tilde{P}_\rho \iff d_0 = d_1^2 \;. \tag{62}$$

The proof of 1.1 amounts to show

$$n_{(1,1,1)} + n_{(2,1)} + n_{(3)} = 120 \;. \tag{63}$$

For $\rho\in\mathcal{R}$, let $\delta_\rho = \sum_{P\in \operatorname{Sing}\Lambda\cap\Lambda_{\xi(\rho)}}\delta_P$, so that $h_0(\mathcal{D}) = 3\,n_{(1,1,1)} + \sum_{\rho\in\mathcal{R}}\delta_\rho$.

As $n_{(2,1)} = \sum_{\rho\in\mathcal{R}} n_{\rho,(2,1)}$ and $n_{(3)} = \sum_{\rho\in\mathcal{R}} n_{\rho,(3)}$, by (58) we are left to show

$$\sum_{\rho\in\mathcal{R}} \left(\delta_\rho + \frac{1}{2}\, r'_\rho - 3\, n_{\rho,(2,1)} - 3\, n_{\rho,(3)}\right) \;=\; 792 - 360 \;=\; 432 \;.$$

Corollary 5.4. *By 4.1 (iii), to prove Theorem 1.1 it suffices check (52)–(57) and show that for every $\rho\in\mathcal{R}$ we have*

$$\delta_\rho + \frac{1}{2}\, r'_\rho - 3\, n_{\rho,(2,1)} - 3\, n_{\rho,(3)} = \begin{cases} 108 & \text{in cases } 1_4,\ 1_6,\ 1_7 \\ 216 & \text{in case } 2 \\ 324 & \text{in case } 3 \\ 432 & \text{in case } 4 \end{cases} \tag{64}$$

where r'_ρ is computed in 5.3. ∎

6 – The computations

In this section we check (52)–(57), proving the irreducibility of Λ, and (64), finishing the proof of 1.1. As in 5.2, we set $\delta^*_{\rho,e_i} = \delta_{P^*_{\rho,i}}$, $\tilde\delta_{\rho,*} = \delta_{\tilde P_{\rho,*}}$, $\tilde\delta_\rho = \delta_{\tilde P_\rho}$. The meaning of 5.4 is that although the number of different global configurations is high, we are left to consider only 6 cases, because in some sense each singularity of the ramification divisor A^* of ϕ_2 (see 4.1, (iv)) gives an independent contribution to the number of (-1)-curves on the surface S.

Case 1_4. We have $r'_\rho = 54$ by 5.3, $n_{(3)} = 0$. Then (64) is $\delta_\rho - 3\, n_{\rho,(2,1)} = 81$.

Remark 6.1. *To satisfy the conditions of 5.4 in case 1_4 it suffices to prove*

$$\textbf{1)}\ \delta^*_{0,e_i} = \begin{cases} 9 & \text{if } d_0 \neq c_1^2/\epsilon_j \\ 10 & \text{if } d_0 = c_1^2/\epsilon_j\ , \end{cases} \qquad \textbf{2)}\ \text{if } d_0 = c_1^2/\epsilon_j \text{ then } \delta_{P_{0,i}} = 2\ , \tag{65}$$

where by (25) we may assume $\rho = 0$, $e_i = 1$; $d_0 = c_1^2/\epsilon_j$ is the condition that $P_{0,i} = \tilde P_{0,j}$ ((61)); ϵ_j are the roots of (30).

Proof: If $P_{0,i} \neq \tilde P_{0,j}$ then $n_{0,(2,1)} = 0$, $\delta_0 = \sum_{i=1}^9 \delta^*_{0,e_i} = 81$ and (64) holds. Each time that $P_{0,i} = \tilde P_{0,j}$, δ_0 grows by $\delta_{P_{0,i}} + 10 - 9 = 3$ and $n_{0,(2,1)}$ grows by 1 hence (64) holds in any case. Moreover (52) is satisfied. ∎

Proof of (65) 1): The parametrization $\lambda^*_{0,e_i}(t)$ centered at P^*_{0,e_i} is given in (22), where $t^*(t) \in \mathbb{K}[[t]]$ is the solution of $\xi(t^*(t)) = \xi(t)$, $t^*(0) = 1$ and as in

(26), $\xi(t) = \sum_{4\le i\le 11} a_i\, t^i + o_{12},\ o_{12} \in (t^{12})$. Solving recursively, we get

$$t^*(t) \;=\; 1 + \Big(g_2(1)/g(1)^2\Big) \sum_{4\le i\le 11} a_i^*\, t^i + o_{12}(t) \tag{66}$$

where $a_i^* = a_i$ for $i \neq 8$ and $a_8^* = a_8 + a_4^2\,(1 + c_1^2 + c_3 + d_1^2 + d_3^2)$. Then by (23)

$$u_2^*(t) \;=\; g_2(1)\left[1 + t^4 + (1{+}a_4)\, t^6 + \Big(1 + g_2(1)\, a_4^2 + a_6\Big)\, t^8 + a_7\, t^9\right] + o_{10}(t)\ .$$

Denote by $\sim$ equality up to an affine coordinate change. We have

(0) $\lambda^*_{0,e_i}{=}(t^*(t), u_2^*(t)){\sim}(p_0, q_0)$ where $p_0 = a_4t^4{+}a_6t^6{+}a_7t^7{+}a_8^*t^8{+}a_9t^9{+}o_{10}$, $q_0 = t^4 + [1 + a_4]\, t^6 + [1 + g_2(1)\, a_4^2 + a_6]\, t^8 + a_7t^9 + o_{10}$.

The initial contribution to δ^*_{0,e_i} is $\delta_0 = 4 \cdot 3/2 = 6$.

(1) After the blow up $(p_0/q_0, q_0) \sim (p_1, q_1)$ we get $q_1 = q_0$ and $p_1 = Ct^2 + a_7t^3 + [C + a_4 + a_8^* + Ca_4 + Cg_2(1) + a_4g_2(1) + a_6g_2(1) + a_4a_6 + a_4a_6g_2(1) + a_4g_2(1)\, C]\, t^4 + [a_7 + a_9]\, t^5 + o_6$, where $C = a_4 + a_4^2 + a^6$.

— Suppose $C = 0$; $a_7 \neq 0$ implies $\delta_1 = 3$ and $(p_1, q_1) \sim (t^3 + o_4,\ t^4 + o_5)$;

(2) blowing up $p_2 = p_1$, $q_2 = q_1/p_1$ we get $q_2 = t + o_2$ hence the resolution stops and $\delta^*_{0,e_i} = 6 + 3 = 9$.

— Suppose $C \neq 0$; then $\delta_1 = 2 \cdot 1/2 = 1$ and we have the following sequence

(2) $(p_1, q_1/p_1) \sim (p_2, q_2)$ where

$$\begin{aligned} p_2 \;&=\; C^{-1}p_1 \;=\; t^2 + C^{-1}a_7t^3 + C^{-1}[\ldots]\, t^4 + C^{-1}[a_7 + a_9]\, t^5 + o_6\ ,\\ q_2 \;&=\; t^2 + C^{-1}a_7t^3 + C^{-1}[a_4 + g_2(1)\, C + a_8^* + a_4g_2(1)\, C + a_6g_2(1) + a_4g_2(1)\\ &\qquad + a_4a_6 + a_7^2\, C^{-1} + a_4a_6g_2(1)]\, t^4 + C^{-1}[a_9 + a_4a_7 + a_7^3\, C^{-2}]\, t^5 + o_6\ ; \end{aligned}$$

then $\delta_2 = 1$.

(3) $(p_2, q_2/p_2) \sim (p_3, q_3)$ where $p_3 = p_2$ and $q_3 = t^2[1 + a_4 + a_7^2\, C^{-2}] + o_3$.

— Suppose $1 + a_4 + a_7^2\, C^{-2} \neq 0$; then $\delta_3 = 1$;

(4) $(p_3/q_3, q_3) \sim (p_4, q_4)$ where $p_4 = t{+}o_2$. The resolution ends and $\delta^*_{0,e_i} = 9$.

— Suppose $1 + a_4 + a_7^2C^{-2} = 0$. We develop p_0, q_0 up to o_{12}:

$$\begin{aligned} p_0 &= a_4t^4 + a_6t^6 + a_7t^7 + a_8^*t^8 + a_9t^9 + a_{10}t^{10} + a_{11}t^{11} + o_{12}\\ q_0 \;&=\; t^4 + [1 + a_4]\, t^6 + [1 + g_2(1)\, a_4^2 + a_6]\, t^8 + a_7t^9 + [1 + a_8^* + a_4^2c_1 + a_4^2d_3^2\\ &\qquad + a_4^2d_1^2 + a_4^2c_1^2]\, t^{10}a_9\, t^{11} + o_{12}. \end{aligned}$$

After the same sequence as before we get (4) $(p_3, q_3/p_3) \sim (p_4, q_4)$ where

$$\begin{aligned} p_4 = t^2 &+ C^{-1}a_7t^3 + C^{-1}[C + g_2(1)\, C + a_4C + a_8^* + a_4 + a_4g_2(1)\, C + a_6g_2(1)\\ &+ a_4g_2(1) + a_4a_6 + a_4a_6g_2(1)]\, t^4 + C^{-1}[a_7 + a_9]\, t^5 + C^{-1}[C^2 + a_6C + a_{10}\\ &+ a_8^* + a_6g_2(1)\, C + a_4a_8^* + a_6^2 + a_4a_6 + a_6^2g_2(1) + a_4a_6g_2(1)]\, t^6 + C^{-1}[a_7C\\ &+ a_{11} + a_9 + a_6a_7 + a_4a_7 + a_6a_7g_2(1) + a_4a_7g_2(1) + a_7g_2(1)]\, t^7 + o_8\ , \end{aligned}$$

$$\begin{aligned}q_4 = {} & C^{-2}[C^1 + C^3 + a_4C^2 + g_2(1)\,C^3 + g_2(1)^2\,C^2 + a_6g_2(1)\,C^2 + a_4g_2(1)\,C^2 \\ & + a_6 + g_2(1)^2\,C + a_6^2\,C + g_2(1)^2\,C^3 + a_6g_2(1)^2\,C^2 + a_4g_2(1)^2\,C^2 + a_8^{*2} \\ & + a_6g_2(1) + a_4g_2(1)^2\,a_6^3 + a_4a_6^2 + a_4 + a_6^2g_2(1)^2\,C + a_6^2g_2(1)^2 + a_6^3g_2(1)^2 \\ & + a_4a_6^2g_2(1)^2]\,t^2 + C^{-3}[a_7g_2(1)\,C^3 + a_7\,C + a_4a_7\,C^2 + a_7g_2(1)^2\,C^3 \\ & + a_7g_2(1)^2\,C^2 + a_6a_7g_2(1)\,C^2 + a_4a_7g_2(1)\,C^2 + a_6a_7 + a_4a_7 + a_7g_2(1)^2\,C \\ & + a_6^2a_7\,C + a_6a_7g_2(1)^2\,C^2 + a_4a_7g_2(1)^2\,C^2 + a_7a_8^{*2} + a_6a_7g_2(1)^2 + a_4a_7g_2(1)^2 \\ & + a_6^3a_7 + a_4a_6^2a_7 + a_6^2a_7g_2(1)^2\,C + a_6^2a_7g_2(1)^2 + a_6^3a_7g_2(1)^2 + a_4a_6^2a_7g_2(1)^2]\,t^3 \\ & + o_4\ .\end{aligned}$$

It follows $\delta_4 = 1$.

(5) $(p_4, q_4/p_4) \sim (p_5, q_5)$ with $q_5 = a_7t + o_2$. The resolution ends, $\delta^*_{0,e_i} = 10$.

By (27), $\delta^*_{0,e_i} = 10 \iff C = a_4 + a_4^2 + a_6 \neq 0$ and $a_7^2 = (1 + a_4)C^2 \iff d_0 + d_0^2 + d_1^2 + c_1^4 = 0 \iff c_1^2/d_0$ is a root of (30). ∎

Proof of (65) 2): By (31), $\tilde\lambda_{0,j}(t) = (t, \tilde u_{2,j}(t))$. $\zeta_j = \epsilon_j\,t + \phi_j\,t^2 + o_3$, where $a_4\epsilon_j^3 + a_6\epsilon_j + a_7 = 0$; to compute ϕ_j, substituting ζ_j in (29), in degree 4 we get $a_4\epsilon_j^2\phi_j + a_6\phi_j + a_7\epsilon_j = 0$. It follows $\tilde u_{2,j}(t) = c_1^4/\epsilon_j^2 + [d_0^2 + c_1^4\phi_j^2/\epsilon_j^4]\,t^2 + o_3$. By (22), (21), (66) $\lambda_{0,e_i}(t) = (t, u_{2,i}(t))$, $u_{2,i}(t) = d_0^2 + d_1^2t^2 + o_3$. As $d_0 = c_1^2/\epsilon_j$ i.e. $\tilde u_{2,j}(0) = u_{2,i}(0)$, we have $\delta_{P_{0,i}} = 2 \Leftrightarrow d_0^2 + c_1^4\phi_j^2/\epsilon_j^4 \neq d_1^2$. But $d_0 + c_1^2\phi_j/\epsilon_j^2 = d_1$ $\Rightarrow d_0 + c_1^2 + d_1 = 0 \Rightarrow a_4 + a_4^2 + a_6 = C = 0$, excluded. ∎

Case 1_6. By (5.3), (64) becomes $\delta_\rho + \frac{1}{2}(r'_{Q_1} + r'_{Q_2}) - 3\,n_{\rho,(2,1)} - 3\,n_{\rho,(3)} = 87$.

Remark 6.2. *To satisfy the conditions of 5.4 in case 1_6 it suffices to prove*

1) $\delta^*_{0,e_i} = \begin{cases} 9 & \text{if } d_0 \neq d_1^2 \\ 10 & \text{if } d_0 = d_1^2\ , \end{cases}$ **2**) *if* $d_0 = d_1^2$ *then* $\delta_{P_{0,i}} = 2$, **3**) $r'_{Q_1} = r'_{Q_2} = 2$,

(67) **4**) *if* $\tilde b_{0,*} = 1$ *then* $\tilde\delta_{0,*} = 3$, **5**) $\tilde\delta_{0,*} = \begin{cases} 22 & \text{if } d_1 + d_2^2 + c_2^4 \neq 0 \\ 25 & \text{if } d_1 + d_2^2 + c_2^4 = 0\ , \end{cases}$

where $\rho = 0$, $e_i = 1$ ((25)); $d_0 = d_1^2$ is the condition that $P_{0,i} = \tilde P_{0,*}$ ((62)).

Proof: (52), (53), (54) follow from 1), 4), 5). It remains to check $\delta_0 - 3\,n_{0,(2,1)} - 3\,n_{0,(3)} = 85$. If $n_{0,(2,1)} = n_{0,(3)} = 0$ then $9 \cdot 7 + 22 = 85$. If $d_0 = d_1^2$, δ_0 increases by 3 and $n_{0,(2,1)}$ increases by 1. If $d_1 + d_2^2 + c_2^4 = 0$, δ_0 increases by 3 and $n_{0,(3)}$ increases by 1. ∎

Case 1_7. By 4.6, 5.3 $n_{\rho,(2,1)} = 0$ and (64) becomes $\delta_\rho - 3\, n_{\rho,(3)} = 90$.

Remark 6.3. *To satisfy the conditions of 5.4 in case 1_7 it suffices to prove*

$$(68) \qquad \mathbf{1})\ \delta^*_{0,e_i} = 9\ , \qquad \mathbf{2})\ \tilde{\delta}_0 = \begin{cases} 36 & \text{if } d_2 + c_2^2 \neq 0 \\ 39 & \text{if } d_2 + c_2^2 = 0\ , \end{cases}$$

where $\rho = 0$, $e_i = 1$ ((25)).

Proof: (52) follows from 1). If $d_2 + c_2^2 \neq 0$ then $n_{0,(3)} = 0$, if $d_2 + c_2^2 = 0$ then $n_{0,(3)} = 1$ and $\delta_0 - 3\, n_{0,(3)} = 90$ follows in both cases. ∎

Case 2. $r'_\rho = 108$ by 5.3, $n_{(3)} = 0$; (64) becomes $\delta_\rho - 3\, n_{\rho,(2,1)} = 162$.

Remark 6.4. *To satisfy the conditions of 5.4 in case 2 it suffices to prove*

$$(69) \qquad \mathbf{1})\ \delta^*_{0,e_i} = \begin{cases} 18 & \text{if } d_0 \neq c_2^2/\omega_j \\ 19 & \text{if } d_0 = c_2^2/\omega_j\ , \end{cases} \qquad \mathbf{2})\ \text{if } d_0 = c_2^2/\omega_j \text{ then } \delta_{P_{0,i}} = 2\ ,$$

where $\rho = 0$, $e_i = 1$ ((25)); $d_0 = c_2^2/\omega_j$ is the condition that $P_{0,i} = \tilde{P}_{0,j}$ ((61)); ω_j are the roots of (45).

Proof: If $P_{0,i} \neq \tilde{P}_{0,j}$, $n_{0,(2,1)} = 0$, $\delta_0 = \sum_{i=1}^{9} \delta^*_{0,e_i} = 162$ and (64) holds. If $P_{0,i} = \tilde{P}_{0,j}$, δ_0 grows by $\delta_{P_{0,i}} + 19 - 18 = 3$ and $n_{0,(2,1)}$ grows by 1; (55) is satisfied. ∎

Case 3. $r'_\rho = 162$ by 5.3, $n_{(3)} = 0$; (64) becomes $\delta_\rho - 3\, n_{\rho,(2,1)} = 243$.

Remark 6.5. *To satisfy the conditions of 5.4 in case 3 it suffices to prove*

$$(70) \qquad \mathbf{1})\ \delta^*_{0,e_i} = \begin{cases} 27 & \text{if } d_0 \neq c_3^2/\omega_j \\ 28 & \text{if } d_0 = c_3^2/\omega_j\ , \end{cases} \qquad \mathbf{2})\ \text{if } d_0 = c_3^2/\omega_j \text{ then } \delta_{P_{0,i}} = 2\ ,$$

where $\rho = 0$, $e_i = 1$ ((25)); $d_0 = c_3^2/\omega_j$ is the condition that $P_{0,i} = \tilde{P}_{0,j}$ ((61)); ω_j are the roots of (47). ∎

Case 4. $r'_\rho = 216$ by 5.3, $n_{(3)} = 0$; (64) becomes $\delta_\rho - 3\, n_{\rho,(2,1)} = 324$.

Remark 6.6. *To satisfy the conditions of 5.4 in case 4 it suffices to prove*

$$(71) \qquad \mathbf{1})\ \delta^*_{0,e_i} = \begin{cases} 36 & \text{if } d_0 \neq 1/\omega_j \\ 37 & \text{if } d_0 = 1/\omega_j\ , \end{cases} \qquad \mathbf{2})\ \text{if } d_0 = 1/\omega_j \text{ then } \delta_{P_{0,i}} = 2\ ,$$

where $\rho = 0$, $e_i = 1$ *((25));* $d_0 = 1/\omega_j$ *is the condition that* $P_{0,i} = \tilde{P}_{0,j}$ *((61));* ω_j *are the roots of (49).* ∎

6.5 and 6.6 are proved as 6.4. The checks of (67), (68), (69), (70), (71) are similar to that of (65) and were done using CoCoA symbolic package on a Sun machine. The proof of Theorem 1.1 is complete. ∎

To finish, table 1 shows the values of $n_{\rho,(2,1)}$, $n_{\rho,(3)}$ and n_ρ, which is the multiplicity of $\rho \in \mathcal{R}$ as root of g and the δ of the singularity $\phi_2^{-1}(\rho)$ of A^* (4.1 (iii), (iv)). The possible values of $n_{(1,1,1)}$, $n_{(2,1)}$, $n_{(3)}$ are defined by

$$(72) \qquad \begin{cases} n_{(2,1)} = \sum\limits_{\rho\in\mathcal{R}} n_{\rho,(2,1)} \ , \\ n_{(3)} = \sum\limits_{\rho\in\mathcal{R}} n_{\rho,(3)} \ , \\ n_{(1,1,1)} = 120 - n_{(2,1)} - n_{(3)} \ , \\ \sum\limits_{\rho\in\mathcal{R}} n_\rho = 4 \ . \end{cases}$$

	$\mathbf{1_4}$	$\mathbf{1_6}$	$\mathbf{1_7}$	**2**	**3**	**4**
$n_{\rho,(2,1)}$	$0,1,2,3$	$0,1$	0	$0,1,2,3$	$0,1,2,3$	$0,1,2,3$
$n_{\rho,(3)}$	0	$0,1$	$0,1$	0	0	0
$n_\rho = \delta_{\phi_2^{-1}(\rho)}$	1	1	1	2	3	4

Table 1 : Possible values of $n_{\rho,(2,1)}$, $n_{\rho,(3)}$ for $\rho \in \mathcal{R} = \phi_2(\mathrm{Sing}\,(A^*))$.

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