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# OPTIMAL CONTROL PROBLEMS WITH WEAKLY CONVERGING INPUT OPERATORS IN A NONREFLEXIVE FRAMEWORK

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**Abstract:** The variational convergence of sequences of optimal control problems with state constraints (namely inclusions or equations) with weakly converging input multi-valued operators is studied in a nonreflexive abstract framework, using Γ-convergence techniques. This allows to treat a lot of situations where a lack of coercivity forces to enlarge the space of states where the limit problem has to be imbedded. Some concrete applications to optimal control problems with measures as controls are given either in a nonlinear multi-valued or nonlocal but single-valued framework.

## 1 – Introduction

This paper deals with sequences of optimal control problems of the form

$$\min\Big\{J_h(u,y)\colon\ A_h(y)\cap B_h(u)\neq\emptyset,\ (u,y)\in U{\times}Y\Big\}\,,\qquad h\in\mathbb{N}\,, \tag{1.1}$$

where the space of controls $U$ and the space of states $Y$ are topological spaces, $J_h\colon U{\times}Y\to(-\infty,+\infty]$ are the cost functionals and the operators $A_h$ and $B_h$ are multi-mappings defined on $Y$ and $U$ respectively and taking values into another topological space $V$, that is

$$A_h\colon Y\to\wp(V)\,,\qquad B_h\colon U\to\wp(V)\,,$$

where $\wp(V)$ denotes the set of all subsets of $V$. If $A_h$ or $B_h$ are single valued then the state constraints in problems (1.1) degenerate to inclusions like

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$A_h(y) \in B_h(u)$, $B_h(u) \in A_h(y)$, or equations. A lot of particular cases have been recently widely studied, from the point of view of variational convergence, by many authors with different techniques (see for instance, [1], [2], [5], [7], [8], [9], [10], [11], [13], [14], [16], [17], [18], [19], [20], [21], [22], [23], [26]). They consist in the identification of a limit problem in the sense of the definition below.

**Definition 1.1.** An optimal control problem

$$(\mathcal{P}_\infty) \qquad \min\Big\{J(u,y)\colon\ A(y)\cap B(u)\neq\emptyset,\ (u,y)\in U\times Y\Big\}$$

is said to be a limit of the sequence (1.1) if it enjoys the following property:

if $(u_h, y_h)$ is an optimal pair for problem (1.1) or, more generally, a sequence such that $A_h(y_h)\cap B_h(u_h)\neq\emptyset$, and there exists the limit

$$\lim_{h\to\infty} J_h(u_h,y_h) = \lim_{h\to\infty} \min\Big\{J_h(u,y)\colon\ A_h(y)\cap B_h(u)\neq\emptyset,\ (u,y)\in U\times Y\Big\}\ ,$$

and if $(u_h, y_h)\to(u,y)$ in $U\times Y$, then $(u,y)$ is an optimal pair for $(\mathcal{P}_\infty)$. □

Sequence (1.1) is equivalent to the following one

$$\min\Big\{J_h(u,y)+\chi_{A_h(y)\cap B_h(u)\neq\emptyset}\colon\ (u,y)\in U\times Y\Big\}$$

where $\chi$ denotes the indicator function taking the value 0 if the subscript condition is satisfied and $+\infty$ otherwise. In this way, the variational convergence problem is leaded to the identification of the Γ-limit of the functionals

$$F_h(u,y) = J_h(u,y)+\chi_{A_h(y)\cap B_h(u)\neq\emptyset}\ . \tag{1.2}$$

Following a fruitful method introduced by Buttazzo in [6] for a single problem (relaxation setting) and extended later to sequences by Buttazzo and Cavazzuti in [7], which consists in introducing an auxiliary variable, and providing that suitable compactness conditions be satisfied (see Section 2) such problem can be splitted into the sub-problems of the identification of the $G$-limit of the inclusions $v\in A_h(y)$ and the calculation of a Γ-limit of the functionals

$$G_h(u,v,y) = J_h(u,y)+\chi_{v\in B_h(u)}\ .$$

The subsequent sections are devoted to the latter. Under a strong enough convergence assumption on the input operators $B_h$, namely the sequential Kuratowski continuous convergence (Section 3), which reduces to the usual continuous convergence in the single-valued case, the limit problem takes the same form of the approximating ones. On the contrary, when such strong assumption is dropped, the limit problem takes a different form. In Section 4 a fundamental duality result is proved and subsequently applied in Section 5 to find the variational limit in the abstract case. In order to provide a concrete application of such abstract framework, Section 6 is devoted to explain the functional tool which will be used in the sequel and to state some technical lemmata. An application to the case of local multi-valued operators between $L^p$ spaces ($p = 1$ included) is the subject of Section 7. An anticipation of the results in that section, but without proofs and in the single-valued case only, appeared in [16]. Section 8 is devoted to the linear, but possibly non-local case and several examples and applications are given. Unfortunately, in the linear case the abstract framework doesn't apply to multi-valued input operators. Hence in Section 8 we are constrained to consider only single-valued operators. The notation of Γ-limits is extensively used but not recalled here. The reader could refer for a general treatment of Γ-convergence to the book of dal Maso [12] and for the application to optimal control problems to [11]. Let us point out moreover that all the Γ-limits used in the paper are of sequential kind.

## 2 – Γ-convergence and $G$-convergence

A first step in the calculation of the Γ-limit of the functionals (1.2) is provided by the following theorem.

**Theorem 2.1** (Buttazzo and Cavazzuti [7], Proposition 2.3)**.** *Let $F_h\colon U\times Y\to \overline{\mathbb{R}}$ be a sequence of functions, and let $\Xi_h\colon U\times Y\to \wp(V)$ be a sequence of multi-mappings. Assume that for every converging sequence $(u_h, y_h)$ with $F_h(u_h, y_h)$ bounded, there exists a sequence $v_h \in \Xi_h(u_h, y_h)$ relatively compact in $V$. If for every $(u, v, y) \in U\times V\times Y$ there exists the Γ-limit*

$$\Gamma\Big(\mathbb{N},\, (U\times V)^-,\, Y^-\Big) \lim_{h\to\infty}\Big[F_h(u, y) + \chi_{\,v\,\in\,\Xi_h(u,y)}\Big]\ ,$$

*then there exists also the Γ-limit $\Gamma(\mathbb{N}, U^-, Y^-) \lim\limits_{h\to\infty} F_h(u, y)$ and coincides with*

$$\inf\Big\{\Gamma\Big(\mathbb{N}, (U\times V)^-, Y^-\Big) \lim_{h\to\infty}\Big[F_h(u, y) + \chi_{\,v\,\in\,\Xi_h(u,y)}\Big] : \ v \in V\Big\}\ . \ \blacksquare$$

Let us set $\Xi_h(u, y) = A_h(y) \cap B_h(u)$, and choose the space $V$, which is not a priori given, in order to satisfy the following compactness condition

(2.1) for every converging sequence $(u_h, y_h)$ such that $A_h(y_h) \cap B_h(u_h) \neq \emptyset$ for every $h \in \mathbb{N}$ and $J_h(u_h, y_h)$ is bounded, there exists a sequence $v_h \in A_h(y_h) \cap B_h(u_h)$ relatively compact in $V$.

As $\chi_{v \in A_h(y) \cap B_h(u)} = \chi_{v \in A_h(y)} + \chi_{v \in B_h(u)}$, by applying the theorem we get

$$\Gamma(\mathbb{N}, U^-, Y^-) \lim_{h \to \infty} F_h(u, y) =$$
$$= \inf_{v \in V} \left\{ \Gamma\Big(\mathbb{N}, (U \times V)^-, Y^-\Big) \lim_{h \to \infty} \Big[ J_h(u, y) + \chi_{v \in A_h(y)} + \chi_{v \in B_h(u)} \Big] \right\} .$$

This fact leads to the very useful possibility of calculate separately the Γ-limits of the two sequences of functionals

$$G_h(u, v, y) = J_h(u, y) + \chi_{v \in B_h(u)} \quad \text{and} \quad \chi_{v \in A_h(y)} \tag{2.2}$$

as the following theorem states. To prove it, is enough to use Corollary 2.1 of Buttazzo and Dal Maso [8] concerning the Γ-limits of sums and to put together with Theorem 2.1 and Corollary 7.17 of Dal Maso [12].

**Theorem 2.2.** *Assume that there exist a multi-mapping $A\colon Y \to \wp(V)$ and a functional $G\colon U \times V \times Y \to \overline{\mathbb{R}}$ such that there exist the following Γ-limits:*

$$\Gamma(\mathbb{N}, V, Y^-) \lim_{h \to \infty} \chi_{v \in A_h(y)} = \chi_{v \in A(y)} , \tag{2.3}$$

$$\Gamma(\mathbb{N}, U \times V^-, Y) \lim_{h \to \infty} G_h(u, v, y) = G(u, v, y) . \tag{2.4}$$

*If the compactness condition (2.1) is satisfied then*

$$\Gamma(\mathbb{N}, U^-, Y^-) \lim_{h \to \infty} F_h(u, y) = \inf\Big\{ G(u, v, y) + \chi_{v \in A(y)} \colon v \in V \Big\}$$

*and a limit problem in the sense of Definition 1.1 is given by*

$$\min\Big\{ \inf_{v \in A(y)} G(u, v, y) \colon (u, y) \in U \times Y \Big\} . \ \blacksquare \tag{2.5}$$

**Definition 2.3.** When condition (2.3) is satisfied we say that the sequence $A_h$ $G$-converges to $A$. □

**Remark 2.4.** Definition 2.3 agrees with the fact that if the operators $A_h$ are single-valued, linear and uniformly elliptic from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ ($\Omega$ bounded open subset of $\mathbb{R}^n$) respectively endowed with the weak and the norm topology, then this definition of $G$-convergence is equivalent to the classical one of Spagnolo [24] (see Buttazzo and Dal Maso [8], Lemma 3.2). It is equivalent to the following two conditions:

**(i)** if $y_h \to y$ in $Y$, $v_h \to v$ in $V$ and $v_h \in A_h(y_h)$ for infinitely many $h \in \mathbb{N}$, then $v \in A(y)$;

**(ii)** if $y \in Y$, $v \in V$ are such that $v \in A(y)$ and $v_h \to v$ in $V$, then there exists $y_h \to y$ in $Y$ such that $v_h \in A_h(y_h)$ for every $h \in \mathbb{N}$ large enough. □

## 3 – Continuously converging operators

Accordingly to the topological definition of Kuratowski convergence of sets and to Proposition 4.15 and Remark 8.2 of Dal Maso [12], let us give the following definition.

**Definition 3.1.** Let $X$ be a topological space and let $(E_h)$ be a sequence of subset of $X$. We say that $(E_h)$ sequentially Kuratowski converges to $E$ if and only if

$$\Gamma(\mathbb{N}, X^-) \lim_{h\to\infty} \chi_{E_h} = \chi_E$$

that is the following two conditions are satisfied:

**(i)** if $x_h \to x$ and $x_h \in E_h$ for infinitely many $h \in \mathbb{N}$ then $x \in E$;

**(ii)** if $x \in E$ then there exists a sequence $x_h \to x$ such that $x_h \in E_h$ for every $h \in \mathbb{N}$ large enough. In this case we use to write $E_h \overset{K_{\rm seq}}{\longrightarrow} E$. □

Coming back to sequences of optimal control problems, the simplest case arises when the input multi-valued operators $B_h$ are sequentially Kuratowski continuously converging to $B$, that is if $u_h \to u$ in $U$ implies $B_h(u_h) \overset{K_{\rm seq}}{\longrightarrow} B(u)$. By using the definition of sequential $\Gamma$-convergence it is immediately seen that

$$(3.1)\qquad \begin{gathered} B_h \overset{K_{\rm seq}}{\longrightarrow} B \text{ continuously} \\ \Longleftrightarrow \\ \Gamma(\mathbb{N}, U, V^-) \lim_{h\to\infty} \chi_{v \in B_h(u)} = \chi_{v\in B(u)} . \end{gathered}$$

In order to characterize condition (2.4), we require some kind of uniform continuity about the cost functionals which is precisely stated in the following theorem.

**Theorem 3.2.** *Assume that the sequence $(A_h)$ G-converges to $A$, that $(B_h)$ sequentially Kuratowski continuously converges to $B$, and that there exist $\Psi : U \to \mathbb{R}$ bounded on the $U$-bounded sets and $\omega : Y\times Y \to \mathbb{R}$ with* $\lim_{z\to y} \omega(y,z) = 0$

*for every* $y \in Y$ *such that* $J_h(u,y) \le J_h(u,z) + \Psi(u)\,\omega(y,z)$ *for every* $u \in U$, $y,z \in Y$ *and* $h \in \mathbb{N}$. *If for every* $y \in Y$ *there exists the* $\Gamma$*-limit* $J(u,y) := \Gamma(\mathbb{N},U^-)\lim_{h\to\infty} J_h(u,y)$ *then, for every* $u \in U$, $v \in V$ *and* $y \in Y$, *there exists also the* $\Gamma$*-limit*

$$\Gamma(\mathbb{N},U\times V^-,Y)\lim_{h\to\infty} G_h(u,v,y) \;=\; J(u,y) + \chi_{\,v\,\in\, B(u)}$$

*and if the compactness condition (2.1) is satisfied then a limit problem in the sense of Definition 1.1 is given by*

$$\min\Big\{J(u,y)\colon\ A(y)\cap B(u)\neq\emptyset,\ (u,y)\in U\times Y\Big\}\ .$$

**Proof:** The proof is a straightforward application of Theorem 2.2. The $\Gamma$-limit (2.4) can be calculated by using Corollary 2.1 of Buttazzo and Dal Maso [8], (3.1) and observing that, by the continuity assumption on the costs, it is $\Gamma(\mathbb{N}^-,U^-,Y)\lim_{h\to\infty} J_h(u,y) = \Gamma(\mathbb{N}^-,U^-)\lim_{h\to\infty} J_h(u,y)$. ∎

**Remark 3.3.** If the input operators are single-valued then the sequential Kuratowski continuous convergence reduces to the pointwise continuous convergence and the result above to the one stated in [11], Theorem 3.6. □

## 4 – Preliminary duality result

This section is devoted to state an abstract theorem which reduces the calculation of the $\Gamma$-limit of a sequence of functionals whose lower semicontinuous envelope is convex to the computation of the pointwise limit of the Fenchel duality transforms.

Let $X$ be a separable Banach space and $X^*$ denotes the topological dual space of $X$. Here and in the sequel the space $X$ will be endowed always with the norm topology and the dual $X^*$ with the weak* topology. Let $G_h\colon X^*\to(-\infty,+\infty]$ be a sequence of proper functionals (i.e. $G_h \not\equiv +\infty$). The dual functionals $G_h^*\colon X\to(-\infty,+\infty]$ defined by $G_h^*(x)=\sup\{\langle x^*,x\rangle - G_h(x^*)\colon\ x^*\in X^*\}$ are proper, convex and strongly lower semicontinuous while $G_h^{**}\colon X^*\to(-\infty,+\infty]$ defined by $G_h^{**}(x)=\sup\{\langle x^*,x\rangle - G_h^*(x)\colon\ x\in X\}$ are proper, convex and weakly* lower semicontinuous.

**Theorem 4.1.** *Assume that*

- **(i)** *the functionals* $G_h^*$ *be locally equi-bounded uniformly with respect to* $h\in\mathbb{N}$;
- **(ii)** *the* $w^*$*-l.s.c. envelopes* $\mathrm{sc}^-(X^*)\,G_h$ *be convex for every* $h\in\mathbb{N}$.

*If there exists the pointwise limit* $\lim_{h\to\infty} G_h^*(x)$ *for every* $x \in X$*, then on* $X^*$ *there exists also the* $\Gamma$*-limit* $\Gamma(\mathbb{N}, X^{*-})\lim_{h\to\infty} G_h$ *and coincides with* $\left(\lim_{h\to\infty} G_h^*\right)^*$.

**Remark 4.2.** Condition (ii) above is equivalent to the equality $\mathrm{sc}^-(X^*)\, G_h = G_h^{**}$ for every $h \in \mathbb{N}$. □

The main tool in the proof of Theorem 4.1 is a theorem which relates the $\Gamma$-convergence of a sequence of convex functionals $F_h\colon\ X \to (-\infty, +\infty]$ to the $\Gamma$-convergence of the Fenchel trasformations $F_h^*(x^*) = \sup\{\langle x^*, x\rangle - F_h(x)\colon x \in X\}$; it has been first proved by Attouch (see [3], Theorem 3.9) when $X$ is a reflexive separable Banach space, and extended later to the nonreflexive framework. Let us recall it for convenience of the reader.

**Theorem 4.3** (see Azé [4], Theorem 3.2.4)**.** *Let* $X$ *be a separable Banach space. Let* $F, F_h\colon\ X \to (-\infty, +\infty]$, $h \in \mathbb{N}$, *be proper, convex, lower semicontinuous functionals. Assume that*

**(a)** *there exists the* $\Gamma(\mathbb{N}, X^-)\lim_{h\to\infty} F_h = F$*;*

**(b)** *the sequence* $(F_h^*)$ *is weakly* sequentially equi-coercive.*

*Then* $\Gamma(\mathbb{N}, X^{*-})\lim_{h\to\infty} F_h^* = F^*$. ■

**Proof of Theorem 4.1:** By uniform local boundedness and convexity, the functionals $G_h^*$ are equi-continuous at every point, hence for every $x \in X$

$$\lim_{h\to\infty} G_h^*(x) \;=\; \Gamma(\mathbb{N}, X^-)\lim_{h\to\infty} G_h^*(x)\;. \tag{4.1}$$

Let us set $F_h := G_h^*$ and $F := \Gamma(\mathbb{N}, X^-)\lim_{h\to\infty} G_h^*$ and observe that $F_h$ and $F$ satisfy the hypotheses of Theorem 4.3. Indeed they are proper, convex and strongly lower semicontinuous. They satisfy condition (b) too, because if $F_h^*(x^*) = \sup\{\langle x^*, x\rangle - G_h^*(x)\colon\ x \in X\} \le L \in \mathbb{R}$ then $\langle x^*, x\rangle \le L + G_h^*(x)$ for every $x \in X$ and $h \in \mathbb{N}$ and, by the hypothesis (i), we have $\|x^*\| = \sup\{\langle x_h^*, x\rangle\colon\ \|x\| \le 1\} \le L + \sup\{G_h^*(x)\colon\ \|x\| \le 1\} \le L + M$. Passing to the Fenchel transformations in formula (4.1), using Theorem 4.3 and the fact that $F_h^* = G_h^{**}$ we have, for every $x^* \in X^*$

$$\left(\lim_{h\to\infty} G_h^*\right)^*(x^*) \;=\; \left(\Gamma(\mathbb{N}, X^-)\lim_{h\to\infty} G_h^*\right)^*(x^*) \;=\; \Gamma(\mathbb{N}, X^{*-})\lim_{h\to\infty} G_h^{**}(x^*)\;.$$

The claim follows by the invariance of $\Gamma$-limits under composition with the lower semicontinuous envelope operator. ■

## 5 – Weak compactness: nonreflexive setting

Theorem 3.2 shows that, if the input operators $B_h$ satisfy the strong assumption of sequential Kuratowski continuous convergence, then the limit control problem takes the same form of the elements of the sequence (1.1). On the contrary, if such assumption is dropped, then the limit problem may have a different form. This section is devoted to study the case when the input operators $B_h$ are only weakly compact, like in Section 4 of [11], but in the more general setting of multi-valued operators and dropping the reflexivity assumption on the Banach space $V$. Precisely, $U$ and $V$ are assumed to be dual of separable Banach spaces $Z$ and $W$ respectively, that is $U = Z^*$ and $V = W^*$. The spaces $Z$ and $W$ are endowed with the usual norm topology, while $U$ and $V$ are endowed with the weak* topology. According to this notation, from now on, $(u, v)$ and $(z, w)$ will be conjugate variables, that is $u = z^*$ and $v = w^*$, and the Fenchel transformations will be taken always with respect to these pairs of variables. Let us denote by $D(B_h) = \{u \in U : B_h(u) \neq \emptyset\}$ the domain of $B_h$. In view of the application of Theorem 2.2 and Theorem 4.1, let us make the following assumptions:

(5.1) $D(B_h) \neq \emptyset$ and $J_h(\cdot, y) \not\equiv +\infty$ on $D(B_h)$ for every $y \in Y$ and $h \in \mathbb{N}$;

(5.2) for every $C > 0$ there exists $L > 0$ such that

$$\|u\|_U \leq C \implies B_h(u) \subseteq \Big\{v \in V : \|v\|_V \leq L\Big\} \quad \forall\, h \in \mathbb{N}\ ;$$

(5.3) there exist $p > 1$, $\alpha > 0$, $\beta \geq 0$ such that $J_h(u, y) \geq \alpha\, \|u\|_U^p - \beta$ for every $u \in D(B_h)$ and $y \in Y$ and

$$\forall\, \varepsilon > 0 \ \ \exists\, R > 0 : \quad \|u\|_U > R \implies \|v\|_V < \varepsilon\, \|u\|_U^p \quad \forall\, v \in B_h(u), \ \ \forall\, h \in \mathbb{N}\ ;$$

(5.4) there exist a function $\Psi : U \to \mathbb{R}$ bounded on the $U$-bounded sets and a function $\omega : Y \times Y \to \mathbb{R}$ with $\lim_{z \to y} \omega(y, z) = 0$ for every $y \in Y$ such that $J_h(u, y) \leq J_h(u, z) + \Psi(u)\, \omega(y, z)$ for every $u \in U$, $y, z \in Y$ and $h \in \mathbb{N}$.

Before going on, let us make some remarks concerning the assumptions. Assumption (5.1) ensures that the functionals $G_h$ defined in (2.2) are proper, and (5.2) implies the compactness condition (2.1), while (5.3) guarantees that the sequence $(G_h^*)$ is locally equi-bounded. Finally (5.4) simplifies the computation of the $\Gamma$-limit by allowing to freeze the variable $y$. To the aim of simplifying notation, from now on the subscript spaces in norms and dualities will be omitted, as they can be deduced by the context.

**Theorem 5.1.** *Let* $(G_h)$ *be the sequence of functionals defined in (2.2). Besides hypotheses (5.1)–(5.4) let us assume that for every* $h \in \mathbb{N}$

$$\mathrm{sc}^-(U \times V)\, G_h(\cdot,\cdot,y) \quad \text{be convex for every } \; y \in Y \; . \tag{5.5}$$

*If, for every* $(z,w,y) \in Z \times W \times Y$ *the pointwise limit*

$$\lim_{h\to\infty} G_h^*(z,w,y) \tag{5.6}$$

*exists, then on* $U \times V \times Y$ *there exists the* $\Gamma$*-limit* $\Gamma(\mathbb{N}, U \times V^-, Y) \lim\limits_{h\to\infty} G_h$ *and coincides with* $\Big(\lim\limits_{h\to\infty} G_h^*\Big)^*$. *If moreover* $(A_h)$ *G-converges to* $A$ *then the limit problem for the sequence (1.1) is given by*

$$\min\Big\{\inf_{v\in A(y)} G(u,v,y):\; (u,y) \in U \times Y\Big\}$$

*where*

$$G(u,v,y) = \Big(\lim_{h\to\infty} G_h^*\Big)^*(u,v,y)$$

*and each polar is taken with respect to* $u$, $v$ *and their dual variables.*

**Proof:** By hypothesis (5.4) we have that

$$\Gamma(\mathbb{N}, U \times V^-, Y) \lim_{h\to\infty} G_h(u,v,y) \;=\; \Gamma(\mathbb{N}, U \times V^-) \lim_{h\to\infty} G_h(u,v,y)$$

for every $(u,v,y) \in U \times V \times Y$. We have then to prove that, setting $X := Z \times W$ (and being then $X^* = U \times V$), the functionals $G_h(\cdot,\cdot,y)$ satisfy, for every $y \in Y$, to condition (i) of Theorem 4.1, condition (ii) coinciding with (5.5). Let us fix $z \in Z$, $w \in W$, $y \in Y$ and $\varepsilon > 0$ small enough. By (5.2) and (5.3) there exists $L > 0$ such that $B_h(u) \subseteq \{v \in V:\; \|v\| \le L + \varepsilon\, \|u\|^p\}$ for every $u \in U$. Therefore, choosing $\varepsilon > 0$ such that $\alpha - \varepsilon\,\|w\| > 0$, we obtain

$$\begin{aligned} G_h^*(z,w,y) \;&=\; \sup\Big\{\langle z,u\rangle + \langle w,v\rangle - J_h(u,y):\; u \in D(B_h),\; v \in B_h(u)\Big\} \\ &\le\; L\|w\| + \beta + \sup\Big\{\|z\|\,\|u\| - \Big(\alpha - \varepsilon\,\|w\|\Big)\,\|u\|^p:\; u \in U\Big\} \\ &=\; L\|w\| + \beta + \frac{\|z\|^{p'}}{p'\Big(p\Big(\alpha - \varepsilon\,\|w\|\Big)\Big)^{1/(p-1)}} \end{aligned}$$

where $p'$ is the conjugate exponent to $p$. Then the functionals $G_h^*$ from $Z \times W$ to $(-\infty,+\infty]$ are locally equi-bounded, and by convexity they are strongly equi-continuous at every point, so that, by Proposition 5.9 of Dal Maso [12], the $\Gamma$-convergence turns out to be equivalent to pointwise convergence. The thesis follows by Theorem (4.1) and Theorem (2.2). ∎

## 6 – The measure framework

To provide a concrete application of the abstract framework of the previous section, we introduce here the functional tool that we are going to use.

Let $\Omega$ be a separable locally compact metric space, $\mathcal{B}$ the Borel $\sigma$-algebra of $\Omega$, and $\mu\colon \mathcal{B} \to [0,+\infty[$ a measure. For every vector-valued measure $\lambda\colon \mathcal{B} \to \mathbb{R}^n$ and every $E \in \mathcal{B}$ let us denote by $|\lambda|(E)$ the variation of $\lambda$ on $E$. The following spaces will be considered.

$C_0(\Omega;\mathbb{R}^n)$, the space of all continuous functions $u\colon\, \Omega \to \mathbb{R}^n$ "vanishing on the boundary", that is, such that for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon$ of $\Omega$ with $|u(x)| < \varepsilon$ for all $x \in \Omega \backslash K_\varepsilon$;

$\mathcal{M}(\Omega;\mathbb{R}^n)$, the space of all vector-valued measures $\lambda\colon\, \mathcal{B} \to \mathbb{R}^n$ with finite variation on $\Omega$;

$L^p_\mu(\Omega;X)$, where $X$ is a normed space and $p \in [1,+\infty)$, the space of functions $u\colon\, \Omega \to X$ such that $\int_\Omega \|u\|_X^p\, d\mu < +\infty$;

$BV(\Omega;\mathbb{R}^n)$ where $\Omega \subseteq \mathbb{R}^n$, the space of functions $u \in L^1(\Omega;\mathbb{R}^n)$ with first distributional derivative $Du \in \mathcal{M}(\Omega;\mathbb{R}^n)$.

If $n = 1$ or $X = \mathbb{R}$ we write $C_0(\Omega)$, $\mathcal{M}(\Omega)$, $L^p_\mu(\Omega)$, $BV(\Omega)$ instead of $C_0(\Omega,\mathbb{R})$, $\mathcal{M}(\Omega,\mathbb{R})$, $L^p_\mu(\Omega,\mathbb{R})$, $BV(\Omega;\mathbb{R})$, and if $\mu$ is the Lebesgue measure, that is $\mu = dx$, we write $L^p(\Omega;X)$ instead of $L^p_{dx}(\Omega,X)$.

**Definition 6.1.** A measure $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^n)$ is said to be absolutely continuous with respect to $\mu$ (shortly $\lambda \ll \mu$) if $|\lambda|(B) = 0$ whenever $B \in \mathcal{B}$ and $\mu(B) = 0$. $\lambda$ is said to be singular with respect to $\mu$ (shortly $\lambda \perp \mu$) if $|\lambda|(\Omega\backslash B) = 0$ for a suitable $B \in \mathcal{B}$ with $\mu(B) = 0$. □

In the sequel, given $u \in L^1_\mu(\Omega;\mathbb{R}^n)$, we denote by $u\cdot\mu$ (or simply by $u$ when no confusion is possible) the measure of $\mathcal{M}(\Omega;\mathbb{R}^n)$ defined by $(u\cdot\mu)(B) = \int_B u\, d\mu$, $B \in \mathcal{B}$. It is well-known that every measure $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^n)$ which is absolutely continuous with respect to $\mu$ is representable in the form $\lambda = u\cdot\mu$ for a suitable $u \in L^1_\mu(\Omega;\mathbb{R}^n)$; moreover, by the Lebesgue–Nikodym decomposition theorem, for every $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^n)$ there exist a unique function $u \in L^1_\mu(\Omega;\mathbb{R}^n)$ and a unique measure $\lambda^s \in \mathcal{M}(\Omega;\mathbb{R}^n)$ such that $\lambda = u\cdot\mu + \lambda^s$ and $\lambda^s$ is singular with respect to $\mu$. The function $u$ is called the Radon–Nikodym derivative of $\lambda$ with respect to $\mu$ and is often indicated by $d\lambda/d\mu$.

It is well-known that $\mathcal{M}(\Omega;\mathbb{R}^n)$ can be identified with the dual space of $C_0(\Omega;\mathbb{R}^n)$ by the duality $\langle\lambda,u\rangle = \int_\Omega u\, d\lambda$, $u \in C_0(\Omega;\mathbb{R}^n)$, $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^n)$, and the dual norm equals the total variation $|\lambda|(\Omega)$. The space $\mathcal{M}(\Omega;\mathbb{R}^n)$ will be endowed with this norm or with the weak* topology deriving from the duality with

$C_0(\Omega;\mathbb{R}^n)$; in particular, a sequence $(\lambda_h)$ in $\mathcal{M}(\Omega;\mathbb{R}^n)$ will be said to weakly*-converge to a measure $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^n)$ if and only if $\langle\lambda_h, u\rangle \to \langle\lambda, u\rangle$ for every $u \in C_0(\Omega;\mathbb{R}^n)$.

**Lemma 6.2** ([10], Proposition 2.1)**.** *Let $(\alpha_h)$ be a bounded sequence of positive measures in $\mathcal{M}(\Omega)$ and $\alpha \in \mathcal{M}(\Omega)$. Then the following conditions are equivalent:*

- **(i)** $\alpha_h \to \alpha \; w^*\mathcal{M}(\Omega)$,
- **(ii)** $\lim\limits_{h\to\infty} \alpha_h(A) = \alpha(A)$ *for every Borel subset $A$ of $\Omega$ with compact closure in $\Omega$ such that $\alpha(\partial A) = 0$.* ∎

Using this lemma we get the following statement concerning sequences of signed measures.

**Proposition 6.3.** *Let $\lambda_h$ be a bounded sequence of measures in $\mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$. If there exists a sequence of positive measures $\alpha_h$ such that $\alpha_h \to \alpha$ weakly* and*

$$\langle\lambda_h, \varphi\rangle \le \langle\alpha_h, \varphi\rangle \qquad \forall\, h \in \mathbb{N}, \;\; \forall\, \varphi \in C_0(\Omega), \;\; \varphi \ge 0 \,, \tag{6.1}$$

*then the following propositions are equivalent:*

- **(i)** $\lambda_h \to \lambda \; w^*\mathcal{M}(\Omega)$;
- **(ii)** $\lim\limits_{h\to\infty} \lambda_h(A) = \lambda(A)$ *for every Borel subset $A$ of $\Omega$ with compact closure in $\Omega$ such that $\lambda(\partial A) = \alpha(\partial A) = 0$.*

**Proof:** It is enough to apply Lemma 6.2 to the sequence of positive measures $\alpha_h$ and $\mu_h = \alpha_h - \lambda_h$. ∎

It is worth notice that the requirement $\alpha(\partial A) = 0$ in (ii) cannot be dropped. Indeed the sequence $\lambda_h = h(1_{]0,1/h[} - 1_{]-1/h,0[})\,dx \in \mathcal{M}(]-1,1[)$ weakly* converges to 0, but $\lambda_h([0,1/2]) \not\to 0$.

## 7 – Local input operators

In this section we apply the abstract framework of Section 5 to the case where the input operators $B_h$ are local, possibly nonlinear, multi-valued, defined on $L^p$ spaces and taking values into the nonreflexive space $L^1$. Precisely, let $\Omega$ be a bounded Borel subset of $\mathbb{R}^n$ having positive measure, let $p \in (1, +\infty)$, and let

$$B_h \colon L^p(\Omega;\mathbb{R}^m) \to \wp(L^1(\Omega;\mathbb{R}^n))$$

be the multi-mapping defined by $B_h(u)(x) = \{v \in L^1(\Omega;\mathbb{R}^n) \colon v(x) \in b_h(x, u(x))$ a.e. $x \in \Omega\}$ where the multi-functions $b_h \colon \Omega\times\mathbb{R}^m \to \wp(\mathbb{R}^n)\backslash\emptyset$ are Borel measurable (i.e. the graphs are Borel sets). Assume that the marginal functions

$$V_h(x, u) = \sup\Big\{|v| \colon v \in b_h(x, u)\Big\}$$

which are measurable, satisfy the following conditions:

(7.1) there exist a constant $N > 0$ and a sequence of functions $(M_h)$ bounded in $L^1(\Omega)$ such that $V_h(x, u) \le M_h(x) + N|u|^p$ for almost every $x \in \Omega$, every $u \in \mathbb{R}^m$ and every $h \in \mathbb{N}$;

(7.2) $V_h(x, u)$ increases at infinity less than the power $p$ with respect to the variable $u$, that is $\lim\limits_{|u|\to+\infty} \dfrac{V_h(x, u)}{|u|^p} = 0$ uniformly with respect to $x \in \Omega$ and $h \in \mathbb{N}$.

In order to find the limit problem we cannot take $V = L^1(\Omega;\mathbb{R}^n)$ because it is not dual of a separable Banach space and the compactness condition (2.1) required by Theorem 5.1 is not satisfied. This difficulty can be overcome by choosing $V = \mathcal{M}(\Omega;\mathbb{R}^n)$. In this way we can take $U = L^p(\Omega;\mathbb{R}^m)$, $Z = L^{p'}(\Omega;\mathbb{R}^m)$ and $W = C_0(\Omega;\mathbb{R}^n)$. Let $Y$ be any space of measurable functions from $\Omega$ to $\mathbb{R}^k$ which is embedded into some $L^s(\Omega;\mathbb{R}^k)$ space with $s \in [1, +\infty]$.

The cost is an integral functional of the form

$$J_h(u, y) = \int_\Omega f_h(x, y, u)\; dx$$

where $f_h \colon \Omega\times\mathbb{R}^k\times\mathbb{R}^m \to ]-\infty, +\infty]$ are Borel functions satisfying

(7.3) there exist $a > 0$ and $b \ge 0$ such that $f_h(x, y, u) \ge a\,|u|^p - b$ for almost every $x \in \Omega$, every $(y, u) \in \mathbb{R}^k\times\mathbb{R}^m$ and $h \in \mathbb{N}$;

(7.4) there exist a function $\sigma \colon \mathbb{R}^k\times\mathbb{R}^k \to [0, +\infty[$, a number $r \in [0, p]$ and a function $\rho \in L^{p/r}(\Omega)$ such that $\sigma(y, \eta) \to 0$ in $L^{p/(p-r)}$ as $\eta \to y$ in $Y$ and $f_h(x, y, u) \le f_h(x, \eta, u) + \sigma(y, \eta)\,(\rho(x) + |u|^r)$ for almost every $x \in \Omega$ and every $u \in \mathbb{R}^m$, $y, \eta \in \mathbb{R}^k$, and $h \in \mathbb{N}$;

(7.5) there exists a control function $u_0 \in L^p(\Omega;\mathbb{R}^m)$ such that for every $y \in L^s(\Omega,\mathbb{R}^k)$ the sequence of functions $(f_h(\cdot, y(\cdot), u_0(\cdot)))$ is bounded in $L^1(\Omega)$.

It is easy to see that conditions (5.1)–(5.4) required by Theorem 5.1 are fulfilled. It remains to check that also condition (5.5) is satisfied and to identify the pointwise limit (5.6). Since the operators $B_h$ are local, setting for every $(x,y,u,v) \in \Omega\times\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R}^n$

$$g_h(x,y,u,v) \;=\; f_h(x,y,u) + \chi_{\,v\,\in\, b_h(x,u)} \tag{7.6}$$

we have

$$G_h(u,v,y) \;=\; \int_\Omega g_h(x,y,u,v)\,dx + \chi_{\,v\ll dx}$$

for any $(u,v,y) \in U\times V\times Y$. With the same arguments of [16] we can prove that $G_h^*(z,w,y) = \int_\Omega g_h^*(x,y,z,w)\,dx$ and $\mathrm{sc}^-(U\times V)\,G_h(u,v,y) = \int_\Omega g_h^{**}(x,y,u,v)\,dx + \chi_{v\ll dx}$, hence Theorem 5.1 applies and to identify the limit problem in an explicit form we have only to calculate the functional

$$G(u,v,y) = \Big(\lim_{h\to\infty} G_h^*\Big)^*(u,v,y) \;. \tag{7.7}$$

The following lemma will be useful.

**Lemma 7.1.** *Under (7.1)–(7.5) there exist a function $\Psi\colon Z\times W \to \mathbb{R}$ bounded on the $Z\times W$-bounded sets and a function $\omega : L^s(\Omega;\mathbb{R}^k)\times L^s(\Omega;\mathbb{R}^k) \to \mathbb{R}$ with $\lim\limits_{\eta\to y} \omega(y,\eta) = 0$ for every $y \in L^s(\Omega;\mathbb{R}^k)$ such that*

$$\int_\Omega g_h^*(x,y,z,w)\,\psi(x)\,dx \;\le\; \int_\Omega g_h^*(x,\eta,z,w)\,\psi(x)\,dx + \Psi(z,w)\,\omega(y,\eta)\,\|\psi\|_\infty$$

*for all $z\in Z$, $w\in L^\infty(\Omega;\mathbb{R}^n)$, $y,\eta\in L^s(\Omega;\mathbb{R}^k)$, $\psi\in L^\infty(\Omega)$, $\psi \ge 0$ and $h\in\mathbb{N}$.*

**Proof:** Let $(x,y,z,w) \in \Omega\times\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R}^n$. By definition of Fenchel transformation

$$g_h^*(x,y,z,w) = \sup\Big\{u\,z + v\,w - f_h(x,y,u)\colon\; u \in \mathbb{R}^m,\; v \in b_h(x,u)\Big\} \;. \tag{7.8}$$

By (7.3) $g_h^*(x,y,z,w)$ is finite, so that, for every $\varepsilon > 0$ there exists $u_\varepsilon = u_\varepsilon(x,y,z,w) \in \mathbb{R}^m$ such that $g_h^*(x,y,z,w) \le u_\varepsilon\,z + \sup\{v\,w:\; v \in b_h(x,u_\varepsilon)\} - f_h(x,y,u_\varepsilon) + \varepsilon$. Using (7.1) and (7.2) we obtain that there exists a decreasing positive function $R$ such that

$$V_h(x,u) \;\le\; |M_h(x)| + N R(\delta) + \delta\,|u|^p \qquad \text{for every} \;\; \delta > 0 \tag{7.9}$$

and therefore, by (7.3) and choosing $\delta = a\,/\,|w|\,p'$ we get (for any $0 < \varepsilon \le 1$)

$$g_h^*(x,y,z,w) \;\le\; |u_\varepsilon|\,|z| + |w|\,|M_h(x)| + |w|\,N R\Big(\frac{a}{|w|\,p'}\Big) - \frac{a}{p}\,|u_\varepsilon|^p + b + 1 \;.$$

To estimate $|u_\varepsilon|$ we observe that, by (7.1)

$$
\begin{aligned}
(7.10)\qquad & g_h^*(x,y,z,w) \geq \\
& \geq -|u_0(x)|\,|z| - |M_h(x)|\,|w| - N\,|w|\,|u_0(x)|^p - |f_h(x,y,u_0(x))| \,.
\end{aligned}
$$

By putting together the last two inequalities, and setting

$$
\gamma_h(x,z,w) = 2\,|M_h(x)|\,|w| + |w|\,NR\left(\frac{a}{|w|\,p'}\right) + b + 1 + |u_0(x)|\,|z| + N\,|w|\,|u_0(x)|^p \,,
$$

then we have $-\frac{a}{p}\,|u_\varepsilon|^p + |z|\,|u_\varepsilon| + \gamma_h(x,z,w) + |f_h(x,y,u_0(x))| \geq 0$ for every $0<\varepsilon\leq 1$, from which we can easily obtain

$$
|u_\varepsilon|^r \leq \left(\frac{p}{a}\,|z|\right)^{\frac{r}{p-1}} + \left(\frac{p}{a}\right)^{\frac{r}{p}} \left(\gamma_h(x,z,w)^{\frac{r}{p}} + |f_h(x,y,u_0(x))|^{\frac{r}{p}}\right) \qquad \forall\, 0<\varepsilon\leq 1 \,.
$$

Therefore, using assumption (7.4), we have

$$
\begin{aligned}
& g_h^*(x,y,z,w) = \\
&\quad = \sup\Big\{ u\,z + \sup\big\{ v\,w\colon\ v \in b_h(x,u)\big\} - f_h(x,y,u) \ : \\
&\qquad\qquad |u|^r \leq \left(\frac{p}{a}\,|z|\right)^{\frac{r}{p-1}} + \left(\frac{p}{a}\right)^{\frac{r}{p}} \left(\gamma_h(x,z,w)^{\frac{r}{p}} + |f_h(x,y,u_0(x))|^{\frac{r}{p}}\right)\Big\} \\
(7.11)\quad &\quad \leq \sup\Big\{ u\,z + \sup\big\{ v\,w\colon\ v \in b_h(x,u)\big\} - f_h(x,\eta,u) + \sigma(y,\eta)\left(\rho(x) + |u|^r\right)\colon \\
&\qquad\qquad |u|^r \leq \left(\frac{p}{a}\,|z|\right)^{\frac{r}{p-1}} + \left(\frac{p}{a}\right)^{\frac{r}{p}} \left(\gamma_h(x,z,w)^{\frac{r}{p}} + |f_h(x,y,u_0(x))|^{\frac{r}{p}}\right)\Big\} \\
&\quad \leq g_h^*(x,\eta,z,w) + \\
&\qquad + \sigma(y,\eta)\left[\rho(x) + \left(\frac{p}{a}\,|z|\right)^{\frac{r}{p-1}} + \left(\frac{p}{a}\right)^{\frac{r}{p}} \left(\gamma_h(x,z,w)^{\frac{r}{p}} + |f_h(x,y,u_0(x))|^{\frac{r}{p}}\right)\right].
\end{aligned}
$$

To conclude is now enough to replace the vectors $y$, $\eta$, $z$, $w$ with functions in the suitable spaces, to multiply by $\psi$, to pass to the integral and to use the Hölder's inequality, the assumptions (7.1) and (7.5) and the fact that $R(a\,/\,|w|\,p')$ is an increasing function of $|w|$. ∎

**Theorem 7.2.** *Under assumptions (7.1), (7.2), (7.3), (7.5) and if there exists a positive measure $\mu \in \mathcal{M}(\Omega)$ and a subsequence $(f_{n_k})$ of $(f_h)$ such that $(|f_{n_k}(\cdot,y,u_0(\cdot))|)$ is weakly converging in $L^1_\mu(\Omega)$ for every $y \in \mathbb{R}^k$ and, denoting by $\lambda$ the weak* limit of a subsequence of $(|M_h|)$ (which always exists), then* *there exist a subsequence* $(g^*_{h_k}(\cdot,y,z,w))$ *and an integrand* $g\colon \Omega\times\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R}^n \to\, ]-\infty,+\infty]$ *such that*

$$g^*_{h_k}(\cdot,y,z,w)\cdot dx \;\to\; g(\cdot,y,z,w)\cdot\nu \qquad \textit{weakly* in } \mathcal{M}(\Omega) \tag{7.12}$$

*for every* $z\in\mathbb{R}^m$, $w\in\mathbb{R}^n$ *and* $y\in\mathbb{R}^k$ *where* $\nu = dx+\mu+\lambda$. *Moreover the integrand* $g$ *turns out to be measurable with respect to* $x$*, continuous with respect to* $y$ *and convex with respect to* $(z,w)$ *for* $\nu$*-a.e.* $x\in\Omega$.

**Proof:** Let $(x,y,z,w)\in\Omega\times\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R}^n$. By definition of Fenchel transformation (see (7.8)) and using (7.3) and (7.9) we get

$$g^*_h(x,y,z,w) \;\le\; \sup_{u\in\mathbb{R}^m}\Big\{z\,u+\Big(|w|\,\delta-a\Big)\,|u|^p\Big\}+|w|\,|M_h(x)|+|w|\,NR(\delta)+b$$

and choosing $\delta = a\,/\,|w|\,p'$ and putting $R=R(a\,/\,|w|\,p')$ we obtain

$$g^*_h(x,y,z,w) \;\le\; a^{1-p'}\,\frac{|z|^{p'}}{p'}+|w|\,|M_h(x)|+|w|\,NR+b\;. \tag{7.13}$$

Putting together (7.10) and (7.13), the following estimate can be obtained for any $x\in\mathbb{R}^n$, $y\in\mathbb{R}^k$, $z\in\mathbb{R}^m$, $w\in\mathbb{R}^n$ and $h\in\mathbb{N}$

$$\begin{aligned}|g^*_h(x,y,z,w)| \;\le\; &|w|\,|M_h(x)|+|f_h(x,y,u_0(x))|+a^{1-p'}\,\frac{|z|^{p'}}{p'}\\ &+|w|\,NE(|w|)+|u_0(x)|\,|z|+N\,|w|\,|u_0(x)|^p+b\end{aligned} \tag{7.14}$$

where $E$ is an increasing positive function. By assumptions (7.1) and (7.5), (7.14) implies that the sequence $(g^*_h(\cdot,y,z,w))$ is bounded in $L^1(\Omega)$ for every $(y,z,w)$. Then we can extract a subsequence, which we continue to denote by $(g^*_h)$, weakly* converging in $\mathcal{M}(\Omega)$ to a measure $\nu_{y,z,w}$ for every $(y,z,w)\in\mathbb{Q}^k\times\mathbb{Q}^m\times\mathbb{Q}^n$. For $(y,z,w)\in\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R}^n$ let us define

$$\nu_{y,z,w} \;=\; w^*-\lim_{j\to\infty}\nu_{y_j,z_j,w_j} \tag{7.15}$$

where $(y_j,z_j,w_j)\in\mathbb{Q}^k\times\mathbb{Q}^m\times\mathbb{Q}^n$ is any sequence converging to $(y,z,w)$. Let us now prove that the definition above is well posed. Let $(y_j,z_j,w_j)$ and $(\overline{y}_j,\overline{z}_j,\overline{w}_j)$ be two sequences in $\mathbb{Q}^m\times\mathbb{Q}^n\times\mathbb{Q}^k$ both converging to $(y,z,w)$ and assume that there exists the weak* limits $\nu_{y,z,w} = w^*-\lim\limits_{j\to\infty}\nu_{y_j,z_j,w_j}$ and

$\overline{\nu}_{y,z,w} = w^* - \lim_{j\to\infty} \nu_{\overline{y}_j,\overline{z}_j,\overline{w}_j}$. Then, for every $\varphi \in C_0(\Omega)$, we have

$$\begin{aligned}\left|\int_\Omega \varphi\, d\nu_{y,z,w} - \int_\Omega \varphi\, d\overline{\nu}_{y,z,w}\right| \le{}& \left|\int_\Omega \varphi\, d\nu_{y,z,w} - \int_\Omega \varphi\, d\nu_{y_j,z_j,w_j}\right| \\ &+ \left|\int_\Omega \varphi\, d\nu_{y_j,z_j,w_j} - \int_\Omega g_h^*(x,y_j,z_j,w_j)\,\varphi(x)\,dx\right| \\ &+ \left|\int_\Omega \Big[g_h^*(x,y_j,z_j,w_j) - g_h^*(x,y,z_j,w_j)\Big]\,\varphi(x)\,dx\right| \\ &+ \left|\int_\Omega \Big[g_h^*(x,y,z_j,w_j) - g_h^*(x,y,\overline{z}_j,\overline{w}_j)\Big]\,\varphi(x)\,dx\right| \\ &+ \left|\int_\Omega \Big[g_h^*(x,y,\overline{z}_j,\overline{w}_j) - g_h^*(x,\overline{y}_j,\overline{z}_j,\overline{w}_j)\Big]\,\varphi(x)\,dx\right| \\ &+ \left|\int_\Omega g_h^*(x,\overline{y}_j,\overline{z}_j,\overline{w}_j)\,\varphi(x)\,dx - \int_\Omega \varphi\, d\nu_{\overline{y}_j,\overline{z}_j,\overline{w}_j}\right| \\ &+ \left|\int_\Omega \varphi\, d\nu_{\overline{y}_j,\overline{z}_j,\overline{w}_j} - \int_\Omega \varphi\, d\overline{\nu}_{y,z,w}\right| .\end{aligned}$$

By splitting $\varphi$ into the sum of its positive and negative parts which are both positive functions in $L^\infty(\Omega)$ and using Lemma 7.1, we obtain

$$\left|\int_\Omega \Big[g_h^*(x,y_j,z_j,w_j)\,x - g_h^*(x,y,z_j,w_j)\Big]\,\varphi(x)\,dx\right| \le 2\,\Psi(z_j,w_j)\,\omega(y,y_j)\,\|\varphi\|_\infty .$$

Being convex and locally uniformly bounded with respect to the variables $z$ and $w$ (see (7.14)) the functionals $(z,w) \to \int_\Omega g_h^*(x,y,z,w)\,\psi(x)\,dx$ ($\psi \in L^\infty(\Omega)$, $\psi \ge 0$) are locally equi-lipschitz, that is, for every $h \in \mathbb{N}$

$$\begin{aligned}\left|\int_\Omega \Big[g_h^*(x,y,z,w)\,\psi(x) - g_h^*(x,y,z_j,w_j)\Big]\,\psi(x)\,dx\right| &\le \\ &\le \alpha(\psi,y)\,\Big(\|z-z_j\|_m + \|w-w_j\|_n\Big)\end{aligned} \tag{7.16}$$

where $\alpha(\psi, y)$ is a constant depending on $\psi$ and $y$. Then, in the same way as before we have

$$\begin{aligned}&\left|\int_\Omega \varphi\, d\nu_{y,z,w} - \int_\Omega \varphi\, d\overline{\nu}_{y,z,w}\right| \le \\ &\le \left|\int_\Omega \varphi\, d\nu_{y,z,w} - \int_\Omega \varphi\, d\nu_{y_j,z_j,w_j}\right| + \left|\int_\Omega \varphi\, d\nu_{y_j,z_j,w_j,y_j} - \int_\Omega g_h^*(x,y_j,z_j,w_j)\,\varphi(x)\,dx\right| \\ &\quad + 2\,\Psi(z_j,w_j)\,\omega(y,y_j)\,\|\varphi\|_\infty + \Big[\alpha(\varphi^+,y) + \alpha(\varphi^-,y)\Big]\Big(\|z_j-\overline{z}_j\|_m + \|w_j-\overline{w}_j\|_n\Big) \\ &\quad + 2\,\Psi(\overline{z}_j,\overline{w}_j)\,\omega(y,\overline{y}_j)\,\|\varphi\|_\infty + \left|\int_\Omega g_h^*(x,\overline{y}_j,\overline{z}_j,\overline{w}_j)\,\varphi(x)\,dx - \int_\Omega \varphi\, d\nu_{\overline{y}_j,\overline{z}_j,\overline{w}_j}\right| \\ &\quad + \left|\int_\Omega \varphi\, d\nu_{\overline{y}_j,\overline{z}_j,\overline{w}_j} - \int_\Omega \varphi\, d\overline{\nu}_{y,z,w}\right| .\end{aligned}$$

Passing to the limit first as $h \to \infty$ and then as $j \to \infty$ the right hand side tends to 0 and $\nu_{y,z,w}$ equals $\overline{\nu}_{y,z,w}$. The existence of the limit (7.15) follows by the facts that, by (7.14), the sequence $\nu_{y_j,z_j,w_j}$ is bounded in $\mathcal{M}(\Omega)$, that the weak* topology is metrizable on bounded sets and that the previous argument applies to every subsequence.

Using the same arguments as before we can easily prove that

$$g_h^*(\cdot, y, z, w) \cdot dx \;\to\; \nu_{z,w,y} \qquad \text{weakly* in } \;\mathcal{M}(\Omega)$$

for every $(y, z, w) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^n$. On the other hand the sequence $(|M_h|)$ admits a subsequence weakly* converging in $\mathcal{M}(\Omega)$ to a measure $\lambda$ while $(|f_h(\cdot, y, u_0(\cdot))|)$ admits, by assumption, a subsequence weakly* converging for every $y \in \mathbb{R}^k$ to measures which are, all togheter, absolutely continuous with respect to a measure $\mu$, so that all the measures $\nu_{y,z,w}$ are absolutely continuous with respect to $\nu = dx + \lambda + \mu$ and, by the Radon–Nikodym theorem, there exists a function $g$ which satisfies (7.12). Moreover it is convex with respect to the two last variables as a straightforward consequence of convexity of the $g_h^*$. Measurability with respect to $x$ is ensured by Radon–Nikodym theorem. The continuity with respect to $y$ can be easily obtained by multiplying (7.11) by a positive $\varphi \in C_0(\Omega)$, managing with Hölder's inequality, passing to the limit as $h \to +\infty$ and getting pointwise extimates on the integrands. ∎

**Theorem 7.3.** *Assume (7.1)–(7.5) and let $(g_h)$ be the sequence of functions defined in (7.6). If there exists a positive measure $\lambda \in \mathcal{M}(\Omega)$ such that*

$$|M_h(\cdot)| \cdot dx \;\to\; \lambda \qquad \textit{weakly* in } \;\mathcal{M}(\Omega)\;, \tag{7.17}$$

*and there exist an integrand $g:\; \Omega \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^n \;\to\; (-\infty, +\infty]$ and a positive measure $\nu \in \mathcal{M}(\Omega)$ with $dx \ll \nu$ such that*

$$g_h^*(\cdot, y, z, w) \cdot dx \;\to\; g(\cdot, y, z, w) \cdot \nu \qquad \textit{weakly* in } \;\mathcal{M}(\Omega) \quad \textit{for every } \;(y, z, w) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^n \tag{7.18}$$

*then*

$$\lim_{h\to\infty} \int_\Omega g_h^*(x, y, z, w)\, dx \;=\; \int_\Omega g(x, y, z, w)\, d\nu \tag{7.19}$$

*for every $(y, z, w) \in Y \times Z \times W$.*

**Proof:** As a first step, let us prove that

$$g_h^*(\cdot, y(\cdot), z, w) \cdot dx \;\to\; g(\cdot, y(\cdot), z, w) \cdot \nu \qquad \text{weakly* in } \;\mathcal{M}(\Omega) \tag{7.20}$$

for every $(y,z,w) \in Y \times \mathbb{R}^m \times \mathbb{R}^n$. To this aim, let us observe that (7.18) implies $\sup_h \int_\Omega |g_h^*(x,y,z,w)|\,dx < +\infty$; hence, as the sequence $(|M_h(\cdot)| \cdot dx)$ weakly* converges to $\lambda$ in $\mathcal{M}(\Omega)$, then by (7.13), for any $y \in \mathbb{R}^k$, $z \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$, the sequence of measures $\lambda_h = g_h^*(\cdot,y,z,w)\cdot dx$ fulfills the assumption (6.1) of Proposition 6.3 with $\alpha = C(z,w)\cdot dx + |w|\cdot\lambda$ (where $C(z,w) = a^{1-p'}|z|^{p'}/p' + |w|\,NR + b$). Using it, assumption (7.18) implies $\int_A g_h^*(x,y,z,w)\,dx \to \int_A g(x,y,z,w)\,d\nu$ for every Borel subset $A$ with compact closure in $\Omega$ such that $\nu(\partial A) = \alpha(\partial A) = 0$. With this remark, (7.20) holds when $y$ is a step function of the form

$$\varphi(t) = \sum_{i=1}^N a_i\, 1_{A_i}(t) \tag{7.21}$$

where $a_i$ are in $\mathbb{R}^k$ and $A_i$ are Borel subsets of $\Omega$ with compact closure in $\Omega$ such that $\nu(\partial A_i) = \alpha(\partial A_i) = 0$. In the general case, for fixed $y \in Y$ there exist step functions $y_k$ of the form (7.21) such that $y_k \to y$ strongly in $L^s_\nu(\Omega;\mathbb{R}^k)$. Moreover, by using (7.14) with $y \in L^s(\Omega;\mathbb{R}^k)$ together with assumption (7.5) and (7.18) then we obtain easily that $\sup_h \int_\Omega |g_h^*(x,y(x),z,w)|\,dx < +\infty$ and $\int_\Omega |g(x,y(x),z,w)|\,d\nu < +\infty$ for every $y \in L^s(\Omega;\mathbb{R}^k)$, $z \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$. For $y \in Y$, $z \in \mathbb{R}^m$, $w \in \mathbb{R}^n$ and $\varphi \in C_0(\Omega)$, by using Lemma 7.1, we have

$$\begin{aligned}
&\left|\int_\Omega g_h^*(x,y(x),z,w)\,\varphi(x)\,dx - \int_\Omega g(x,y(x),z,w)\,\varphi(x)\,d\nu\right| \le \\
&\qquad \le \left|\int_\Omega g_h^*(x,y(x),z,w)\,\varphi(x)\,dx - \int_\Omega g_h^*(x,y_k(x),z,w)\,\varphi(x)\,dx\right| \\
&\qquad\quad + \left|\int_\Omega g_h^*(x,y_k(x),z,w)\,\varphi(x)\,dx - \int_\Omega g(x,y_k(x),z,w)\,\varphi(x)\,d\nu\right| \\
&\qquad\quad + \left|\int_\Omega g(x,y_k(x),z,w)\,\varphi(x)\,d\nu - \int_\Omega g(x,y(x),z,w)\,\varphi(x)\,d\nu\right| \\
&\qquad \le 2\,\Psi(z,w)\,\omega(y,y_k) \\
&\qquad\quad + \left|\int_\Omega g_h^*(x,y_k(x),z,w)\,\varphi(x)\,dx - \int_\Omega g(x,y_k(x),z,w)\,\varphi(x)\,d\nu\right| .
\end{aligned}$$

By choosing $k$ large enough that $2\,\Psi(z,w)\,\omega(y,y_k) < \varepsilon$ and passing to the limit as $h \to +\infty$ we obtain (7.20). To prove (7.19), let us observe that, by (7.13), for any fixed $y \in Y$, $z \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$, the sequence of measures $\lambda_h = g_h^*(\cdot,y(\cdot),z,w)\cdot dx$ fulfills the assumption (6.1) of Proposition 6.3 with the same $\alpha$ as before. By Proposition 6.3, (7.20) is equivalent to $\int_A g_h^*(x,y(x),z,w)\,dx \to \int_A g(x,y(x),z,w)\,d\nu$ for every Borel subset $A$ with compact closure in $\Omega$ such that

$\nu(\partial A) = \alpha(\partial A) = 0$. With this remark, (7.19) holds when $y \in Y$ while $z$ and $w$ are step functions of the form (7.21) where $a_i$ are in $\mathbb{R}^m$ or $\mathbb{R}^n$ respectively. In the general case, for fixed $z \in L^{p'}(\Omega;\mathbb{R}^m)$, $w \in C_0(\Omega;\mathbb{R}^n)$, for every $\varepsilon > 0$ there exist step functions of the form (7.21), $z_\varepsilon$ and $w_\varepsilon$, such that $\|z_\varepsilon - z\|_{p'} < \varepsilon$, $\|w_\varepsilon - w\|_\infty < \varepsilon$. Let now $y \in Y$, $z \in L^{p'}(\Omega;\mathbb{R}^m)$ and $w \in C_0(\Omega;\mathbb{R}^n)$. By (7.14), the functionals $\int_\Omega g_h^*(x,y,z,w)\,dx$ take finite values and, using the lower semicontinuity of the total variation and the absolute continuity of the Lebesgue measure with respect to $\nu$, it is easy to see that also $\int_\Omega g(x,y,z,w)\,d\nu$ takes finite values, hence

$$
\begin{aligned}
\left|\int_\Omega g_h^*(x,y,z,w)\,dx - \int_\Omega g(x,y,z,w)\,d\nu\right| &\le \\
&\le \left|\int_\Omega g_h^*(x,y,z,w)\,dx - \int_\Omega g_h^*(x,y,z_\varepsilon,w_\varepsilon)\,dx\right| \\
&\quad + \left|\int_\Omega g_h^*(x,y,z_\varepsilon,w_\varepsilon)\,dx - \int_\Omega g(x,y,z_\varepsilon,w_\varepsilon)\,d\nu\right| \\
&\quad + \left|\int_\Omega g(x,y,z_\varepsilon,w_\varepsilon)\,d\nu - \int_\Omega g(x,y,z,w)\,d\nu\right| .
\end{aligned} \tag{7.22}
$$

Being convex and locally uniformly bounded with respect to the variables $z$ and $w$ (see (7.14)) the functionals $\int_\Omega g_h^*(x,y,z,w)\,dx$ are locally equi-lipschitz, that is, for every $h \in \mathbb{N}$ an inequality like (7.16) holds with $\psi = 1$. By (7.14) and the lower semicontinuity of the total variation, the convex functional $\int_\Omega g(x,y,z,w)\,d\nu$ is locally bounded with respect to $(z,w)$ and an inequality like (7.16) still holds for it. Therefore, by (7.22) we have

$$
\begin{aligned}
\left|\int_\Omega g_h^*(x,y,z,w)\,dx - \int_\Omega g(x,y,z,w)\,dv\right| &\le \\
&\le 2\,\alpha(1,y)\,\varepsilon + \left|\int_\Omega g_h^*(x,y,z_\varepsilon,w_\varepsilon)\,dx - \int_\Omega g(x,y,z_\varepsilon,w_\varepsilon)\,d\nu\right| .
\end{aligned}
$$

The conclusion is now achieved by passing to the limit first as $h \to +\infty$ and then as $\varepsilon \to 0$. ∎

We can now calculate the functional $G$ in (7.7) by using a theorem of Valadier [25]. We have

$$
G(u,v,y) = \int_\Omega g^*\left(x,y,u,\frac{dv}{d\nu}\right) d\nu + \chi_{v \ll \nu}(v)
$$

where $dv/d\nu$ denotes the Radon–Nikodym derivative of the measure $v$ with respect to $\nu$.

Summarizing, (7.17) and (7.18) together with assumptions (7.1)–(7.5) and (2.3) give to the limit problem (2.5) the form

$$\min_{(u,y)\in U\times Y} \inf\left\{ \int_\Omega g^*\left(x, y, u, \frac{dv}{d\nu}\right) d\nu\colon\ v \in A(y),\ v \ll \nu \right\} . \tag{7.23}$$

**Example 7.4.** Consider the sequence of optimal control problems

$$\min_{\substack{u\in L^2(0,1)\\ y\in W^{1,1}(0,1)}} \left\{ \int_0^1 \left(\frac{u^2}{2}\, f(x,y) + \varphi(x,y)\right) dx :\ \ |y' - a_h(x,y)| \le b_h(x)\,|u|,\ y(0) = \xi_h \right\}$$

where $f, \varphi\colon (0,1)\times\mathbb{R}^k \to (-\infty, +\infty]$ are Carathéodory integrands satisfying:

- **(i)** $f$ is locally Lipschitz in $y$ uniformly with respect to $x$;
- **(ii)** $f(x,y) \ge c > 0$ for every $(x,y) \in (0,1)\times\mathbb{R}^k$;
- **(iii)** there exists a function $\psi : (0,1)\times[0,+\infty[\ \to [0,+\infty[$, $\psi = \psi(x,t)$, increasing in $t$ and integrable in $x$ such that $0 \le \varphi(x,y) \le \psi(x,|y|)$ for every $(x,y) \in (0,1)\times\mathbb{R}^k$.

Let $b_h \in L^2(0,1)$, and about the coefficients $a_h$ assume that:

- **(i)** there exist two sequences of functions $(C_h)$, $(D_h)$ bounded in $L^1(0,1)$ such that $|a_h(x,s)| \le C_h(x)\,|s| + D_h(x)$ for every $(x,s) \in (0,1)\times\mathbb{R}$, $\forall\, h \in \mathbb{N}$;
- **(ii)** for every $x \in (0,1)$, $r \ge 0$, $y_1, y_2 \in \mathbb{R}$ with $|y_1|, |y_2| \le r$ it is $|a_h(x,y_1) - a_h(x,y_2)| \le \alpha_h(x,r)\,|y_1 - y_2|$ where $\|\alpha_h(\cdot,r)\|_{L^1(0,1)} \le \alpha(r) < +\infty$ for every $h \in \mathbb{N}$, and the sequence $(\alpha_h(\cdot,r))_h$ be uniformly integrable.

Let us make the following convergence assumptions on data

$$\begin{aligned}
&a_h(\cdot,y) \to a(\cdot,y) \quad \text{weakly in } L^1(0,1) \text{ for every } y \in \mathbb{R}\ ,\\
&b_h \to b \quad \text{weakly in } L^2(0,1)\ ,\\
&b_h^2 \to \mu \quad \text{weakly* in } \mathcal{M}([0,1))\ ,\\
&\xi_h \to \xi \quad \text{in } \mathbb{R}\ .
\end{aligned}$$

Assume moreover that $b(x) > 0$ for almost every $x \in (0,1)$.

Our abstract construction suggests to take $U = L^2(0,1)$, $Y = W^{1,1}(0,1)$, $V = \mathcal{M}([0,1))\times\mathbb{R}$, $A_h\colon Y \to V$ defined by $A_h(y) = (y' - a_h(x,y),\ y(0) - \xi_h)$ and

$B_h : U \to V$ defined by $B_h(u) = (b_h|u|, 0)$. But in this way condition (2.3) of $G$-convergence is not satisfied because if $v \in \mathcal{M}([0,1))\backslash L^1(0,1)$ then a function $y \in W^{1,1}(0,1)$ such that $y' - a(x,y) = v$ does not exist. To recover this property we have to enlarge the space of states by taking $Y$ the space $BV(0,1)$ of functions with bounded variation. The initial condition $y(0) = \xi_h$ has to be intended in the sense of the right limit in 0, that is $y(0^+) = \xi_h$. It is now very easy to see that assumptions (7.1)–(7.5) are satisfied with $p = 2$ and $M_h(x) = b_h^2(x)$. By Proposition 4.1 of [10] the limit operator $A\colon Y \to V$ is $A(y) = (y' - a(x,y),\ y(0^+) - \xi - y'(\{0\}))$. By continuity of the functional $\int_0^1 \varphi(x,y)\,dx$ with respect to the weak* convergence in $BV(0,1)$ and the stability of $\Gamma$-convergence with respect to continuous additive perturbations, we can calculate the limit problem by neglecting this term, and add it to the limit cost at the end of computation.

As in (7.6), let us define

$$g_h(x,y,u,v) \;=\; \frac{u^2}{2}\, f(x,y) + \chi_{\,|v| \le b_h(x)|u|}\;.$$

According to Theorem 7.3 and to (7.23), we have to calculate $g_h^*$ which turns out to be equal to $g_h^*(x,y,z,w) = (|z| + b_h(x)\,|w|)^2 / \,2f(x,y)$; then we have to calculate the weak* limit in $\mathcal{M}([0,1))$ which is $g(\cdot,y,z,w)\cdot\nu$ with

$$g(x,y,z,w) = \begin{cases} \dfrac{|z|^2 + 2\,b(x)\,|w|\,|z| + \mu^a(x)\,|w|^2}{2\,f(x,y)} & \text{in } (0,1)\backslash\Omega^s\,, \\[2ex] \dfrac{|w|^2}{2\,f(x,y)} & \text{in } \Omega^s\,, \end{cases}$$

and $\nu = dx + \mu^s$, where $\mu^a(x)\cdot dx$ and $\mu^s$ denote respectively the absolutely continuous and the singular part of $\mu$ with respect to the Lebesgue measure and $\Omega^s$ denotes the support of $\mu^s$. By taking the polar again we have

$$g^*(x,y,u,v) \;=\; f(x,y)\cdot\begin{cases} \dfrac{|u|^2}{2} + \dfrac{\Big(|v| + b(x)\,|u|\Big)^2}{2\Big(\mu^a(x) - b^2(x)\Big)} & \text{if } b(x)|u| < |v| < \dfrac{\mu^a(x)}{b(x)}\,|u|\,, \\[2ex] \dfrac{|u|^2}{2} & \text{if } |v| \le b(x)\,|u|\,, \\[2ex] \dfrac{|v|^2}{2\,\mu^a(x)} & \text{if } |v| \ge \dfrac{\mu^a(x)}{b(x)}\,|u|\,, \end{cases}$$

and the limit problem turns out to be

$$\min\Big\{\int_{\Omega_1}\frac{|u|^2}{2}\,f(x,y)\,dx+\int_{\Omega_2}\Big[\frac{|u|^2}{2}+\frac{\Big(|y'^a-a(x,y)|-b(x)\,|u|\Big)^2}{2\Big(\mu^a(x)-b^2(x)\Big)}\Big]\,f(x,y)\,dx\;+$$

$$+\int_{\Omega_3}\frac{|y'^a-a(x,y)|^2}{2\,\mu^a(x)}\,f(x,y)\,dx+\int_{\Omega^s}\frac{1}{2}\left|\frac{d|y'^s|}{d\mu^s}\right|^2 f(x,y)\,d\mu^s+\int_{\Omega}\varphi(x,y)\,dx\;:$$

$$y'-a(x,y)\ll dx+\mu^s,\;\; u\in L^2(0,1),\;\; y\in BV(0,1)\Big\}$$

where

$$\Omega_1=\Big\{x\in(0,1)\colon\;\Big|(y')^a(x)-a(x,y(x))\Big|\le b(x)\,|u(x)|\Big\}\;,$$

$$\Omega_2=\Big\{x\in(0,1)\colon\;b(x)\,|u(x)|\le\Big|(y')^a(x)-a(x,y(x))\Big|\le\frac{\mu^a(x)}{b(x)}\,|u(x)|\Big\}\;,$$

$$\Omega_3=\Big\{x\in(0,1)\colon\;\Big|(y')^a(x)-a(x,y(x))\Big|\ge\frac{\mu^a(x)}{b(x)}\,|u(x)|\Big\}\;.\;\square$$

**Example 7.5.** A rather particular and concrete case is when $f(x,y)=1+y^2$, $\varphi(x,y)=|y-y_0(x)|^2$ where $y_0$ is given in $L^2(0,1)$, $a_h=0$, and the coefficient $b_h$ is sum of three functions $b_h=1+r_h+\delta_h$ where $\delta_h=\mathbf{1}_{[0,1/h[}(t)\,\sqrt{h}$ while $r_h$ is the $h$-th Rademacher function. The sequence of control problems is then

$$\min_{\substack{u\in L^2(0,1)\\ y\in W^{1,1}(0,1)}}\left\{\int_0^1\Big(\frac{u^2}{2}\,(1+y^2)+|y-y_0|^2\Big)\,dt:\;\;|y'|\le b_h(t)|u|,\;\,y(0)=\xi_h\right\}.$$

As $b_h\to1$ weakly in $L^2(0,1)$ while $b_h^2\to2\,dx+\delta_0$ weakly* in $\mathcal{M}([0,1[)$ then the limit problem turns out to be

$$\min_{\substack{u\in L^2(0,1)\\ y\in W^{1,1}(0,1)}}\Bigg\{\int\limits_{\{x:\;|y'|\le|u|\}}\frac{|u|^2}{2}\,(1+y^2)\,dx\;+\int\limits_{\{x:\;|u|\le|y'|\le2|u|\}}\Big[\frac{|u|^2}{2}+\frac{(|y'|-|u|)^2}{2}\Big]\,(1+y^2)\,dx\;+$$

$$+\int\limits_{\{x:\;|y'|\ge2|u|\}}\frac{|y'|^2}{4}\,(1+y^2)\,dx\;+\;\frac{|y(0)-\xi|^2}{2}\Big(1+y(0)^2\Big)\;+\int_0^1|y-y_0|^2\,dx\Bigg\}\;.\;\square$$

## 8 – Linear input operators

In this section we consider the case when the input operators $B_h : U \to V$ are linear, but they are not required to be local. Moreover, for simplicity, assume that also the operators $A_h$ are single-valued (see [15] for the extension to the multi-valued setting). According to Theorem 5.1, the crucial step is to characterize the pointwise limit

$$\lim_{h\to\infty} G_h^*(z,w,y)$$

for every $z \in Z$, $w \in W$ and $y \in Y$. Let us denote by $B_h^*$ the adjoint operator of $B_h$ with respect to the duality pairs $(U,Z)$ and $(V,W)$. We recall that these operators are defined between the spaces $W$ and $Z$ regarded as duals of $U$ and $V$ with respect to the weak* topology, that is $B_h^*\colon W \to Z$ and fulfill the relations $\langle B_h u, w\rangle = \langle u, B_h^* w\rangle$. With this simple remark the arguments of [11] can be repeated without substantial modifications. Here we synthesize the results for convenience of the reader and because the subsequent examples do not fall into the previous framework. We have

$$G_h^*(z,w,y) = {J_h}^*(z + B_h^* w,\, y)\ . \tag{8.1}$$

Assume that (5.1)–(5.4) hold for a suitable $p > 1$. Let us observe however that the growth condition on $B_h$ imposed by (5.3) is implied by the linearity and that such assumptions take the simpler form (4.1)–(4.4) of [11]. If $A_h$ $G$-converge to $A$ and for every $z \in Z$, $w \in W$ and $y \in Y$

$$\lim_{h\to\infty} {J_h}^*(z + B_h^* w,\, y) \;=\; \Phi(z,w,y)\ ,$$

and if assumption (5.5) is satisfied then, by Theorem 5.1, the limit problem is given by

$$\min\Big\{\Phi^*(u, A(y), y)\colon\ (u,y) \in U\times Y\Big\}\ .$$

In order to make more explicit computations, let us assume that $U = Z^*$ be a Hilbert space, and that the cost functionals be of the form

$$J_h(u,y) = J(u,y) = \frac{\langle u,\, J(y)\, u\rangle}{2} + \Psi(y) \qquad \forall\, h \in \mathbb{N} \tag{8.2}$$

where $\Psi\colon\, Y \to \mathbb{R}$ is a continuous functional and, for every $y \in Y$, $J(y)\colon\, Z^* \to Z$ is a linear selfadjoint isomorphism which is continuous in the norm of linear bounded operators, and such that the quadratic form in (8.2) be positive.

The following lemmata generalizes Lemma 6.1 and 6.2 of [11] (the proofs are simple adaptations).

**Lemma 8.1.** *Let $X$ be a Banach space, $E\colon X\to X^*$ be a linear, bounded, positive, selfadjoint operator and $\varphi\in X^*$. With the usual convention* $\inf\emptyset=+\infty$, *we have* $\sup\Big\{2\,\langle\varphi,x\rangle-\langle Ex,x\rangle\colon\ x\in X\Big\}=\inf\Big\{\langle\varphi,x\rangle\colon\ x\in E^{-1}(\varphi)\Big\}$. ∎

**Lemma 8.2.** *Let $X$ be a Banach space, $E\colon X^*\to X$ a linear, bounded, positive selfadjoint operator and $\varphi\in X$. Then* $\sup\Big\{2\,\langle x^*,\varphi\rangle-\langle Ex^*,x^*\rangle\colon x^*\in X^*\Big\}=\inf\Big\{\langle x^*,\varphi\rangle\colon\ x^*\in E^{-1}(\varphi)\Big\}$. ∎

By Lemma 8.2 and formula (8.1), we have

$$G_h^*(z,w,y)+\Psi(y)\;=\;\frac{\langle J(y)^{-1}z,\,z\rangle}{2}+\langle B_h\,J(y)^{-1}z,\,w\rangle+\frac{\langle B_h\,J(y)^{-1}\,B_h^*\,w,\,w\rangle}{2}\;.$$

Assume that there exist operators $B\colon U\to V$ and $C(y)\colon W\to V$ such that

$$B_h\to B\,,\qquad B_h\,J(y)^{-1}B_h^*\to C(y)\quad\forall\,y\in Y$$

in the weak* pointwise sense. Then, for every $z\in Z$, $w\in W$ and $y\in Y$

$$G_h^*(z,w,y)\;\to\;\Phi(z,w,y)=\frac{\langle J(y)^{-1}z,\,z\rangle}{2}+\langle BJ(y)^{-1}z,\,w\rangle+\frac{\langle C(y)w,\,w\rangle}{2}-\Psi(y)\;.$$

By definition of Fenchel's transform we have

$$\Phi^*(u,v,y)-\Psi(y)\;=$$

$$=\sup_{(z,w)\,\in\,Z\times W}\left\{\langle u,z\rangle+\langle v,w\rangle-\frac{\langle J(y)^{-1}z,\,z\rangle}{2}-\langle BJ(y)^{-1}z,\,w\rangle-\frac{\langle C(y)w,\,w\rangle}{2}\right\}$$

$$=\sup_{w\in W}\left\{\langle v,w\rangle-\frac{\langle C(y)w,\,w\rangle}{2}+\sup_{z\in Z}\left\{\langle u-J^{-1}(y)\,B^*w,\,z\rangle-\frac{\langle J(y)^{-1}z,\,z\rangle}{2}\right\}\right\}$$

and by applying Lemma 8.1 we obtain

$$\Phi^*(u,v,y)\;=\;J(u,y)+\sup\left\{\langle v-Bu,w\rangle-\frac{\langle E(y)\,w,\,w\rangle}{2}\colon\ w\in W\right\}$$

where $E(y)=C(y)-B\,J(y)^{-1}B^*$. For every $y\in Y$ the operator $E(y)$ is linear, positive, bounded and selfadjoint. A further application of Lemma 8.1 gives to the limit problem the form

$$\min_{U\times Y}\left\{J(u,y)+\frac{1}{2}\inf\Big\{\langle A(y)-Bu,\,w\rangle\colon\ E(y)\,w=A(y)-Bu,\ w\in W\Big\}\right\}\;.$$

A first interesting case arises when $C(y) - B\,J(y)^{-1}B^* = 0$ for every $y \in Y$ which happens when $B_h^* \to B^*$ strongly. In this case, according to the results of Section 3, the limit problem becomes

$$\min\Big\{J(u,y)\colon\ A(y) = Bu,\ (u,y) \in U\times Y\Big\}\ .$$

Another case is when the operator $E$ is invertible, and the limit problem is

$$\min\left\{J(u,y) + \frac{\Big\langle A(y)-Bu,\ E(y)^{-1}\Big(A(y)-Bu\Big)\Big\rangle}{2}\colon\ (u,y) \in U\times Y\right\}\ .$$

If the isomorphism $J$ does not depend on $y$, identifying $Z$ with its dual $U$, our result can be summarized in the following statement.

**Corollary 8.3.** *Consider the optimal control problem*

$$\min\left\{\frac{\|u\|^2}{2} + \Psi(y)\colon\ A_h(y) = B_h\,u,\ (u,y) \in U\times Y\right\} \tag{8.3}$$

*where $U$ is a Hilbert space and $\Psi\colon Y \to \mathbb{R}$ is continuous. If*

$$\begin{aligned} &A_h \xrightarrow{G} A \\ &B_h \to B \quad \textit{weakly* pointwise} \\ &B_h\,B_h^* \to C \quad \textit{weakly* pointwise} \end{aligned} \tag{8.4}$$

*then a limit problem for the sequence (8.3) is given by*

$$\begin{aligned} \min_{U\times Y}\bigg\{&\frac{\|u\|^2}{2} + \Psi(y)\ + \\ &+\frac{1}{2}\inf\Big\{\langle A(y)-Bu,\ w\rangle\colon\ Ew = A(y)-Bu,\ w \in W\Big\}\bigg\} \end{aligned} \tag{8.5}$$

*where $E = C - BB^*$.* ∎

**Remark 8.4.** This setting includes the case of optimal control problems for state equations with deviating arguments, where the input operator are nonlocal. In fact it has been used in [9]. □

**Remark 8.5.** The case when the input multi-valued operators $B_h\colon U \to \wp(V)$ are linear in the sense that their graphs are vector subspaces of $U\times V$, but they are not required to be local cannot be treated here. Indeed in this setting assumption (5.2) cannot be satisfied unless the operators are single-valued.

It could be modified by requiring that it holds for a linear selection, but also in this case the equicoercivity required in (b) of Theorem 4.3 is not fulfilled unless the operators are single-valued. Hence, the general case of multi-valued linear input operators remains still open. □

**Example 8.6.** Let us give an application of Corollary 8.3 by considering the sequence of optimal control problems (8.3) in the concrete case where $U = L^2(\Omega)$ and the input operators $B_h$ have the general form

$$B_h \colon L^2(\Omega) \to \mathcal{M}(\Omega)\,, \qquad \langle B_h u, w\rangle = \int_\Omega \langle \beta_h(z), w\rangle\, u(z)\, dz, \quad w \in C_0(\Omega)\ ,$$

where $\beta_h$ belongs to the space $L^2(\Omega; \mathcal{M}(\Omega))$ (see Section 6 for notation). Assume that the sequence $A_h$ $G$-converges to a limit operator $A$. To characterize the other limits in (8.4), on the sequence $\beta_h$ we require that

(8.6) $\beta_h \to \beta$ weakly* in $L^2(\Omega; \mathcal{M}(\Omega))$,

(8.7) there exist a measure $\mu \in \mathcal{M}(\Omega)$ and a function $\gamma \in L^1_\mu(\Omega; \mathcal{M}(\Omega\times\Omega))$ such that $\int_\Omega \langle \beta_h(z) \otimes \beta_h(z),\ \varphi \otimes w\rangle\, dz \to \int_\Omega \langle \gamma(z),\ \varphi \otimes w\rangle\, d\mu(z)$ for every $\varphi, w \in C_0(\Omega)$.

About the second limit in (8.4) we immediately get $B_h \to B$ weakly* where $B\colon L^2(\Omega) \to \mathcal{M}(\Omega)$ is defined by $\langle Bu, w\rangle = \int_\Omega \langle \beta(z), w\rangle\, u(z)\, dz$. Concerning the third limit we have $B_h B_h^*\colon C_0(\Omega) \to \mathcal{M}(\Omega)$,

$$\langle B_h B_h^* w,\ \varphi\rangle = \int_\Omega \langle \beta_h(z), \varphi\rangle\, \langle \beta_h(z), w\rangle\, dz = \int_\Omega \Big\langle \beta_h(z) \otimes \beta_h(z),\ \varphi \otimes w \Big\rangle\, dz\ .$$

Denoting by $C\colon C_0(\Omega) \to \mathcal{M}(\Omega)$ the operator $\langle Cw, \varphi\rangle = \int_\Omega \langle \gamma(z), \varphi \otimes w\rangle\, d\mu(z)$, we obtain $B_h B_h^* \to C$ weakly* while $BB^*\colon C_0(\Omega) \to \mathcal{M}(\Omega)$, $\langle BB^* w, \varphi\rangle = \int_\Omega \langle \beta(z) \otimes \beta(z),\ \varphi \otimes w\rangle\, dz$. Then the operator $E$ which appears in the limit problem (8.5) is given by

$$\langle Ew, \varphi\rangle = \int_\Omega \Big\langle \gamma(z),\ \varphi \otimes w \Big\rangle\, d\mu(z) - \int_\Omega \Big\langle \beta(z) \otimes \beta(z),\ \varphi \otimes w \Big\rangle\, dz\ .$$

The limit problem takes the form

$$\min\Big\{ J(u, y) + I(u, y)\colon\ (u, y) \in U \times Y \Big\} \tag{8.8}$$

where

$$
\begin{aligned}
I(u,y) &= \\
&= \frac{1}{2} \inf_{w\in C_0(\Omega)} \Big\{ \langle A(y), w\rangle - \int_\Omega \langle \beta(z), w\rangle\, u(z)\, dz : \\
&\qquad \int_\Omega \Big\langle \gamma(z), \varphi\otimes w\Big\rangle\, d\mu(z) \;-\; \int_\Omega \Big\langle \beta(z)\otimes\beta(z),\, \varphi\otimes w\Big\rangle\, dz \;= \\
&\qquad\qquad = \langle A(y), \varphi\rangle - \int_\Omega \langle \beta(z), \varphi\rangle\, u(z)\, dz \quad \forall\, \varphi\in C_0(\Omega) \Big\} = \\
&= \frac{1}{2} \inf_{w\in C_0(\Omega)} \Big\{ \int_\Omega \Big\langle \gamma(z), w\otimes w\Big\rangle\, d\mu(z) - \int_\Omega \Big\langle \beta(z)\otimes\beta(z), w\otimes w\Big\rangle\, dz : \\
&\qquad \int_\Omega \Big\langle \gamma(z), \varphi\otimes w\Big\rangle\, d\mu(z) - \int_\Omega \Big\langle \beta(z)\otimes\beta(z),\, \varphi\otimes w\Big\rangle\, dz \;= \\
&\qquad\qquad = \langle A(y), \varphi\rangle - \int_\Omega \langle \beta(z), \varphi\rangle\, u(z)\, dz \quad \forall\, \varphi\in C_0(\Omega) \Big\}\;. \;\square
\end{aligned}
$$

The additional contribution $I$ to the limit cost becomes easier in some particular interesting cases. Let us look at them in the following examples.

**Example 8.7** (see also [10], Example 4.4)**.** Let us consider first the case of linear local input operators. It occurs when $\beta_h(z) = b_h(z)\cdot\delta_z$ where $b_h$ are $L^2$ functions and $\delta_z$ is the Dirac mass at $z$. With this choice we have $B_h u(z) = b_h(z)\, u(z)$, that is $B_h$ is the product operator with coefficient $b_h$. The convergence conditions (8.6) and (8.7) are satisfied by assuming that

$$
\begin{aligned}
&b_h \to b \quad \text{weakly in } \; L^2(\Omega)\;, \\
&b_h^2 \to \mu \quad \text{weakly* in } \; \mathcal{M}(\Omega)
\end{aligned}
$$

and taking $\gamma(z) = \delta_z\otimes\delta_z$. The limit problem becomes then

$$
\min_{U\times Y} \Big\{ J(u,y) + \frac{1}{2} \inf_{w\in C_0(\Omega)} \Big\{ \int_\Omega w^2\, d\mu(z) - \int_\Omega w^2\, b^2\, dz : \\
w\cdot\mu - w\, b^2\cdot dz = A(y) - ub\cdot dz \;\text{ in } \mathcal{M}(\Omega) \Big\} \Big\}\;.
$$

By splitting $\mu$ and $A(y)$ into their absolutely continuous and singular parts with respect to the Lebesgue measure $dz$ we obtain $A(y)^a - b\,u \;=\; w(\mu^a - b^2)$ and

$A(y)^s = w \cdot \mu^s$. Therefore it is easy to calculate the infimum and the limit problem takes the form

$$\min_{U\times Y}\left\{ J(u,y) + \frac{1}{2}\int_\Omega \frac{|A(y)^a - b\,u|^2}{\mu^a - b^2}\,dz + \frac{1}{2}\int_\Omega \left|\frac{dA(y)^s}{d\mu^s}\right|^2 d\mu^s :\ A(y) \ll \mu + dz \right\}.$$

In the regions where the equality $\mu^a - b^2 = 0$ holds we recover the limit state equation constraint again. It is worth notice that the equality $\mu - b^2 = 0$ holds on the whole space $\Omega$ if and only if $b_h \to b$ strongly.

In the particular case

$$\min_{\substack{u\in L^2(0,1)\\ y\in W^{1,1}(0,1)}} \left\{ \int_0^1 \left[\frac{u^2}{2} + |y - y_0|^2\right] dx :\ \ y' = a_h(x,y) + b_h(x)\,u,\ \ y(0) = \xi_h \right\}$$

where $a_h$, $b_h$ and $\xi_h$ satisfy the same assumptions of Example 7.4, the limit problem turns out to be

$$\min_{\substack{u\in L^2(0,1)\\ y\in BV(0,1)}} \left\{ \int_0^1 \left[\frac{u^2}{2} + |y - y_0|^2\right] dx \ +\ \frac{1}{2}\int_0^1 \frac{|y'^a - a(x,y) - b\,u|^2}{\mu^a - b^2}\,dx\ + \right.$$
$$\left. +\ \frac{1}{2}\int_{(0,1)} \left|\frac{dy'^s}{d\mu^s}\right|^2 d\mu^s + \frac{|y(0^+) - \xi|^2}{2\,\mu(\{0\})} :\quad y' \ll \mu + dx \right\}.$$

For instance, if $b_h(x) = \sin(hx) + \sqrt{h}\,1_{]0,1/h[}(x)$ we have $b_h \to 0$ weakly in $L^2(0,1)$ while $b_h^2 \to \frac{1}{2}\,dx + \delta_0$ weakly* in $\mathcal{M}([0,1[)$ and the limit problem takes the following form

$$\min_{\substack{u\in L^2(0,1)\\ y\in W^{1,1}(0,1)}} J(u,y) + \int_0^1 |y' - a(x,y)|^2\,dx + \frac{|y(0^+) - \xi|^2}{2}\ . \ \square$$

**Example 8.8.** Another case where we are able to make explicit computations is when $\beta_h$ is the tensor product of an $L^2$ function and a measure, that is $\beta_h(z) = b_h(z)\,\mu_h$ where $b_h$ is an $L^2$ function and $\mu_h \in \mathcal{M}(\Omega)$. With this choice we have

$$B_h :\ L^2(\Omega) \to \mathcal{M}(\Omega)\,, \qquad B_h\,u = \int_\Omega b_h(z)\,u(z)\,dz\ \cdot\ \mu_h\ .$$

Let us assume that

$$\mu_h \to \mu \quad \text{weakly* in } \mathcal{M}(\Omega)\ ,$$
$$b_h \to b \quad \text{weakly in } L^2(\Omega)\ ,$$
$$\|b_h\|_2^2 \to \gamma\ .$$

Then assumptions (8.6) and (8.7) are satisfied with $\mu$ as above, $\beta(x) = b(x)\,\mu$ and $\gamma(x) = \gamma$ and the limit problem 8.8 becomes in this case

$$\min_{U\times Y} J(u,y) + \frac{\left(c - \int_\Omega bu\,dz\right)^2}{2\left(\gamma - \|b\|_2^2\right)} + \chi_{A(y)=c\cdot\mu,\ c\in\mathbb{R}} \quad . \square$$

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## REFERENCES

**[1]** Aizicovici, S. and Papageorgiou, N.S. – A sensitivity analysis of volterra integral inclusions with applications to optimal control problems, *J. Math. Anal. Appl.,* 186 (1994), 97–119.

**[2]** Artstein, Z. – Chattering variational limits of control systems, *Forum Math.,* 5(4) (1993), 369–403.

**[3]** Attouch, H. – *Variational convergence for functions and operators*, Appl. Math. Ser., Pitman, Boston, 1984.

**[4]** Azé, D. – Convergence des variables duales dans des problèmes de transmission à travers des couches minces par des methodes d'épi convergence, *Ricerche Mat.,* 35 (1986), 125–159.

**[5]** Briani, A. – Convergence of Hamilton–Jacobi equations for sequences of optimal control problems, *Commun. Appl. Anal.,* to appear.

**[6]** Buttazzo, G. – Some relaxation problems in optimal control theory, *J. Math. Anal. Appl.,* 125 (1987), 272–287.

**[7]** Buttazzo, G. and Cavazzuti, E. – Limit problems in optimal control theory, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* 6 (1989), 151–160.

**[8]** Buttazzo, G. and Dal Maso, G. – Γ-convergence and optimal control problems, *J. Optim. Theory Appl.,* 38 (1982), 385–407.

**[9]** Buttazzo, G.; Drakhlin, M.E.; Freddi, L. and Stepanov, E. – Homogenization of optimal control problems for functional differential equations, *J. Optim. Theory Appl.,* 93(1) (1997), 103–119.

**[10]** Buttazzo, G. and Freddi, L. – Sequences of optimal control problems with measures as controls, *Adv. Math. Sci. Appl.,* 2(1) (1993), 215–230.

**[11]** Buttazzo, G. and Freddi, L. – Optimal control problems with weakly converging input operators, *Discrete Contin. Dynam. Systems,* 1(3) (1995), 401–420.

**[12]** Dal Maso, G. – *An introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[13] Denkowski, Z. and Migorski, S. – Control problems for parabolic and hyperbolic equations via the theory of $G$ and $\Gamma$-convergence, *Ann. Mat. Pura Appl.,* 149 (1987), 23–39.

[14] Denkowski, Z. and Mortola, S. – Asymptotic behaviour of optimal solutions to control problems for systems described by differential inclusions corresponding to partial differential equations, *J. Optim. Theory Appl.,* 78(2) (1993), 365–391.

[15] Freddi, L. – *Limits of control problems with weakly converging nonlocal input operators*, Preprint Dip. Matematica e Informatica, Udine, no. 12, 1998.

[16] Freddi, L. – *Optimal control problems in a nonreflexive framework and application to weakly converging nonlinear input operators*, International Conference on Differential Equations (EQUADIFF 95), World Scientific, 1995, 343–349.

[17] Kesavan, S. and Saint Jean Paulin, J. – Homogenization of an optimal control problem, *SIAM J. Control Optim.,* 35 (1997), 1557–1573.

[18] Migorski, S. – Asymptotic behaviour of optimal solutions in control problems for elliptic equations, *Riv. Mat. Pura Appl.,* 11 (1992), 7–28.

[19] Migorski, S. – On asymptotic limits of control problems with parabolic and hyperbolic equations, *Riv. Mat. Pura Appl.,* 12 (1992), 33–50.

[20] Migorski, S. – Sensitivity analysis of distributed parameter optimal control problems for nonlinear parabolic equations, *J. Optim. Theory Appl.,* 87 (1995), 595–613.

[21] Papageorgiou, N.S. – A convergence result for a sequence of distributed parameter optimal control problems, *J. Optim. Theory Appl.,* 68 (1991), 305–320.

[22] Papageorgiou, N.S. – On the variational stability of a class of nonlinear parabolic optimal control problems, *Z. Anal. Anwendungen,* 15 (1996), 245–262.

[23] Papageorgiou, N.S. – Optimal control, sensitivity analysis and relaxation of maximal monotone integrodifferential inclusions in $\mathbb{R}^n$, *Bull. Inst. Math. Acad. Sinica,* 25(1) (1997), 45–69.

[24] Spagnolo, S. – *Convergence in energy for elliptic operators*, Numerical Solutions of Partial Differential Equations III, Synspade 1975 (New York), Academic Press, 1976, 469–498.

[25] Valadier, M. – Fonctions et opérateurs sur les mesures, *C. R. Acad. Sci. Paris Ser. I,* 304 (1987), 135–137.

[26] Zolezzi, T. – Characterizations of some variational perturbations of the abstract linear-quadratic problem, *SIAM J. Control Optim.,* 16 (1978), 106–121.

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