PORTUGALIAE MATHEMATICA
Vol. 57 Fasc. 2 – 2000

FIBONACCI AND LUCAS NUMBERS WITH ONLY ONE DISTINCT DIGIT *

F. Luca

1 – Introduction

Let $(F_n)_n$ and $(L_n)_n$ be the Fibonacci and the Lucas sequence, respectively. The Fibonacci sequence $(F_n)_n$ is given by $F_0 = 0,\ F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for} \ \ n \geq 0 \ .$$

The Lucas sequence $(L_n)_n$ is given by $L_0 = 2,\ L_1 = 1$ and

$$L_{n+2} = L_{n+1} + L_n \quad \text{for} \ \ n \geq 0 \ .$$

In this note, we show that 55 is the largest member of the Fibonacci sequence whose decimal expansion has only one distinct digit. We also show that 11 is the largest member of the Lucas sequence whose decimal expansion has only one distinct digit.

More precisely, we have the following results:

Theorem 1. *If*

$$F_n = a \cdot \frac{10^m - 1}{9} \quad \textit{for some} \ \ 0 \leq a \leq 9 \ , \tag{1}$$

then $n = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 10$.

Theorem 2. *If*

$$L_n = a \cdot \frac{10^m - 1}{9} \quad \textit{for some} \ \ 0 \leq a \leq 9 \ , \tag{2}$$

then $n = 0,\ 1,\ 2,\ 3,\ 4,\ 5$.

Received: December 11, 1998.
AMS Subject Classification: 11A63, 11B39, 11D61.
* Financial support from the Alexander von Humboldt Foundation is gratefully acknowledged.

The results are not surprising. In fact, Mignotte (see [3] and [4]) showed that if $(u_n)_n$ and $(v_n)_n$ are two lineary recurrence sequences then, under some weak technical assumptions, the equation

$$u_n = v_m \quad \text{for some} \ \ m \ge 0 \ \ \text{and} \ \ n \ge 0$$

has only finitely many solutions and all such solutions are effectively computable. For example, if $(u_n)_n$ and $(v_n)_n$ are both nondegenerate binary recurrence sequences whose characteristic equations have real roots, then it suffices that the logarithms of the absolute values of the two largest roots to be linearly independent over $\mathbb{Q}$. This is certainly the case for our equations. However, finding all such solutions for two particular binary recurrence sequences has been, in general, a difficult problem.

Baker and Davenport (see [1]) have determined all the integer solutions of the system

$$\begin{cases} 3\,x^2 - 2 \;=\; y^2 \\ 8\,x^2 - 7 \;=\; z^2 \end{cases}$$

and Kanagasabapathy and Ponnudurai (see [2]) have solved the slightly modified version of the above system, namely

$$\begin{cases} y^2 - 3\,x^2 \;=\; -2 \\ z^2 - 8\,x^2 \;=\; -7 \end{cases}$$

Sansone (see [5]) has determined the solutions of the diophantine system

$$\begin{cases} N + 1 \;=\; x^2 \\ 3\,N + 1 \;=\; y^2 \\ 8\,N + 1 \;=\; z^2 \end{cases}$$

and the system

$$\begin{cases} z^2 - 3\,y^2 \;=\; -2 \\ z^2 - 6\,x^2 \;=\; -5 \end{cases}$$

was solved by Velupillai in [6].

What all these problems have in common is the fact that each one of them reduces to solving one or more equations of the type $u_n = v_m$ where $(u_n)_n$ and $(v_n)_n$ are certain binary recurrence sequences arising from the integer solutions of an appropriate Pell equation. In order to solve some of these problems some authors have employed very technical methods such as linear forms in logarithms followed by heavy computations.

In this paper, we show that determining all Fibonacci and Lucas numbers having only one distinct digit in their decimal representation can be done in an elementary way.

2 – The Proofs

We use the well-known relations

$$F_{2n} = F_n L_n \quad \text{for } n \geq 0 , \tag{3}$$

$$L_{2n} = L_n^2 + 2 \cdot (-1)^{n+1} \quad \text{for } n \geq 0 , \tag{4}$$

and

$$L_n^2 - 5 F_n^2 = 4 \cdot (-1)^n \quad \text{for } n \geq 0 . \tag{5}$$

Proof of Theorem 1: One can check that the values $n = 0, 1, 2, 3, 4, 5, 6, 10$ are the only values of $n \leq 20$ for which

$$F_n = a \cdot \frac{10^m - 1}{9} \quad \text{for some } m \geq 1 \text{ and } 1 \leq a \leq 9 .$$

From now on assume that $n \geq 21$. In particular,

$$F_n \geq F_{21} = 10946 > 10^5$$

so $m \geq 5$.

We first treat the case $a = 5$.

Case $a = 5$.

In this case

$$F_n = 5 \cdot \frac{10^m - 1}{9} \equiv 3 \pmod{16} .$$

It follows easily that $n \equiv 4 \pmod{24}$. In particular, $4 \mid n$. Since $5 \mid F_n$, it follows that $5 \mid n$. Hence, $20 \mid n$. Thus

$$11 \mid F_{10} \mid F_n = 5 \cdot \frac{10^m - 1}{9}$$

so

$$11 \Bigm| \frac{10^m - 1}{9} .$$

It follows that $2 \mid m$. Moreover,

$$3 \mid F_4 \mid F_n = 5 \cdot \frac{10^m - 1}{9}$$

so

$$3 \,\Big|\, \frac{10^m - 1}{9} \ .$$

It follows that $3 \mid m$. Hence, $6 \mid m$. It now follows that

$$7 \,\Big|\, \frac{10^6 - 1}{9} \,\Big|\, \frac{10^m - 1}{9} \,\Big|\, F_n$$

so $8 \mid n$. This contradicts the fact that $n \equiv 4 \pmod{24}$.

From now on we assume that $a \neq 5$. We first show that m is odd. Indeed, assume that m is even. Then,

$$11 \,\Big|\, \frac{10^2 - 1}{9} \,\Big|\, \frac{10^m - 1}{9} \,\Big|\, F_n \ .$$

Since $11 \mid F_n$, it follows that $10 \mid n$. Hence,

$$5 \mid F_{10} \mid F_n = a \cdot \frac{10^m - 1}{9}$$

which is impossible if $a \neq 5$.

We are now ready to treat the remaining cases.

Case $a = 1$.

In this case

$$F_n = \frac{10^m - 1}{9} \ .$$

Since $m \geq 5 > 4$, it follows that $F_n \equiv 7 \pmod{16}$. This implies $n \equiv 10 \pmod{24}$. Write $n = 2\,k$ where $k \equiv 5 \pmod{12}$. We have

$$F_n = F_{2k} = F_k L_k = \frac{10^m - 1}{9} \ .$$

Assume that p is a prime divisor of L_k. Clearly, $p \neq 2$. Since

$$L_k^2 - 5\,F_k^2 = -4 \ ,$$

it follows that

$$-5\,F_k^2 \equiv -4 \pmod{p} \ .$$

In particular, $\left(\frac{5}{p}\right) = 1$ so $\left(\frac{p}{5}\right) = 1$. Since p was an arbitrary prime dividing L_k, follows that

$$\left(\frac{L_k}{5}\right) = 1 \ . \tag{6}$$

Let now p be a prime divisor of F_k. Since

$$L_k^2 - 5\,F_k^2 = -4 \ ,$$

it follows that $L_k^2 \equiv -4 \pmod{p}$. In particular, $\left(\frac{-1}{p}\right) = 1$. Hence, $p \equiv 1 \pmod{4}$. Write

$$F_k = \prod_{i=1}^{t} p_i^{\alpha_i} \prod_{j=1}^{s} q_j^{\beta_j} \tag{7}$$

for some $t \geq 0$ and $s \geq 0$ where p_i and q_j are distinct prime numbers such that $p_i \equiv 1 \pmod{8}$ for all $i = 1, ..., t$ and $q_j \equiv 5 \pmod{8}$ for all $j = 1, ..., s$.

Let again p be a prime divisor of F_k. Since $p \mid (10^m - 1)$, it follows that

$$10^m \equiv 1 \pmod{p}$$

or

$$10 \cdot (10^{(m-1)/2})^2 \equiv 1 \pmod{p} \ .$$

Hence, $\left(\frac{10}{p}\right) = 1$.

If $p = p_i \equiv 1 \pmod{8}$, then we know that $\left(\frac{2}{p_i}\right) = 1$. We conclude that $\left(\frac{5}{p_i}\right) = 1$ so $\left(\frac{p_i}{5}\right) = 1$.

If $p = q_j \equiv 5 \pmod{8}$, then we know that $\left(\frac{2}{q_j}\right) = -1$. We conclude that $\left(\frac{5}{q_j}\right) = -1$ so $\left(\frac{q_j}{5}\right) = -1$.

The above argument and equations (6) and (7) show that

$$\left(\frac{F_n}{5}\right) = \left(\frac{F_k\,L_k}{5}\right) = \left(\frac{F_k}{5}\right) = (-1)^{\sum_{j=1}^{s} \beta_j} \ .$$

However,

$$F_n = \frac{10^m - 1}{9} \equiv 1 \pmod{5} \ .$$

Thus, $\left(\dfrac{F_n}{5}\right) = 1$. It follows that $\sum_{j=1}^{s} \beta_j$ is even. Since $p_i \equiv 1 \pmod 8$ for all $i = 1, ..., t$, $q_j \equiv 5 \pmod 8$ for all $j = 1, ..., s$ and $\sum_{j=1}^{s} \beta_j$ is even, it follows, by formula (7), that $F_k \equiv 1 \pmod 8$. On the other hand, $k \equiv 5 \pmod{12}$. The period of $(F_n)_n$ modulo 8 is 12. Hence, one should have

$$F_k \equiv F_5 \equiv 5 \pmod 8 .$$

This gives the desired contradiction.

Case $a = 2$.

We have

$$F_n = 2 \cdot \frac{10^m - 1}{9} \equiv -2 \pmod{16} .$$

The period of $(F_n)_n$ modulo 16 is 24. One can check that there is no k, $0 \le k \le 24$ such that $F_k \equiv -2 \pmod{24}$. Thus, this case is impossible.

Case $a = 3$.

We have $3 \mid F_n$; hence $4 \mid n$. We also have

$$F_n = 3 \cdot \frac{10^m - 1}{9} \equiv 5 \pmod{16} .$$

The only value of n modulo 24 such that $F_n \equiv 5 \pmod 8$ and $4 \mid n$ is $n = 8$. Thus, $n \equiv 8 \pmod{24}$. It follows that $8 \mid n$. It now follows that

$$7 \mid F_8 \mid F_n = 3 \cdot \frac{10^m - 1}{9}$$

so $7 \mid (10^m - 1)$. It follows that $6 \mid m$. This contradicts the fact that m is odd.

Case $a = 4$.

In this case, we have $4 \mid F_n$, therefore $6 \mid n$. Hence,

$$8 = F_6 \mid F_n = 4 \cdot \frac{10^m - 1}{9}$$

which is impossible.

Case $a = 6$.

In this case $6 \mid F_n$, therefore $12 \mid n$. Thus,

$$16 \mid F_{12} \mid F_n = 6 \cdot \frac{10^m - 1}{9}$$

which is impossible.

Case $a = 7$.

In this case $7 \mid F_n$ which implies that $8 \mid n$. On the other hand,

$$F_n = 7 \cdot \frac{10^m - 1}{9} \equiv 1 \pmod{16} .$$

The period of $(F_n)_n$ modulo 16 is 24. One can check that there is no k, $1 \le k \le 24$, such that $8 \mid k$ and $F_k \equiv 1 \pmod{16}$.

Case $a = 8$.

In this case $8 \mid F_n$ therefore $6 \mid n$. Moreover,

$$F_n = 8 \cdot \frac{10^m - 1}{9} \equiv -8 \pmod{32} .$$

The period of $(F_n)_n$ modulo 32 is 48. Since $6 \mid n$, one concludes that $n \equiv 18, 42 \pmod{48}$.

Suppose first that $n \equiv 18 \pmod{48}$. It follows that $n \equiv 2 \pmod{16}$. The sequence $(F_n)_n$ is periodic modulo 7 with period 16. Hence,

$$8 \cdot \frac{10^m - 1}{9} = F_n \equiv F_2 \equiv 1 \pmod 7$$

or

$$\frac{10^m - 1}{9} \equiv 1 \pmod 7 .$$

This implies $m \equiv 1 \pmod 6$. On the other hand, $n \equiv 2 \pmod 8$. The sequence $(F_n)_n$ is periodic modulo 3 with period 8. Hence,

$$8 \cdot \frac{10^m - 1}{9} = F_n \equiv F_2 \equiv 1 \pmod 3$$

or

$$\frac{10^m - 1}{9} \equiv -1 \pmod 3 .$$

This gives $m \equiv 2 \pmod 3$. Since m is odd, it follows that $m \equiv 5 \pmod 6$. This contradicts the fact that $m \equiv 1 \pmod 6$.

Assume now that $n \equiv 42 \pmod{48}$. It follows that $n \equiv 10 \pmod{16}$. The sequence $(F_n)_n$ is periodic modulo 7 with period 16. Thus

$$8 \cdot \frac{10^m - 1}{9} = F_n \equiv F_{10} \equiv -1 \pmod 7$$

or

$$\frac{10^m - 1}{9} \equiv -1 \pmod 7 .$$

This implies that $m \equiv 3 \pmod 6$. In particular, $3 \mid m$. Hence,

$$3 \;\Big|\; \frac{10^3-1}{9} \;\Big|\; \frac{10^m-1}{9} = F_n \;,$$

which implies that $3 \mid F_n$. This implies $4 \mid n$ which contradicts the fact that $n \equiv 42 \pmod{48}$.

Case $a = 9$.

In this case $9 \mid F_n$, therefore $12 \mid n$. It follows that

$$16 \mid F_{12} \mid F_n = 9 \cdot \frac{10^m-1}{9}$$

which is impossible.

Theorem 1 is therefore completely proved. ∎

Proof of Theorem 2: One can check that the values $n = 0, 1, 2, 3, 4, 5$ are the only values of $n \le 19$ such that

$$L_n = a \cdot \frac{10^m-1}{9} \quad \text{for some} \;\; m \ge 1 \;\; \text{and} \;\; 0 \le a \le 9 \;.$$

From now on assume that $n \ge 20$. In particular,

$$L_n \;\ge\; L_{20} \;=\; 15127 \;.$$

Hence, $m \ge 5$.

All cases except for $a = 1$ or 4 will follow immediately from the following lemma:

Lemma. *Assume*

$$L_n = a \cdot \frac{10^m-1}{9} \quad \textit{for some} \;\; a \ge 1 \;\; \textit{and} \;\; m \ge 3 \;.$$

Then n is odd.

Proof of the Lemma: Assume that n is even. We first show that m is odd. Indeed, assume that m is even. In this case,

$$11 \;\Big|\; \frac{10^2-1}{9} \;\Big|\; \frac{10^m-1}{9} \;\Big|\; L_n \;.$$

Hence, $11 \mid L_n$. It follows that $5 \mid n$ and n is odd which contradicts the assumption that n is even.

Assume now that $n = 2\,k$ where k is odd. Then,

$$L_n \;=\; L_{2k} \;=\; L_k^2 + 2 \;=\; a \cdot \frac{10^m - 1}{9} \;.$$

Let p be an arbitrary prime divisor of $(10^m - 1)/9$. Clearly, p is odd. Since

$$L_k^2 + 2 \;=\; 0 \pmod{p} \;,$$

it follows that $\left(\dfrac{-2}{p}\right) = 1$. Hence, $p \equiv 1, 3 \pmod 8$. Since this is true for all primes p dividing $(10^m - 1)/9$, it follows that

$$\frac{10^m - 1}{9} \;\equiv\; 1, 3 \pmod 8 \;.$$

On the other hand, since $m \geq 3$, it follows that

$$\frac{10^m - 1}{9} \;\equiv\; -1 \pmod 8 \;.$$

This contradiction disposes of this case.

Assume now that $n = 2\,k$ where k is even. Then,

$$L_n \;=\; L_{2k} \;=\; L_k^2 - 2 \;=\; a \cdot \frac{10^m - 1}{9} \;.$$

Let again p be any prime divisor of $(10^m - 1)/9$. Clearly, p is odd. Since

$$L_k^2 - 2 \;\equiv\; 0 \pmod{p} \;,$$

it follows that $\left(\dfrac{2}{p}\right) = 1$. Since $10^m - 1 \equiv 0 \pmod{p}$ and m is odd, it follows that

$$10\left(10^{(m-1)/2}\right)^2 \;\equiv\; 1 \pmod{p} \;.$$

Hence, $\left(\dfrac{10}{p}\right) = 1$. Since we already know that $\left(\dfrac{2}{p}\right) = 1$, we conclude that $\left(\dfrac{5}{p}\right) = 1$. Finally, we use the fact that

$$L_n^2 - 5\,F_n^2 \;=\; 4 \;.$$

Since

$$p \;\Big|\; \frac{10^m - 1}{9} \;\Big|\; L_n \;,$$

it follows that

$$-5\,F_n^2 \;\equiv\; 4 \pmod{p} \;.$$

Thus, $\left(\frac{-5}{p}\right)=1$. Since $\left(\frac{5}{p}\right)=1$, it follows that $\left(\frac{-1}{p}\right)=1$. Hence, $p \equiv 1 \pmod 4$. Since we also know that $\left(\frac{2}{p}\right)=1$, it follows that $p \equiv 1 \pmod 8$. Since the above argument applies to all prime divisors of $(10^m-1)/9$, we conclude that

$$\frac{10^m-1}{9} \equiv 1 \pmod 8 .$$

On the other hand,

$$\frac{10^m-1}{9} \equiv -1 \pmod 8$$

for $m \geq 3$. This gives the desired contradiction.

The lemma is therefore proved. ∎

We are now ready to prove Theorem 2.

Case $a = 1$.

We get

$$L_n = \frac{10^m-1}{9} \equiv 7 \pmod{16} .$$

It follows that $n \equiv 4,\ 11,\ 20 \pmod{24}$. Since n cannot be even, it follows that $n \equiv 11 \pmod{24}$. Hence, $n \equiv 3 \pmod 8$. The sequence $(L_n)_n$ is periodic modulo 3 with period 8. It follows that

$$\frac{10^m-1}{9} = L_n \equiv L_3 \equiv 1 \pmod 3 .$$

This implies $m \equiv 1 \pmod 3$. We distinguish two situations:

1. m is odd. In this case, $m \equiv 1 \pmod 6$. Thus,

$$L_n = \frac{10^m-1}{9} \equiv 1 \pmod 7 .$$

The sequence $(L_n)_n$ is periodic modulo 7 with period 16. Since $L_n \equiv 1 \pmod 7$ and n is odd, we get $n \equiv 1,\ 7 \pmod{16}$. In particular, $n \equiv 1,\ 7 \pmod 8$ which contradicts the fact that $n \equiv 3 \pmod 8$.

2. m is even. In this case, $m \equiv 4 \pmod 6$. Thus,

$$L_n = \frac{10^m-1}{9} \equiv 5 \pmod 7 .$$

One can check that there is no odd value of k, $1 \leq k < 16$ such that $L_k \equiv 5 \pmod 7$.

Case $a = 2$.

It follows that $L_n \equiv 2 \pmod 4$. This implies that $6 \mid n$ which contradicts the lemma.

Case $a = 3$.

In this case L_n is an odd multiple of 3. It follows that $n = 2k$ where k is odd. In particular, n is even which contradicts the lemma.

Case $a = 4$.

In this case L_n is a multiple of 4. It follows that $n = 3k$ where k is odd. Moreover,

$$L_n = 4 \cdot \frac{10^m - 1}{9} \equiv -4 \pmod{16} .$$

It follows that $n \equiv 9, 21 \pmod{24}$. In particular, $n \equiv 1 \pmod 4$. However,

$$L_n = 4 \cdot \frac{10^m - 1}{9} \equiv 4 \pmod 5 .$$

The sequence $(L_n)_n$ is periodic modulo 5 with period 4. Since $L_n \equiv 4 \pmod 5$, it follows that $n \equiv 3 \pmod 4$. This contradicts the fact that $n \equiv 1 \pmod 4$.

Case $a = 5$.

This is impossible because $5 \not| L_k$ for any $k \geq 0$.

Case $a = 6$.

In this case we have $6 \mid L_n$. This implies $6 \mid n$ which contradicts the lemma.

Case $a = 7$.

In this case we have $7 \mid L_n$. This implies that $4 \mid n$ which contradicts the lemma.

Case $a = 8$.

This is impossible because $8 \not| L_k$ for any $k \geq 0$.

Case $a = 9$.

In this case $9 \mid L_n$. This implies $6 \mid n$ which contradicts the lemma.

Theorem 2 is therefore completely proved. ∎

ACKNOWLEDGEMENTS – We would like to thank professor Andreas Dress and his research group in Bielefeld for their hospitality during the period when this paper was written.

REFERENCES

[1] Baker, A. and Davenport, H. – The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. of Math. Ser. (2),* 20 (1969), 129–137.

[2] Kanagasabapathy, P. and Ponnudurai, T. – The simultaneous Diophantine equations $y^2 - 3x^2 = -2$, $z^2 - 8x^2 = -7$, *Quart. J. of Math. Ser. (2),* 26 (1975), 275–278.

[3] Mignotte, M. – Intersection des images de certaines suites récurrentes linéaires, *Theor. Comp. Sci.,* 7 (1978), 117–121.

[4] Mignotte, M. – Une extension du théoreme de Skolem–Mahler, *C. R. Acad. Sci. Paris,* 288 (1979), 233–235.

[5] Sansone, G. – Il sistema diophanteo $N+1 = x^2$, $3N+1 = y^2$, $8N+1 = z^2$, *Ann. Mat. Pura Appl.,* 111(4) (1976), 125–151.

[6] Velupillai, M. – *The equations* $z^2 - 3y^2 = -2$ *and* $z^2 - 6x^2 = -5$, in "A Collection of Manuscripts Related to The Fibonacci Sequence" (V.E. Hoggatt and M. Bicknell-Johnson, Eds.), The Fibonacci Association, Santa Clara, 1980, 71–75.

Florian Luca,
Mathematics Department, Universität Bielefeld,
Postfach 10 01 31, 33501 Bielefeld – GERMANY
E-mail: fluca@Mathematik.Uni-Bielefeld.DE