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EXPONENTIAL STABILITY FOR THE WAVE EQUATION WITH WEAK NONMONOTONE DAMPING *

P. Martinez and J. Vancostenoble

Abstract: We consider the wave equation with a weak nonlinear internal damping. First for a weak monotone damping in dimension 2, we prove that the energy of strong solutions decays exponentially to zero. This improves earlier results of Komornik and Nakao.

Then we consider a class of nonmonotone dampings. For strong solutions, we give new results of strong asymptotic stability and we prove that the energy decays to zero with an explicit decay rate estimate.

1 – Introduction

In this paper, we consider the wave equation in a smooth bounded domain Ω of $\mathbb{R}^N$, $N \geq 1$. A control is exerced by means of a force which is a nonlinear function of the observed velocity. The system is the following:

$$\begin{cases} u'' - \Delta u + g(u') \,=\, 0 \quad \text{ in } \ \Omega \times \mathbb{R}_+ \ , \\ u = 0 \quad \text{ on } \ \partial\Omega \times \mathbb{R}_+ \ , \\ u(0) = u^0, \quad u'(0) = u^1 \ , \end{cases} \tag{1.1}$$

with $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and $g(0) = 0$.

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* This work was done while the authors were working in the Institut de Recherche Mathématique Avancée, Université Louis Pasteur Strasbourg I et CNRS, 7 rue René Descartes, 67 084 Strasbourg Cédex, France.

As usual, define the energy of the system by

$$E(t) = \frac{1}{2}\int_\Omega \Big(u'^2 + |\nabla u|^2\Big)\, dx \ .$$

First we study the case when g is monotone increasing, and then the nonmonotone case. We recall briefly some known results on these cases:

• *When g is increasing*, Dafermos [7] and Haraux [11] proved strong asymptotic stability for this problem i.e.

$$E(t) \to 0 \quad \text{when} \quad t \to +\infty \ ,$$

using the compactness of the trajectories in the energy space and LaSalle's invariance principle. Aassila [1] extended their results on some unbounded domains. See also Conrad and Pierre [6] for strong asymptotic stability results in an abstract framework.
Moreover, when the feedback term satisfies

$$\forall\, x \in \mathbb{R}, \quad \alpha|x| \le |g(x)| \le \beta|x| \tag{1.2}$$

for some positive constants α and β, it is easy to see that the energy decays exponentially to zero.

Komornik [14] and Nakao [17] extended some results of Haraux and Zuazua [12], of Conrad, Leblond and Marmorat [5] and of Zuazua [23] studying the case of increasing dampings that have a polynomial growth in zero and at infinity with different methods. In dimension $N \ge 2$, they proved that the energy decays polynomially to zero with an explicit decay rate estimate, even when the dissipation is *weak* at infinity, that means when

$$\frac{g(v)}{v} \to 0 \quad \text{as} \quad |v| \to +\infty \ .$$

In particular, Nakao [17] considered the function

$$g(v) = \frac{v}{\sqrt{1+v^2}} \ , \tag{1.3}$$

which has finite limits at infinity. He noted that, in one space dimension, the energy decays exponentially; in dimension 2, he proved that the energy decays faster than t^{-m} for all $m \in \mathbb{N}$:

$$\forall\, t \in \mathbb{R}_+ \,, \qquad E(t) \le \frac{C(m)}{t^m} \quad \text{for all } m \in \mathbb{N} \ ,$$

with $C(m)$ depending on the norm of the initial conditions in $H^2(\Omega) \times H_0^1(\Omega)$; in higher dimensions, he proved that the energy decays polynomially.
Their proofs are based on the boundedness of the trajectories in $H^2(\Omega) \times H_0^1(\Omega)$, on the theorem of Gagliardo–Nirenberg and on the polynomial form of the dissipation.

In this paper, we adapt their methods to study the case of weak dissipations. In dimension 2, we show that if g is increasing, the behavior of g at infinity has no real effect on the decay rate of the energy of strong solutions (Theorem 2): if $g'(0) \neq 0$ (for example in the case (1.3)), we show that the energy decays exponentially (the decay rate depending on the norm of the initial conditions in $H^2(\Omega) \times H_0^1(\Omega)$).

• *When g is nonmonotone*, few results seem to be known. We assume that

$$x\, g(x) \geq 0 \quad \text{ for all } \; x \in \mathbb{R}\ ,$$

which implies that the energy is nonincreasing and that the trajectories are bounded in the energy space. To our knowledge, the trajectories are not compact in general. If g is globally Lipschitz, Slemrod [20] proved weak asymptotic stability for the problem (1.1) i.e.

$$(u(t), u'(t)) \rightharpoonup (0,0) \quad \text{ when } \; t \to +\infty$$

weakly in $H_0^1(\Omega) \times L^2(\Omega)$. One of the authors proved that this result still occurs for all global solutions of (1.1) even if g is not globally Lipschitz (see [22]). See also Feireisl [8] for a strong stability result in the one dimensional case (and [19] for a similar result in the case of a boundary feedback).

In the particular case (1.2), it is easy to see that the classical results of exponential stability, obtained for g monotone, remain valid for g nonmonotone once the problem is well posed.

When (1.2) is replaced by a weaker assumption, the proofs of Komornik [14] and Nakao [17] for g monotone cannot be extended to the nonmonotone case. Indeed, they are based on the boundedness of strong solutions in $H^2(\Omega) \times H_0^1(\Omega)$, provided by the monotonicity. See also Aassila [2] for nonmonotone feedback with the hypothesis that u' is bounded.

In this paper, we consider nonmonotone functions g of class $\mathcal{C}^1$, satisfying

$$\forall\, x \in \mathbb{R}, \quad g'(x) \geq -m\ ,$$

$$\forall\, x \in \mathbb{R}, \quad c_1 \,\frac{|x|}{\Big(\ln(2+|x|)\Big)^k} \leq |g(x)| \leq c_2\, |x|^q\ ,$$

with $c_1 > 0$, $c_2 > 0$, $m \geq 0$, $q \geq 0$ and $k \in [0,1]$. We prove that the problem (1.1) is well posed and we estimate the norm of strong solutions in $H^2(\Omega) \times H_0^1(\Omega)$. Then we show that the energy of strong solutions decays to zero with an explicit decay rate estimate, (Theorem 3 for $N=2$ and Theorem 4 for $N \geq 3$). The proof is based on a method recently introduced by one of the authors in [16], that allows one to compensate the lack of a priori uniform bound of strong solutions in $H^2(\Omega) \times H_0^1(\Omega)$ and on a new nonlinear integral inequality (Lemma 6) that generalizes a result of Haraux [9].

We make precise our results in Section 2 (see Theorem 2 for the monotone case and Theorem 3 and Theorem 4 for the nonmonotone case) and we apply them on some examples. We establish the well posedness of the problem in Section 3. We prove Theorem 2 in Section 4, Theorem 3 in Section 5, and Theorem 4 in Section 6.

2 – Statement of the problem and main results

Let Ω be a bounded open set of $\mathbb{R}^N$ of class $\mathcal{C}^2$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^1$. We denote $\mathbb{R}_+ := [0, +\infty)$ and we consider the evolutionary problem

$$u'' - \Delta u + g(u') = 0 \quad \text{in} \;\; \Omega \times \mathbb{R}_+ \;, \tag{2.1}$$

$$u = 0 \quad \text{on} \;\; \partial\Omega \times \mathbb{R}_+ \;, \tag{2.2}$$

$$u(0) = u^0, \;\; u'(0) = u^1 \;, \tag{2.3}$$

where (u^0, u^1) is given in $\mathcal{Z}$, which is the subset of $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by

$$\mathcal{Z} := \Big\{ (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega),\; -\Delta u + g(v) \in L^2(\Omega) \Big\} \;. \tag{2.4}$$

We will denote

$$C(u^0, u^1) := \| - \Delta u^0 + g(u^1)\|^2_{L^2(\Omega)} + \|u^1\|^2_{H_0^1(\Omega)} \;. \tag{2.5}$$

As usual, we define the energy of the solution u by

$$\forall\, t \in \mathbb{R}_+, \quad E(t) = \frac{1}{2} \int_\Omega \Big(u'^2 + |\nabla u|^2 \Big)\, dx \;. \tag{2.6}$$

2.1. Well posedness

Assume that g is a function of class $\mathcal{C}^1$ that satisfies

$$\forall\, x \in \mathbb{R}, \quad x\, g(x) \geq 0 \ , \tag{2.7}$$

$$\forall\, x \in \mathbb{R}, \quad g'(x) \geq -m \ , \tag{2.8}$$

with $m \geq 0$. Then existence and regularity of the solution u of (2.1)–(2.3) are given by the

Theorem 1.

(i) *Assume (2.8). Then the problem (2.1)–(2.3) is well posed: for any $(u^0, u^1) \in \mathcal{Z}$ such that $-\Delta u^0 + g(u^1) \in L^2(\Omega)$, there exists a unique strong solution $u(t)$ satisfying*

$$\forall\, t \in \mathbb{R}_+, \quad \Big(u(t), u'(t)\Big) \in \mathcal{Z} \ ,$$

and, for any $T > 0$,

$$\Big(u(\cdot), u'(\cdot)\Big) \in W^{1,\infty}\Big([0,T];\, H_0^1(\Omega) \times L^2(\Omega)\Big) \ .$$

(ii) *Moreover, if we also assume (2.7), then we have the following energy estimate:*

$$\forall\, t \in \mathbb{R}_+, \quad \|u(t)\|^2_{H_0^1(\Omega)} + \|u'(t)\|^2_{L^2(\Omega)} \leq \|u^0\|^2_{H_0^1(\Omega)} + \|u^1\|^2_{L^2(\Omega)} \ , \tag{2.9}$$

and so

$$\Big(u(\cdot), u'(\cdot)\Big) \in W^{1,\infty}\Big(\mathbb{R}_+;\, H_0^1(\Omega) \times L^2(\Omega)\Big) \ .$$

(iii) *We also have the following estimate:*

$$\forall\, t \in \mathbb{R}_+, \quad \|-\Delta u(t) + g(u'(t))\|^2_{L^2(\Omega)} + \|u'(t)\|^2_{H_0^1(\Omega)} \leq C(u^0, u^1)\, e^{2mt} \ . \tag{2.10}$$

Remark. In the monotone case ($m = 0$), Theorem 1 gives a classical result of existence and regularity of theory of maximal monotone operators. In particular, part (iii) implies

$$u' \in L^\infty(\mathbb{R}_+, H_0^1(\Omega)) \ . \tag{2.11}$$

This estimate is strictly provided by the monotonicity. It is essential in the proofs in [14] and in [17]. Our proof of exponential stability (Theorem 2) will also be based on this estimate. □

Remark. In the classical case

$$\forall\, x \in \mathbb{R}, \quad |g(x)| \le \beta |x| \ , \tag{2.12}$$

it is easy to check that Theorem 1 gives that, for any $(u^0, u^1) \in H^2(\Omega) \times H_0^1(\Omega)$,

$$\forall\, t \in \mathbb{R}_+, \quad \Big(u(t), u'(t)\Big) \in H^2(\Omega) \times H_0^1(\Omega) \ ,$$

and

$$\forall\, t \in \mathbb{R}_+, \quad \|u(t)\|^2_{H^2(\Omega)} + \|u'(t)\|^2_{H_0^1(\Omega)} \ \le\ \Big(\|u\|^2_{H^2(\Omega)} + \|u^1\|^2_{H_0^1(\Omega)}\Big)\, e^{2mt} \ . \ \square$$

2.2. Exponential stability when g is increasing

We already know from the principle of LaSalle that the energy of these solutions decays to zero at infinity. Our main result is the following decay rate estimate:

Theorem 2. *Assume that $N = 2$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing function of class $\mathcal{C}^1$ such that $g(0) = 0$, $g'(0) \ne 0$ and*

$$\forall\, |x| \ge 1, \quad |g(x)| \le c\, |x|^q \tag{2.13}$$

with $c \ge 0$ and $q \ge 0$.
Given $(u^0, u^1) \in \mathcal{Z}$, the energy of the solution $u(t)$ of (2.1)–(2.3) decays exponentially: there exists an explicit constant ω, depending on $C(u^0, u^1)$ such that

$$\forall\, t \ge 0, \quad E(t) \le E(0)\, e^{1-\omega t} \ . \tag{2.14}$$

Remarks.

1. Theorem 2 improves earlier results of [14] and of [17] who showed that the energy decays faster than t^{-m} for all $m \in \mathbb{N}$.

2. In fact the weakness of g at infinity has no real effect on the decreasingness of the energy of *strong* solutions: we find the same estimate on the energy as if g would satisfy

$$\alpha |v| \le |g(v)| \le \beta |v| \quad \text{for all} \ \ v, \quad \text{with } \alpha > 0 \ .$$

The only difference comes from the fact that the decay rate depends on $C(u^0, u^1)$. $\square$

Example. Consider

$$g(v) = \frac{v}{\sqrt{1+v^2}} \qquad \text{for all} \quad v \in \mathbb{R} ,$$

that satisfies (2.12). Then (2.14) gives the estimate

$$E(t) \;\leq\; E(0)\, e^{1-\omega t} ,$$

and the proof of Theorem 2 gives

$$\omega \;=\; \frac{c}{1 + C'(u^0, u^1)} ,$$

where $C'(u^0,u^1) = \|(u^0,u^1)\|_{H^2(\Omega)\times H^1_0(\Omega)}$ and c is a constant that depends on Ω. □

2.3. Exponential stability for a class of nonmonotone dampings

Assume that g is a function of class $\mathcal{C}^1$ that satisfies

$$\forall\, x \in \mathbb{R}, \quad x\, g(x) \geq 0 , \tag{2.15}$$

$$\forall\, x \in \mathbb{R}, \quad g'(x) \geq -m , \tag{2.16}$$

$$\forall\, |x| \geq 1, \quad c_1 \frac{|x|}{\Big(\ln(2+|x|)\Big)^k} \leq |g(x)| \leq c_2\, |x|^q , \tag{2.17}$$

with $c_1 > 0$, $c_2 > 0$, $m \geq 0$, $q \geq 0$ and $k \in [0,1]$.

Theorem 3. *Assume $N = 2$ and let g be a function satisfying (2.15)–(2.17) such that $g'(0) \neq 0$. Given $(u^0,u^1) \in \mathcal{Z}$, there exists an explicit positive constant ω, depending on $C(u^0,u^1)$ such that the energy of the solution $u(t)$ of (2.1)–(2.3) satisfies the following estimate*

$$\forall\, t \geq 0, \quad E(t) \leq E(0)\, e^{1+\omega}\, e^{-\omega(1+t)^{1-k}} \qquad \text{if} \;\; k \in [0,1) , \tag{2.18}$$

$$\forall\, t \geq 0, \quad E(t) \leq \frac{e\, E(0)\, 2^\omega}{(2+t)^\omega} \qquad \text{if} \;\; k = 1 . \tag{2.19}$$

Remark. Theorem 3 implies strong asymptotic stability results: the energy of strong solutions decays to zero, with an explicit decay rate estimate. □

Example. Theorem 3 can be applied to the odd function defined on $\mathbb{R}_+$ by

$$\forall\, x \geq 0, \quad g(x) \;=\; \Big(\sin(\theta(x))\Big)^2\, x \Big(\ln(x+2)\Big)^q + \Big(\cos(\theta(x))\Big)^2 \frac{x}{\Big(\ln(x+2)\Big)^k} ,$$

with $q \in [0,1]$, $k \in [0,1]$ and

$$\theta(x) = \Big(\ln(x+2)\Big)^{1-q} \quad \text{if } \ q \in [0,1) \ ,$$
$$\theta(x) = \ln\Big(\ln(x+2)\Big) \quad \text{if } \ q = 1 \ .$$

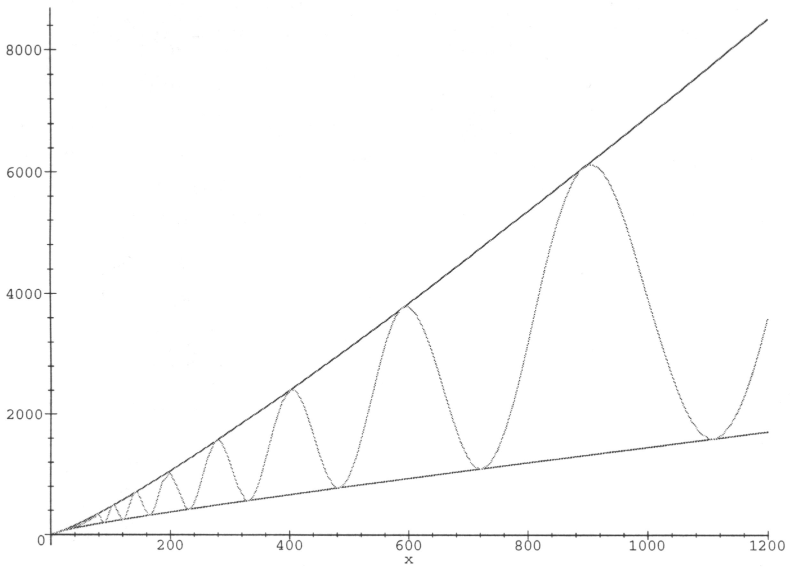
(Note that θ is strictly increasing and $\theta(x) \to +\infty$ as $x \to +\infty$. One can check that g is nonmonotone, g' is bounded if $q = 0$ and just bounded from below if $q > 0$.)
If $k = \frac{1}{2}$, the energy of strong solutions decays as

$$\forall\, t \geq 0, \quad E(t) \leq C\, E(0)\, e^{-\omega\sqrt{t}} \ ,$$

(C and ω depending on $C(u^0, u^1)$).
For example, for

$$\begin{aligned} g(x) \ = \ & \Big(\sin\Big(50\ln\Big(\ln(x+2)\Big)\Big)\Big)^2 \, x \, \ln(x+2) \\ & + 10\,\Big(\cos\Big(50\ln\Big(\ln(x+2)\Big)\Big)\Big)^2 \, \frac{x}{\ln(x+2)} \ , \end{aligned}$$

then, the graph of g is



2.4. Comments and extensions

1. Our method is not specific to dimension $N = 2$: in higher dimension, we also obtain

Theorem 4. *Assume $N \geq 3$ and let g be a function satisfying (2.15)–(2.17) with*
$$0 \leq k \leq 1 \quad \text{and} \quad 1 \leq q \leq \frac{N+2}{N-2} \ ,$$
and such that $g'(0) \neq 0$.

Given $(u^0, u^1) \in \mathcal{Z}$, there exists two positive constants C, ω such that the energy of the solution $u(t)$ of (2.1)–(2.3) satisfies the following estimate

$$\forall\, t \geq 0, \quad E(t) \leq C E(0)\, e^{-\omega(1+t)^{1-k}} \quad \text{if} \ \ k \in [0,1) \ , \tag{2.20}$$

$$\forall\, t \geq 0, \quad E(t) \leq \frac{C\, E(0)\, 2^{\omega}}{(2+t)^{\omega}} \quad \text{if} \ \ k = 1 \ . \tag{2.21}$$

Remark. If g is increasing ($m = 0$) and $N \leq 3$, then the same estimate holds for all $q \geq 1$. This improves earlier results of [17]. □

2. Applying the method described in [16], we could eliminate the assumption $g'(0) \neq 0$ and we would still obtain decay rate estimates, even when g has not a polynomial growth in zero.

3. We can extend the previous results to a control force exerced on a part of Ω. We consider the equation
$$u'' - \Delta u + a(x)\, g(u') = 0 \quad \text{in} \ \ \Omega \times \mathbb{R}_+ \ ,$$
where $a\colon \Omega \to \mathbb{R}$ is continuous positive function such that, for example, the region where $a(x) \geq \alpha > 0$, contains a neighbourhood of $\partial\Omega$ or at least a neighbourhood of
$$\Gamma(x^0) := \Big\{ x \in \partial\Omega, \ (x - x^0) \cdot \nu(x) \geq 0 \Big\} \ ,$$
where ν is the outward unit normal to Ω and $x^0 \in \mathbb{R}^2$ (see Zuazua [24]) or even more general conditions, (see [16]). See also Nakao [18] and Tcheugoue Tebou [21].

4. All the previous results are still true if we just assume that $g\colon \mathbb{R} \to \mathbb{R}$ is continuous such that
$$\forall\, x_1, x_2 \in \mathbb{R}, \ x_1 \neq x_2, \quad \frac{g(x_1) - g(x_2)}{x_1 - x_2} \geq -m \ ,$$
and g is of class $\mathcal{C}^1$ in a neighborhood of 0 such that $g'(0) \neq 0$.

3 – Well posedness and a priori estimates

3.1. Well posedness

Letting $v = u'$, we may rewrite (2.1)–(2.3) in the form

$$\text{(3.1)} \qquad \begin{cases} u' - v = 0 \ , \\ v' - \Delta u + g(v) = 0 \ , \\ u(0) = u^0 \ , \\ v(0) = u^1 \ . \end{cases}$$

We introduce the Hilbert space $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ and write (3.1) as the one order evolution equation in $\mathcal{H}$ for the vector $U(t) = (u(t), v(t))$:

$$\text{(3.2)} \qquad \begin{cases} \dfrac{dU}{dt}(t) + \mathcal{A}\, U(t) + \mathcal{L}\, U(t) = 0 \ , \\ U(0) = (u^0, u^1) \ . \end{cases}$$

We denote $\tilde{g}$ the monotone increasing function

$$\forall\, x \in \mathbb{R}, \quad \tilde{g}(x) = g(x) + m\, x \ .$$

We define the nonlinear operator $\mathcal{A}$ by

$$\begin{cases} D(\mathcal{A}) = \Big\{ (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid -\Delta u + \tilde{g}(v) \in L^2(\Omega) \Big\} \ , \\ \forall\, (u,v) \in D(\mathcal{A}), \quad \mathcal{A}(u,v) = \Big(-v, \ -\Delta u + \tilde{g}(v) \Big) \ . \end{cases}$$

$\mathcal{A}$ is a maximal monotone operator in $\mathcal{H}$, (see Haraux [10], Theorem 45, p. 90). Note that $D(\mathcal{A}) = \mathcal{Z}$.

Then we define $\mathcal{L} \colon \mathcal{H} \to \mathcal{H}$ by

$$\forall\, (u,v) \in \mathcal{H}, \quad \mathcal{L}(u,v) = (0, -m\, v) \ .$$

Clearly, $\mathcal{L}$ is Lipschitzian.

So we can apply the following theorem about Lipschitz perturbations of a maximal monotone operator:

Theorem 5 (Brézis [4], Theorem 3.17 and Remark 3.14)**.** *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} \colon D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be a maximal monotone operator and $\mathcal{L} \colon \mathcal{H} \to \mathcal{H}$ be a Lipschitzian operator.*

Then, for all $U_0 \in D(\mathcal{A})$, there exists a unique $U\colon [0,+\infty) \to \mathcal{H}$ such that:

$$\begin{gathered}
U(0) = U_0 \ , \\
\forall\, t \geq 0, \quad U(t) \in D(\mathcal{A}) \ , \\
\forall\, T > 0, \quad U(\cdot) \in W^{1,\infty}((0,T);\mathcal{H}) \ , \\
\frac{dU}{dt}(t) + \mathcal{A}\, U(t) + \mathcal{L}\, U(t) = 0 \qquad p.p. \;\; t \in (0,+\infty) \ . \blacksquare
\end{gathered}$$

This proves the first part of Theorem 1. ∎

3.2. A priori estimates

Note that since we consider only *strong* solutions, the previous regularity results allow us to justify the following computations, where we omit to write the differential elements in order to simplify the expressions. We will denote by c all the constants that depend only on the structure of the problem (Ω, g) and by C all the constants that depend also on

$$\| -\Delta u^0 + g(u^1)\|^2_{L^2(\Omega)} + \|u^1\|^2_{H^1_0(\Omega)} \ .$$

First we verify that the energy is nonincreasing:

Lemma 1. *Assume (2.7) and (2.8). Then*

$$\forall\, 0 \leq S < T < +\infty, \quad E(T) - E(S) = -\int_S^T \int_\Omega u'\, g(u')\, dx\, dt \ \leq 0 \ . \tag{3.3}$$

Remark. Since $x\, g(x) \geq 0$ for all $x \in \mathbb{R}$, it follows that the energy is nonincreasing, locally absolutely continuous and

$$E'(t) = -\int_\Omega u'\, g(u')\, dx \qquad \text{a.e. in} \ \ \mathbb{R}_+ \ . \ \square$$

Proof of Lemma 1: We multiply (2.1) by u' and we integrate by parts on $\Omega \times [S,T]$:

$$-\int_S^T \int_\Omega u'\, g(u') \ = \int_S^T \int_\Omega u'(u'' - \Delta u) \ = \ \left[\frac{1}{2}\int_\Omega u'^2 + |\nabla u|^2\right]_S^T = \ E(T) - E(S) \ . \blacksquare$$

This proves the part (ii) of Theorem 1. ∎

Next we prove the part (iii) of Theorem 1:

Lemma 2. *Assume (2.8). Then*

$$\forall\, t \geq 0, \quad \| - \Delta u(t) + g(u'(t))\|^2_{L^2(\Omega)} + \|u'(t)\|^2_{H^1_0(\Omega)} \;\leq\; C(u^0, u^1)\, e^{2mt}\ . \tag{3.4}$$

Proof of Lemma 2: Denote $w := u'$. Then differentiating (2.1)–(2.3) with respect to time, we see that w satisfies

$$w'' - \Delta w + g'(w)\, w' = 0 \quad \text{in} \;\; \Omega \times \mathbb{R}_+\ , \tag{3.5}$$

$$w = 0 \quad \text{on} \;\; \partial\Omega \times \mathbb{R}_+\ , \tag{3.6}$$

$$w(0) = u^1, \;\; w'(0) = \Delta u^0 - g(u^1)\ . \tag{3.7}$$

We multiply (3.5) by w' and we integrate by parts on $\Omega \times [S, T]$:

$$-\int_0^t \int_\Omega g'(w)\, w'^2 \;=\; \int_0^t \int_\Omega w'(w'' - \Delta w) \;=\; \left[\frac{1}{2}\int_\Omega w'^2 + |\nabla w|^2\right]_0^t\ .$$

So

$$\left[\int_\Omega u''^2 + |\nabla u'|^2\right]_0^t \leq\; 2\, m \int_S^T \int_\Omega u''^2\, dx\, d\tau\ ,$$

i.e.

$$\| - \Delta u(t) + g(u'(t))\|^2_{L^2(\Omega)} + \|u'(t)\|^2_{H^1_0(\Omega)} \;\leq$$
$$\leq\; C(u^0, u^1) + 2\, m \int_S^T \| - \Delta u(\tau) + g(u'(\tau))\|^2_{L^2(\Omega)}\, d\tau\ .$$

We apply Gronwall's lemma to get (3.4). ∎

3.3. Inequality given by the multiplier method

Lemma 3. *Assume (2.8). Let Ω be a bounded domain of class $\mathcal{C}^2$ in $\mathbb{R}^N$. Let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing concave function of class $\mathcal{C}^2$. Set $\sigma \geq 0$. Assume that g is a function of class $\mathcal{C}^1$ that satisfies $g'(0) \neq 0$ and*

$$\forall\, |x| \geq 1, \quad |g(x)| \leq c|x|^q \quad \textit{with} \;\; 1 \leq q \leq \frac{N+2}{\max(0, N-2)}\ .$$

Given $(u^0, u^1) \in \mathcal{Z}$, there exists $c > 0$ that depends on Ω such that the solution $u(t)$ of (2.1)–(2.3) satisfies

$$\int_S^T E(t)^{1+\sigma}\,\phi'(t)\,dt \;\le\; c\,E(S)^{1+\sigma} + c\int_S^T E(t)^{\sigma}\,\phi'(t)\int_\Omega u'^2\,dx\,dt\;. \tag{3.8}$$

Note that, if $N = 2$, (3.8) is true for all $q \ge 1$. Note also that, if g is increasing ($m = 0$) and $N \le 3$, then we can prove that the same estimate holds for all $q \ge 1$.

Remark. The proof of (3.8) is based on multiplier techniques; the constant c is explicit. (3.8) is classical when $\phi'(t) = 1$ (see, e.g., [14]). ϕ' will be closely related on the behavior of g at infinity. □

Proof of Lemma 3: First integrate by parts the following expression:

$$\begin{aligned}
0 &= \int_S^T E^\sigma \phi' \int_\Omega u\,(u'' - \Delta u + g(u'))\\
&= \left[\int_\Omega (E^\sigma \phi'\,u)\,u'\right]_S^T - \int_S^T\int_\Omega (E^\sigma\phi'\,u)'\,u'\\
&\quad - \int_S^T E^\sigma\phi' \int_{\partial\Omega} u\,\partial_\nu u + \int_S^T E^\sigma\phi'\int_\Omega |\nabla u|^2 + u\,g(u')\\
&= \left[E^\sigma\phi'\int_\Omega u\,u'\right]_S^T - \int_S^T \Big(\sigma\,E' E^{\sigma-1}\,\phi' + E^\sigma\phi''\Big)\int_\Omega u\,u'\\
&\quad - \int_S^T E^\sigma\phi'\int_\Omega 2\,u'^2 + \int_S^T E^\sigma\phi'\int_\Omega u'^2 + |\nabla u|^2 + u\,g(u')\;.
\end{aligned}$$

So

$$\begin{aligned}
2\int_S^T E^{1+\sigma}\phi' &= -\left[E^\sigma\phi'\int_\Omega u\,u'\right]_S^T + \int_S^T\Big(\sigma\,E'E^{\sigma-1}\,\phi' + E^\sigma\phi''\Big)\int_\Omega u\,u'\\
&\quad + \int_S^T E^\sigma\phi'\int_\Omega 2\,u'^2 - u\,g(u')\;.
\end{aligned} \tag{3.9}$$

Since ϕ' is nonnegative and nonincreasing, ϕ' is bounded on $\mathbb{R}_+$ and we have

$$\left|E^\sigma\phi'(t)\int_\Omega u\,u'\,dx\right| \;\le\; c\,E(t)^{1+\sigma}\;,$$

and

$$\left|\int_S^T \sigma\,E'E^{\sigma-1}\,\phi'\int_\Omega u\,u'\,dx\,dt\right| \;\le\; c\,E(S)^{1+\sigma}\;.$$

Since ϕ'' is nonpositive,

$$\left|\int_S^T E^\sigma \phi'' \int_\Omega u\,u'\,dx\,dt\right| \le c\,E(S)^{1+\sigma} \int_S^T -\phi''(t)\,dt$$
$$\le c\,E(S)^{1+\sigma} \phi'(S) \le c\,E(S)^{1+\sigma} .$$

It remains to estimate the last term of (3.9):

Lemma 4. *There exists $c > 0$ depending on Ω such that, for all $\varepsilon > 0$, we have*

$$\int_S^T E^\sigma \phi' \int_\Omega u\,g(u')\,dx\,dt \le \tag{3.10}$$
$$\le c_\varepsilon\,E(S)^{1+\sigma} + \frac{c}{\varepsilon}\int_S^T E^\sigma \phi' \int_\Omega u'^2\,dx\,dt + \varepsilon \int_S^T E^{1+\sigma}\phi'\,dt$$

Proof of Lemma 4: There exists $\lambda > 0$ such that

$$|g(x)| \le \lambda |x| \quad \text{if} \;\; |x| \le 1 .$$

Then set $\eta > 0$.

$$\int_S^T E^\sigma \phi' \int_{|u'|\le 1} u\,g(u')\,dx\,dt \le \int_S^T E^\sigma \phi' \int_{|u'|\le 1} \frac{\eta}{2}\,u^2 + \frac{1}{2\,\eta}\,g(u')^2$$
$$\le \frac{c\,\eta}{2}\int_S^T E^{1+\sigma}\phi' + \int_S^T E^\sigma \phi' \int_{|u'|\le 1} \frac{1}{2\,\eta}\,g(u')^2$$
$$\le \frac{c\,\eta}{2}\int_S^T E^{1+\sigma}\phi' + \int_S^T E^\sigma \phi' \int_\Omega \frac{\lambda^2}{2\,\eta}\,u'^2 .$$

Next we look at the part $|u'| > 1$: since $q \le \frac{N+2}{\max(0,N-2)}$,

$$H^1(\Omega) \subset L^{q+1}(\Omega) ,$$

and so

$$\|u\|_{L^{q+1}(\Omega)} \le c\,\|u\|_{H^1(\Omega)} \le c\sqrt{E} .$$

Then

$$\int_S^T E^\sigma \phi' \int_{|u'|>1} u\, g(u')\, dx\, dt \le$$

$$\begin{aligned}
&\le \int_S^T E^\sigma \phi' \Big(\int_\Omega |u|^{q+1}\Big)^{1/(q+1)} \Big(\int_{|u'|>1} |g(u')|^{(q+1)/q}\Big)^{q/(q+1)} \\
&\le c \int_S^T E^{\sigma+\frac{1}{2}}\, \phi' \Big(\int_{|u'|>1} u'\, g(u')\Big)^{q/(q+1)} \\
&\le c \int_S^T \phi' E^{\sigma+\frac{1}{2}} (-E')^{q/(q+1)} \\
&\le c \int_S^T \phi' \Big(E^{\sigma+\frac{1}{2}-\frac{q\sigma}{q+1}}\Big)\Big((-E')^{q/(q+1)} E^{\frac{q\sigma}{q+1}}\Big) \\
&\le c\, \eta^{q+1} \int_S^T \phi' E^{(q+1)(\sigma+\frac{1}{2}-\frac{q\sigma}{q+1})} + \frac{c}{\eta^{(q+1)/q}} \int_S^T \phi' (-E' E^\sigma) \\
&\le c\, \eta^{q+1} E(0)^{(q-1)/2} \int_S^T \phi' E^{1+\sigma} + \frac{c}{\eta^{(q+1)/q}}\, E(S)^{1+\sigma}\ .
\end{aligned}$$

Thus we get (3.10) by choosing η small enough. ∎

Therefore we deduce from the three last estimates that

$$2 \int_S^T E(t)^{1+\sigma}\, \phi'(t)\, dt \le$$

$$\le c\, E(S)^{1+\sigma} + \frac{c}{\varepsilon} \int_S^T E(t)^\sigma\, \phi'(t) \int_\Omega u'^2\, dx\, dt + \varepsilon \int_S^T E(t)^{1+\sigma}\, \phi'(t)\, dt\ .$$

We get (3.8) choosing ε small enough. ∎

Remark. When g is increasing ($m = 0$) and $N \le 3$, we use the fact that $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. So $u \in L^\infty(\mathbb{R}_+, L^\infty(\Omega))$. Then,

$$\begin{aligned}
\Big|\int_S^T E^\sigma \phi' \int_{|u'|>1} u\, g(u')\, dx\, dt\Big| &\le c\, \|u\|_{L^\infty(\mathbb{R}_+, L^\infty(\Omega))} \int_S^T E^\sigma \int_{|u'|>1} |g(u')|\, dx\, dt \\
&\le c\, \|u\|_{L^\infty(\mathbb{R}_+, L^\infty(\Omega))} \int_S^T E^\sigma \int_{|u'|>1} u'\, g(u')\, dx\, dt \\
&\le c\, \|u\|_{L^\infty(\mathbb{R}_+, L^\infty(\Omega))}\, E(S)^{1+\sigma}\ . \ \square
\end{aligned}$$

4 – Proof of Theorem 2

In this Section, we study the decay rate of the energy when g is monotone increasing. This allows to introduce in a well-known case the ideas we will use in the next part to study the nonmonotone case.

We choose $\phi(t) = t$ for all $t \in \mathbb{R}_+$. With $\sigma = 0$, Lemma 3 gives that

$$\int_S^T E(t)\,dt \;\leq\; c\,E(S) + c\int_S^T\int_\Omega u'^2\,dx\,dt\;.$$

Our goal is to estimate

$$\int_S^T\int_\Omega u'^2\,dx\,dt\;.$$

Set $R > 0$ and fix $t \geq 0$. Define

$$\Omega_1^t := \Big\{x \in \Omega\colon\; |u'| \leq R\Big\}\;, \tag{4.1}$$

$$\Omega_2^t := \Big\{x \in \Omega\colon\; R < |u'|\Big\}\;. \tag{4.2}$$

Remark. Komornik [14] used this partition with $R = 1$, and obtained a polynomial decay rate estimate. We will choose R depending on the norm of the initial data. A suitable choice of R will lead us to exponential decay rate estimate. □

First we look at the part on Ω_2^t. In order to study the term

$$\int_{\Omega_2^t} u'^2\,dx\,dt\;,$$

we will use the regularity of u and the injections of Sobolev. We recall the interpolation inequality:

Lemma 5 (Gagliardo–Nirenberg). *Let $1 \leq r < p \leq \infty$, $1 \leq q \leq p$ and $m \geq 0$. Then the inequality*

$$\|v\|_p \leq c\,\|v\|_{m,q}^{\theta}\,\|v\|_r^{1-\theta} \qquad \textit{for}\;\; v \in W^{m,q} \cap L^r \tag{4.3}$$

holds with some $c > 0$ and

$$\theta \;=\; \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1} \tag{4.4}$$

provided that $0 < \theta \leq 1$ ($0 < \theta < 1$ if $p = \infty$ and $mq = N$). ∎

(Here $\|\cdot\|_p$ denotes the usual $L^p(\Omega)$ norm and $\|\cdot\|_{m,q}$ the norm in $W^{m,q}(\Omega)$.) As a consequence, in dimension $N=2$ we get that there exists a positive constant c that depends on Ω such that

$$\forall\, v \in H^1(\Omega), \quad \|v\|_{L^3(\Omega)} \le c\,\|v\|_{H^1(\Omega)}^{1/3}\,\|v\|_{L^2(\Omega)}^{2/3} \tag{4.5}$$

(we used (4.3) with $p=3$, $m=1$, $q=r=2$, $N=2$ and $\theta=\frac{1}{3}$.)

Using Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_{\Omega_2^t} u'^2\,dx &= \int_{\Omega_2^t} u'^{1/2}\,u'^{3/2}\,dx \le \\ &\le \Big(\int_{\Omega_2^t}|u'|\Big)^{1/2}\Big(\int_{\Omega_2^t}|u'|^3\Big)^{1/2} \le \Big(\int_{\Omega_2^t}|u'|\Big)^{1/2}\|u'\|_{L^3(\Omega)}^{3/2}\,.\end{aligned}$$

Since

$$R\int_{\Omega_2^t}|u'| \le \int_{\Omega_2^t} u'^2\,, \tag{4.6}$$

we obtain

$$\int_{\Omega_2^t} u'^2\,dx \le \frac{1}{R}\,\|u'\|_{L^3(\Omega)}^3\,.$$

Then, since u is a strong solution, we can apply (4.5) with $v=u'$ to get

$$\|u'\|_{L^3(\Omega)}^3 \le c\,\|u'\|_{H^1(\Omega)}\,\|u'\|_{L^2(\Omega)}^2 \le c\,\|u'\|_{H^1(\Omega)}\,E(t)\,.$$

Consequently,

$$\int_{\Omega_2^t} u'^2\,dx \le \frac{c}{R}\,\|u'\|_{H^1(\Omega)}\,E(t)\,. \tag{4.7}$$

Since g is increasing, we can apply Lemma 2 with $m=0$ and we get

$$\forall\, t\in\mathbb{R}_+, \quad \|u'\|_{H^1(\Omega)} \le \sqrt{C(u^0,u^1)}\,.$$

Thus

$$\begin{aligned}\int_S^T E\,dt &\le c\,E(S) + c\int_S^T\int_\Omega u'^2\,dx\,dt \\ &\le c\,E(S) + c\int_S^T\int_{\Omega_1^t} u'^2\,dx\,dt + \frac{c}{R}\int_S^T \|u'\|_{H^1(\Omega)}\,E(t)\,dx\,dt \\ &\le c\,E(S) + c\int_S^T\int_{\Omega_1^t} u'^2\,dx\,dt + \frac{c}{R}\sqrt{C(u^0,u^1)}\int_S^T E(t)\,dx\,dt\end{aligned}$$

Now we choose $R>0$ such that

$$\frac{c}{R}\sqrt{C(u^0,u^1)} \le \frac{1}{2}\,. \tag{4.8}$$

Then

$$\frac{1}{2}\int_S^T E\,dt \;\le\; c\,E(S) + c\int_S^T\int_{\Omega_1^t} u'^2\,dx\,dt\;.$$

Next we look at the part on Ω_1^t: since $g'(0) \neq 0$, we can choose $r > 0$ such that

$$\forall\, v\in[-r,r], \quad |g(v)| \ge \alpha_1\,|v|\;,$$

for some $\alpha_1 > 0$. Then we define

$$\alpha_2 := \inf\left\{\left|\frac{g(v)}{v}\right|\!:\; r\le|v|\le R\right\} > 0\;.$$

With $\alpha := \min(\alpha_1,\alpha_2)$, we have

$$|g(v)| \ge \alpha|v| \quad \text{ if } \ |v|\le R\;.$$

So

$$\begin{aligned}\int_S^T\int_{\Omega_1^t} u'^2\,dx\,dt \;&=\; \int_S^T\int_{\Omega_1^t} u'\,g(u')\,\frac{u'}{g(u')}\;dx\,dt \;\le\\ &\le\; \frac{1}{\alpha}\int_S^T\int_{\Omega_1^t} u'\,g(u')\;dx\,dt \;=\; \frac{1}{\alpha}\Big(E(S)-E(T)\Big)\;.\end{aligned} \tag{4.9}$$

Finally, we get

$$\frac{1}{2}\int_S^T E(t)\,dt \;\le\; c\,E(S) + \frac{c}{\alpha}\Big(E(S)-E(T)\Big) \;\le\; \Big(c+\frac{c}{\alpha}\Big)\,E(S)\;. \tag{4.10}$$

Letting T go to infinity, we get

$$\int_S^{+\infty} E(t)\,dt \;\le\; \frac{1}{\omega}\,E(S) \tag{4.11}$$

with $\frac{1}{\omega} = 2\,c(1+\frac{1}{\alpha})$. Since E is nonincreasing and nonnegative, a well-known Gronwall type inequality (see, e.g., [13]) gives

$$E(t) \;\le\; E(0)\,e^{1-\omega t}\;. \tag{4.12}$$

We recall the proof of this inequality briefly: set $h(t) = \int_t^{+\infty} E(\tau)\,d\tau$. h satisfies the differential inequality

$$\forall\, t\ge 0, \quad h'(t) + \omega\,h(t) \le 0\;.$$

So

$$\forall\, t \geq 0, \quad h(t) \leq h(0)\, e^{-\omega t} \leq \frac{1}{\omega}\, E(0)\, e^{-\omega t} \ .$$

Then since E is nonnegative and nonincreasing, for all $\varepsilon > 0$ we have

$$E(t) \ \leq\ \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} E(\tau)\, d\tau \ \leq\ \frac{1}{\varepsilon}\, h(t-\varepsilon) \ \leq\ \frac{1}{\omega\,\varepsilon}\, E(0)\, e^{\omega\varepsilon}\, e^{-\omega t} \ ,$$

and the best estimate is obtained for $\omega\varepsilon = 1$.

The proof of Theorem 2 is completed. ∎

5 – Proof of Theorem 3

Note that the proof of the exponential stability when g is monotone does not use the monotonicity of g in itself, but only the regularity of the solution u provided by the monotonicity. When g is nonmonotone, we use the same strategy than in Section 4. We will use the only estimate that we have on the second order energy (see Lemma 2). The choice of R and ϕ will be related to that estimate.

Our goal is to estimate

$$\int_S^T \int_\Omega u'^2\, dx\, dt \ .$$

Set $R_0 \geq 1$ and define

$$\forall\, t \geq 0, \quad R(t) = R_0\, e^{mt} \ , \tag{5.1}$$

and

$$\forall\, t \geq 0, \quad \phi(t) = (1+t)^{1-k} - 1 \quad \text{ if } \ k \in [0,1) \ , \tag{5.2}$$

$$\forall\, t \geq 0, \quad \phi(t) = \ln(2+t) - \ln 2 \quad \text{ if } \ k = 1 \ . \tag{5.3}$$

Note that ϕ is an increasing concave function of class $\mathcal{C}^2$ on $\mathbb{R}_+$ (and $\phi(0) = 0$).

Fix $t \geq 0$ and define

$$\Omega_0^t := \Big\{ x \in \Omega\colon\ |u'| \leq R_0 \Big\} \ , \tag{5.4}$$

$$\Omega_1^t := \Big\{ x \in \Omega\colon\ R_0 < |u'| \leq R(t) \Big\} \ , \tag{5.5}$$

$$\Omega_2^t := \Big\{ x \in \Omega\colon\ R(t) < |u'| \Big\} \ . \tag{5.6}$$

Note that this partition generalizes the one we constructed in the monotone case: if $m = 0$, $R(t) = R_0$ and $\Omega_1^t = \emptyset$. As in Section 4, R_0 will depend $C(u^0, u^1)$.

First we look at the part on Ω_2^t. We have already shown in Section 4 that

$$\int_{\Omega_2^t} u'^2\, dx \ \leq\ \frac{c}{R(t)}\, \|u'\|_{H^1(\Omega)}\, E(t) \ . \tag{5.7}$$

Using the estimate given by Lemma 2, we get

$$\int_{\Omega_2^t} u'^2\,dx \;\le\; \frac{c}{R(t)}\,\sqrt{C(u^0,u^1)}\,e^{mt}\,E(t) \;=\; \frac{c}{R_0}\,\sqrt{C(u^0,u^1)}\,E(t)\;. \tag{5.8}$$

Next we look at the part on Ω_1^t:

$$\begin{aligned}\int_S^T \phi'(t)\int_{\Omega_1^t} u'^2\,dx\,dt \;&=\; \int_S^T \phi'(t)\int_{\Omega_1^t} u'\,g(u')\,\frac{u'}{g(u')}\;dx\,dt\\ &\le\; c\int_S^T \phi'(t)\int_{\Omega_1^t} u'\,g(u')\,\Big(\ln(2+|u'|)\Big)^k\,dx\,dt\\ &\le\; c\int_S^T \phi'(t)\,\Big(\ln(2+R(t))\Big)^k\int_{\Omega_1^t} u'\,g(u')\,dx\,dt\;.\end{aligned} \tag{5.9}$$

Remark that thanks to the definitions of R and ϕ, the function $t\mapsto\phi'(t)\,(\ln(2+R(t)))^k$ is bounded on $\mathbb{R}_+$: if $k\in[0,1)$,

$$\forall\, t\ge 0,\quad \phi'(t)\,\Big(\ln(2+R(t))\Big)^k \;=\; (1-k)\,(1+t)^{-k}\,\Big(\ln(2+R_0\,e^{mt})\Big)^k \;\le\; M\;,$$

and if $k=1$,

$$\forall\, t\ge 0,\quad \phi'(t)\,\Big(\ln(2+R(t))\Big) \;\le\; \frac{1}{2+t}\,\Big(\ln(2+R_0\,e^{mt})\Big) \;\le\; M\;.$$

So

$$\begin{aligned}\int_S^T \phi'(t)\int_{\Omega_1^t} u'^2\,dx\,dt \;&\le\; \int_S^T \phi'(t)\Big(\ln(2+R(t))\Big)^k\int_{\Omega_1^t} u'\,g(u')\,dx\,dt\\ &\le\; M\,E(S)\;.\end{aligned} \tag{5.10}$$

At last, we look at the part on Ω_0^t: since $g'(0)\neq 0$, we have

$$|g(v)|\ge \alpha\,|v| \quad \text{ if } \; |v|\le R_0$$

for some $\alpha>0$. So

$$\begin{aligned}\int_S^T \phi'(t)\int_{\Omega_0^t} u'^2\,dx\,dt \;&\le\; \frac{1}{\alpha}\int_S^T \phi'(t)\int_{\Omega_0^t} u'\,g(u')\,dx\,dt\\ &\le\; \frac{\phi'(S)}{\alpha}\,E(S) \;\le\; c\,E(S)\;.\end{aligned} \tag{5.11}$$

Thus we deduce from the inequality (3.8) and the estimates (5.8), (5.10) and (5.11) that

$$\int_S^T E(t)\,\phi'(t)\,dt \;\le\; 2\,c\,E(S)+M\,E(S)+\frac{c}{R_0}\,\sqrt{C(u^0,u^1)}\int_S^T E(t)\,\phi'(t)\,dt\;.$$

Define R_0 by

$$R_0 := \max\Big\{1,\, 2\,c\,\sqrt{C(u^0,u^1)}\Big\}\ .$$

Then we obtain

$$\frac{1}{2}\int_S^T E(t)\,\phi'(t)\,dt \;\le\; C\,E(S)\ .$$

(Note that C depends on R_0, so depends on $C(u^0,u^1)$.)
Letting go T to infinity, we get

$$\forall\, S\ge 0,\quad \int_S^{+\infty} E(t)\,\phi'(t)\,dt \;\le\; C\,E(S)\ . \tag{5.12}$$

With the change of variable defined by $\tau=\phi(t)$, we see that the nonincreasing function $F(\tau) := E(\phi^{-1}(\tau))$ satisfies

$$\forall\,\sigma\ge 0,\quad \int_\sigma^{+\infty} F(\tau)\,d\tau \;\le\; C\,F(\sigma) \;=\; \frac{1}{\omega}\,F(\sigma)\ ,$$

therefore

$$\forall\,\tau\ge 0,\quad F(\tau)\le F(0)\,e^{1-\omega\tau}\ ,$$

i.e.

$$\forall\, t\ge 0,\quad E(t)\le E(0)\,e^{1-\omega\phi(t)}\ . \tag{5.13}$$

The proof of Theorem 3 is completed. ∎

6 – Proof of Theorem 4

When $N\ge 3$, we cannot absorb the term on Ω_2^t like we did in Sections 4 and 5. Our result is based on a new nonlinear integral inequality (Lemma 6), that generalizes a result from A. Haraux [9].

Assume that g satisfies (2.15)–(2.17). With $\sigma=1$, Lemma 3 gives that

$$\int_S^T E(t)^2\,\phi'(t)\,dt \;\le\; c\,E(S)^2 + c\int_S^T E(t)\,\phi'(t)\int_\Omega u'^2\,dx\,dt\ . \tag{6.1}$$

We use the same strategy than in dimension 2: define

$$\forall\, t\ge 0,\quad R(t) = R_0\,e^{\gamma t}\ , \tag{6.2}$$

with $\gamma>0$ (that we will choose later later), and

$$\forall\, t\ge 0,\quad \phi(t) = (1+t)^{1-k}-1 \quad\text{ if }\; k\in(0,1)\ . \tag{6.3}$$

Consider the partition of Ω defined by (5.4)–(5.6), where $R(t)$ is given by (6.2).

First we look at the part on Ω_2^t. Set $p = \frac{N+2}{4} > 1$ and q' its conjugate exponent:

$$\frac{1}{p} + \frac{1}{q'} = 1 \ .$$

Then

$$\begin{aligned}\int_{\Omega_2^t} u'^2 \, dx &= \int_{\Omega_2^t} u'^{1/p} \, u'^{2-1/p} \, dx \le \Big(\int_{\Omega_2^t} |u'|\Big)^{1/p} \Big(\int_{\Omega_2^t} |u'|^{q'(2-1/p)}\Big)^{1/q'} \\ &\le \Big(\frac{1}{R(t)}\Big)^{1/p} \Big(\int_{\Omega_2^t} u'^2\Big)^{1/p} \|u'\|_{(2p-1)/(p-1)}^{(2p-1)/p} \ .\end{aligned}$$

So

$$\int_{\Omega_2^t} u'^2 \, dx \le \Big(\frac{1}{R(t)}\Big)^{1/(p-1)} \|u'\|_{(2p-1)/(p-1)}^{(2p-1)/(p-1)} \ .$$

We use Lemma 5 to get

$$\|u'\|_{(2p-1)/(p-1)} \le c \, \|u'\|_{H^1}^{\theta} \, \|u'\|_2^{1-\theta} \le C \, e^{\theta m t} \, E(t)^{(1-\theta)/2}$$

with

$$\theta = \frac{N}{2} \, \frac{1}{2\,p-1} = 1 \ ,$$

and C depending on $C(u^0, u^1)$. So

$$\|u'\|_{2N/(N+2)}^{2N/(N+2)} \le C \, e^{\frac{2Nm}{N+2} t} \ . \tag{6.4}$$

(Note that we would have obtained a better estimate by choosing a larger p, but this does not change the result.) Define γ such that

$$\frac{2\,Nm}{N+2} - \frac{\gamma}{p-1} = -1 \ .$$

Hence

$$\begin{aligned}\int_S^T E(t) \, \phi'(t) \int_{\Omega_2^t} u'^2 \, dx \, dt &\le C \int_S^T E(t) \, \phi'(t) \, \frac{e^{2Nmt/(N+2)}}{\Big(R(t)\Big)^{1/(p-1)}} \, dt \\ &\le C \, E(S) \int_S^T e^{-t} \, dt \le C \, E(S) \, e^{-S} \ .\end{aligned} \tag{6.5}$$

Next we look at the part on Ω_1^t: since the function $t \mapsto \phi'(t) \, (\ln(2 + R(t))^k$ is bounded on $\mathbb{R}_+$, we have

$$\int_S^T E(t) \, \phi'(t) \int_{\Omega_1^t} u'^2 \, dx \, dt \le M \, E(S)^2 \ ,$$

and as usual,

$$\int_S^T E(t)\,\phi'(t)\int_{\Omega_0^t} u'^2\,dx\,dt \;\le\; c\,E(S)^2\ ,$$

Thus

$$\forall\, S\ge 0,\quad \int_S^T E(t)^2\,\phi'(t)\,dt \;\le\; c\,E(S)^2 + C\,E(S)\,e^{-S}\ . \tag{6.6}$$

With the change of variable defined by $\tau=\phi(t)$ and the change of function $F(\tau):=E(t)$, we get

$$\forall\, S\ge 0,\quad \int_{\phi(S)}^{\phi(T)} F(\tau)^2\,d\tau \;\le\; c\,F(\phi(S))^2 + C\,F(\phi(S))\,e^{-S}\ .$$

The nonincreasing function F satisfies, if $k\in[0,1)$,

$$\begin{aligned}\forall\, y\ge 1,\quad \int_y^{+\infty} F(\tau)^2\,d\tau \;&\le\; c\,F(y)^2 + C\,F(y)\,e^{-\phi^{-1}(y)}\\ &\le\; c\,F(y)^2 + C\,F(y)\,e^{-y^{1/(1-k)}}\ ,\end{aligned} \tag{6.7}$$

and, if $k=1$,

$$\begin{aligned}\forall\, y\ge 1,\quad \int_y^{+\infty} F(\tau)^2\,d\tau \;&\le\; c\,F(y)^2 + C\,F(y)\,e^{-e^y}\\ &\le\; c\,F(y)^2 + C\,F(y)\,e^{-y^2}\ .\end{aligned}$$

Remark that if $C=0$, we deduce from (4.12) that F decays exponentially to zero. Since $0<k<1$, $\frac{1}{1-k}>1$. We show that the term $F(y)\,e^{-y^{1/(1-k)}}$ has a negligible effect in front of $F(y)^2$:

Lemma 6. *Set $\gamma\ge 1$. Assume that F satisfies*

$$\forall\, t\ge 1,\quad \int_t^{+\infty} F(\tau)^2\,d\tau \;\le\; c\,F(t)^2 + c\,F(1)\,F(t)\,e^{-t^\gamma}\ . \tag{6.8}$$

Then there exists C_γ such that F satisfies the decay property:

$$\forall\, t\ge 1,\quad F(t)\le C_\gamma\,F(1)\,e^{-\omega t}\quad \textit{with}\ \ \omega=\frac{1}{2\,c+\frac{2}{\gamma}}\ . \tag{6.9}$$

Remark. In fact, the decay rate estimate (6.9) is not optimal: if F satisfies (6.8), then it is easy to prove that for all $\varepsilon>0$, there exists $C_{\gamma,\varepsilon}$ such that:

$$\forall\, t\ge 1,\quad F(t)\le C_{\gamma,\varepsilon}\,F(1)\,e^{-\omega_\varepsilon t}\quad \text{with}\ \ \omega_\varepsilon=\frac{1}{2\,c+\varepsilon}\ .\ \square$$

Assume Lemma 6 is true. Then the proof of Theorem 4 is completed: we deduce from (6.9) that

$$\forall\, t \geq 0, \quad E(t) \leq C_k\, E(0)\, e^{-\omega\phi(t)} \qquad \text{with} \;\; \omega = \frac{1}{2\,c + 2(1-k)}\ . \ \blacksquare$$

Proof of Lemma 6: In order to simplify the notations, we prove Lemma 6 when $\gamma = 2$. The proof is the same for $\gamma \geq 1$. Define

$$\forall\, t \geq 1, \quad G(t) := F(t)^2 + F(1)\, F(t)\, e^{-t^2}\ .$$

Then G is nonincreasing and satisfies

$$\begin{aligned}\forall\, t \geq 1, \quad \int_t^{+\infty} G(\tau)\, d\tau \;&\leq\; c\, G(t) + F(1)\, F(t) \int_t^{+\infty} e^{-\tau^2}\, d\tau \\ &\leq\; c\, G(t) + F(1)\, F(t) \int_{t^2}^{+\infty} e^{-y}\, \frac{dy}{2\sqrt{y}} \\ &\leq c\, G(t) + \frac{1}{2}\, F(1)\, F(t)\, e^{-t^2} \\ &\leq \left(c + \frac{1}{2}\right) G(t)\ .\end{aligned}$$

So we deduce from the computations that we made to get (4.12) that

$$\forall\, t \geq 1, \quad G(t) \leq G(1)\, e^{1-(t-1)/(c+\frac{1}{2})}\ .$$

So

$$\forall\, t \geq 1, \quad F(t) \leq \sqrt{1+e}\, F(1)\, e^{-(t-1)/(2c+1)}\ . \ \blacksquare$$

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Patrick Martinez,
M.I.P. Université Paul Sabatier Toulouse III,
118 route de Narbonne, 31 062 Toulouse Cedex 4 – FRANCE
E-mail: martinez@mip.ups-tlse.fr

and

Judith Vancostenoble,
M.I.P. Université Paul Sabatier Toulouse III,
118 route de Narbonne, 31 062 Toulouse Cedex 4 – FRANCE
E-mail: vancoste@mip.ups-tlse.fr