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ENTROPY NUMBERS OF EMBEDDINGS BETWEEN LOGARITHMIC SOBOLEV SPACES

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Abstract: Let Ω be a bounded domain in $\mathbb{R}^n$ and id be the natural embedding

$$H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \;\to\; H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)$$

between these logarithmic Sobolev spaces, where $-\infty < s_2 < s_1 < \infty$, $0 < p_1 < p_2 < \infty$, with $s_1 - n/p_1 = s_2 - n/p_2$, and $-\infty < a_2 \le a_1 < \infty$. We show that if the real numbers a_1 and a_2 satisfy the conditions $a_1 > 0$, $a_1 \notin \big]1/\min\{1,p_2\},\, 2(s_1-s_2)/n + 1/\min\{1,p_2\}\big]$ and $a_2 < a_1 - 2(s_1-s_2)/n - 1/\min\{1,p_2\}$ then there exist $c_1, c_2 > 0$ such that, for all $k \in \mathbb{N}$,

$$c_1\, k^{-(s_1-s_2)/n} \;\le\; e_k(id) \;\le\; c_2\, k^{-(s_1-s_2)/n} \;,$$

where the e_k stand for entropy numbers. This improves earlier results of Edmunds and Triebel [4].

1 – Introduction

The present work was prompted by an open problem mentioned in [4], namely in their Remark 5.1/2 (p. 364). We recall that in [4] Edmunds and Triebel estimated the entropy numbers of some embeddings between logarithmic Sobolev spaces, with a view to apply the results to the study of spectral properties of some degenerate elliptic operators. We recall here one of their more interesting results with respect to those entropy numbers:

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Given a bounded C^∞-domain Ω in $\mathbb{R}^n$, $-\infty < s_2 < s_1 < \infty$, $0 < p_1 < p_2 < \infty$, $s_1 - n/p_1 = s_2 - n/p_2$, $a_1 \leq 0$ and $a_2 < a_1 - 2(s_1 - s_2)/n$, there exists $c_1, c_2 > 0$ such that, for all $k \in \mathbb{N}$,

$$
\begin{aligned}
c_1\, k^{-(s_1-s_2)/n} &\leq e_k\Big(id\colon\ H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \to H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)\Big) \\
&\leq c_2\, k^{-(s_1-s_2)/n}\ .
\end{aligned}
\tag{1}
$$

Here the notation $H^s_p(\log H)_a(\Omega)$ stands for logarithmic Sobolev spaces, which we will define below (see subsection 2.4).

Besides this very complete result in the case $a_1 \leq 0$, Edmunds and Triebel also have tried to obtain a corresponding result for $a_1 > 0$, but then they had to leave out the case when p_1 or p_2 lie in $]0, 1]$. In Remark 5.1/2 of [4] they have clearly mentioned that it seemed likely that the restriction to p_1 and p_2 in $]1, \infty[$ was only due to their technique, which used duality arguments, and that it should be possible to remove this restriction.

We tackle this problem in this paper.

First of all, we didn't find it necessary to restrict Ω to be a bounded C^∞-domain, as we can derive our results merely assuming that Ω is a bounded domain. This is in contrast with the technique of duality just mentioned, where it seems necessary to have some amount of smoothness on $\partial\Omega$ in order that the argument runs. Further, we can in fact deal with any p_1, p_2 in $]0, \infty[$ in the case $a_1 > 0$, though the result we obtain is not as complete as the one mentioned above in the case $a_1 \leq 0$: we arrive at (1), but with some extra restrictions imposed on the parameters a_1 and a_2 (for the full assertion, see Theorem 4.3.1 below). It should, however, also be remarked that, as we learned after finishing the present work, even the less restrictive assumptions on a_1 and a_2 made in [4] seem to be excessive: see the recent results of Edmunds and Netrusov [2] when the parameters of type s and p in the spaces considered above belong, respectively, to $\mathbb{N}_0$ and $]1, \infty[$.

The plan of the paper is the following:

In section 2 we start by briefly recalling some basic definitions and properties related to Triebel–Lizorkin spaces F^s_{pq}, either in $\mathbb{R}^n$ or in other domains Ω in $\mathbb{R}^n$. Then we draw attention to the usefulness of interpolation arguments to control constants in multiplier assertions and embedding theorems. Finally, we define the logarithmic Sobolev spaces for any bounded domain Ω in $\mathbb{R}^n$ and discuss some of their elementary properties.

The long section 3 is devoted to control constants in the estimates for entropy numbers of compact embeddings (between some Triebel–Lizorkin spaces) approaching a limiting situation. A great part of the proof is modelled (now with extra care, because of the need to control the dependence of the constants on the parameters) on the proof of the sharp asymptotic estimates for the entropy numbers of embeddings between Besov spaces (cf. [3]), though some modifications are in order, due to the shifting from B^s_{pq}- to F^s_{pq}-spaces. We also take a slightly different point of view, as can be seen by comparison with [3].

The final section 4 gives the result we announced above, taking advantage of the estimates obtained in the preceding section.

All positive constants, the precise value of which is unimportant for us, are denoted by lowercase c, occasionally with additional subscripts within the same formula or the same step of a proof.

2 – General function spaces and embeddings

Fix $n \in \mathbb{N}$ and the Euclidean n-space $\mathbb{R}^n$ with norm $|\cdot|$.

Denote by $\mathcal{S} \equiv \mathcal{S}(\mathbb{R}^n)$ the (complex) Schwartz space and by $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of tempered distributions. Furthermore, given a domain (i.e., a non-empty open set) Ω in $\mathbb{R}^n$, denote by $\mathcal{D}(\Omega)$ the usual space of (complex) test functions and by $\mathcal{D}'(\Omega)$ its topological dual, the space of distributions on Ω. Let $L_p(\mathbb{R}^n)$, for $p \in \,]0,\infty]$, denote the usual (complex) Lebesgue spaces.

Our option for the definition of the Fourier transform of $\varphi \in \mathcal{S}$ is

$$\hat{\varphi}(\xi) \;\equiv\; (\mathcal{F}\varphi)(\xi) \;\equiv\; (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\, \varphi(x)\, dx \;.$$

From this we take the usual procedure in order to extend the Fourier transformation to $\mathcal{S}'$ and notice that $\check{}$ or $\mathcal{F}^{-1}$ will be used to denote the inverse Fourier transformation.

Throughout all the paper, the word embedding will always be used in the sense of continuous embedding.

2.1. The case $\Omega = \mathbb{R}^n$

Let $\varphi \in \mathcal{S}$ with $\varphi(x) = 1$ if $|x| \le 1$ and $\varphi(x) = 0$ if $|x| \ge 3/2$. Put $\varphi_0 = \varphi$, $\varphi_1(x) = \varphi(x/2) - \varphi(x)$ and $\varphi_j(x) = \varphi_1(2^{-j+1}x)$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}$, so that

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1\,, \qquad \forall\, x \in \mathbb{R}^n \tag{2}$$

($\{\varphi_j\}_{j\in\mathbb{N}_0}$ form a so-called dyadic partition of unity).

Recall that, given $f \in \mathcal{S}'$, $(\varphi_j \hat{f})\check{}$ is an entire analytic function on $\mathbb{R}^n$ (Paley–Wiener–Schwartz theorem). In particular, it makes sense pointwise and it is measurable (the concepts of measurability, measure and integration we will consider are always Lebesgue's).

Given $s \in \mathbb{R}$ and $0 < p, q < \infty$, $F^s_{pq}(\mathbb{R}^n)$ is defined as the set of $f \in \mathcal{S}'$ such that

$$\|f \,|\, F^s_{pq}(\mathbb{R}^n)\|_\varphi \equiv \left(\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{jsq} \,|(\varphi_j \hat{f})\check{}(x)|^q\right)^{p/q} dx\right)^{1/p} < \infty \,. \tag{3}$$

We have the following properties of $F^s_{pq}(\mathbb{R}^n)$:

- **(i)** It is a quasi-Banach space, taking the expression in (3) as the quasi-norm (which can, in fact, be easily seen to be a t-norm, with $t = \min\{1, p, q\}$).
- **(ii)** The definition of the space is independent of the φ chosen in (3) in accordance with the considerations leading to (2) (though for two different choices of φ the corresponding quasi-norms can be different, they are equivalent). In the sequel we assume that one such φ has been chosen once and for all and will most of the time omit the reference to it in the quasi-norm.
- **(iii)** If $s_1, s_2 \in \mathbb{R}$ and $0 < p_1, p_2, q_1, q_2 < \infty$ are such that $s_1 - n/p_1 \geq s_2 - n/p_2$ and $p_1 < p_2$ ($\Rightarrow s_1 > s_2$), then there exists the embedding

$$F^{s_1}_{p_1 q_1}(\mathbb{R}^n) \hookrightarrow F^{s_2}_{p_2 q_2}(\mathbb{R}^n) \,.$$

For proofs, and also for connections with classical spaces and the study of the diversity of these spaces, please refer to [8]. Here we mention only that, when $p > 1$, $F^s_{p2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$ (equivalent norms), the Bessel-potential spaces. We shall use this result later.

2.2. The case of any domain Ω in $\mathbb{R}^n$

Let Ω be a domain in $\mathbb{R}^n$.

Given $s \in \mathbb{R}$ and $0 < p, q < \infty$, $F^s_{pq}(\Omega)$ is defined as the set of $f \in \mathcal{D}'(\Omega)$ which can be considered as $f = g|_\Omega$ for some $g \in F^s_{pq}(\mathbb{R}^n)$, quasi-normed by

$$\|f \,|\, F^s_{pq}(\Omega)\|_\varphi \equiv \inf_{g \in F^s_{pq}(\mathbb{R}^n),\, g|_\Omega = f} \|g \,|\, F^s_{pq}(\mathbb{R}^n)\|_\varphi \,. \tag{4}$$

We have the following properties of $F^s_{pq}(\Omega)$:

(i) It is a quasi-Banach space (and the expression in (4) can, in fact, be easily seen to be a t-norm, with $t = \min\{1, p, q\}$).

(ii) The definition of the space is, of course, independent of the φ chosen as in 2.1, being equivalent any quasi-norms defined by means of two different choices of φ.

(iii) If $s_1, s_2 \in \mathbb{R}$ and $0 < p_1, p_2, q_1, q_2 < \infty$ are such that $s_1 - n/p_1 \geq s_2 - n/p_2$ and $p_1 < p_2$ ($\Rightarrow s_1 > s_2$), then there exists the embedding

$$F^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow F^{s_2}_{p_2 q_2}(\Omega).$$

For a proof of statement (i) the reader can consult [8] (though there only bounded C^∞-domains are considered, it is easily seen that it also works in as broader a context as ours). As to assertion (iii), it is a direct consequence of 2.1(iii) and the following result (which, in turn, follows easily from the definitions — cf. also with the proof of Corollary 2.3.7 below).

Proposition 2.2.1. *If for some choice of the parameters* $s_1, s_2 \in \mathbb{R}$ *and* $0 < p_1, p_2, q_1, q_2 < \infty$ *there is an embedding*

$$F^{s_1}_{p_1 q_1}(\mathbb{R}^n) \hookrightarrow F^{s_2}_{p_2 q_2}(\mathbb{R}^n) \ ,$$

then there is also, for each domain Ω *in* $\mathbb{R}^n$*, a corresponding embedding*

$$F^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow F^{s_2}_{p_2 q_2}(\Omega) \ .$$

Moreover, the quasi-norm of the first embedding is an upper estimate for the quasi-norm of the second. ∎

Remark 2.2.2. If we define, for $p > 1$, $H^s_p(\Omega)$ from $H^s_p(\mathbb{R}^n)$ (a Bessel-potential space) by the same procedure used to define $F^s_{pq}(\Omega)$ from $F^s_{pq}(\mathbb{R}^n)$, i.e., by restriction, then it follows from the equality $F^s_{p2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$ given in 2.1 and as easily as for the proposition above that, for $p > 1$, $F^s_{p2}(\Omega) = H^s_p(\Omega)$ with equivalent norms. □

2.3. Interpolation, multipliers and embeddings

2.3.1. Interpolation

We would like to take advantage of complex interpolation in order to control constants. Since we want to deal with the spaces $F^s_{pq}(\mathbb{R}^n)$, introduced in 2.1, and these are not always normed, but merely quasi-normed, the method of complex interpolation we are referring to is the one presented in [8, 2.4.4 to 2.4.7], which is denoted by $(\cdot,\cdot)_\theta$.

The problem with this method, in contrast with the method of complex interpolation in the framework of general Banach spaces, is that we don't know *a priori* whether an interpolation property holds or not: it depends on the operator in question. In order to facilitate our task of checking whether we can use an interpolation property in each specific situation, we present below a general result in that direction (see Proposition 2.3.2). We begin with some definitions, though.

We recall the concept of $\mathcal{S}'$-analytic function in $A \equiv \{z \in \mathbb{C}\colon\ 0 < \Re z < 1\}$ from [8, p. 67], namely that f is such a function if

- **(i)** $f\colon\ \overline{A} \to \mathcal{S}'$;
- **(ii)** for all $\varphi\in\mathcal{D}(\mathbb{R}^n)$, $(x,z) \mapsto (\varphi\widehat{f(z)})\check{}(x)$ is uniformly continuous in $\mathbb{R}^n\times\overline{A}$;
- **(iii)** for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$, $z \mapsto (\varphi\widehat{f(z)})\check{}(x)$ is analytic in A.

Definition 2.3.1. Given a linear operator $T:\ \mathcal{S}' \to \mathcal{S}'$, the set of all $\mathcal{S}'$-analytic functions in A is said to be invariant under T if whenever f is $\mathcal{S}'$-analytic in A then the same happens to $T \circ f$. □

Proposition 2.3.2. *Given $s_0, s_1, \sigma_0, \sigma_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1, \pi_0, \pi_1, \chi_0, \chi_1 < \infty$ and $0 < \theta < 1$, let $T:\ \mathcal{S}' \to \mathcal{S}'$ be a linear operator such that T is bounded linear from $F^{s_l}_{p_l q_l}(\mathbb{R}^n)$ into $F^{\sigma_l}_{\pi_l\chi_l}(\mathbb{R}^n)$, $l = 0, 1$. If the set of all $\mathcal{S}'$-analytic functions in A is invariant under T, then T is also a bounded linear operator from $(F^{s_0}_{p_0q_0}(\mathbb{R}^n), F^{s_1}_{p_1q_1}(\mathbb{R}^n))_\theta$ into $(F^{\sigma_0}_{\pi_0\chi_0}(\mathbb{R}^n), F^{\sigma_1}_{\pi_1\chi_1}(\mathbb{R}^n))_\theta$, the quasi-norm of which is bounded above by*

$$\left\|T\colon\ F^{s_0}_{p_0q_0}(\mathbb{R}^n) \to F^{\sigma_0}_{\pi_0\chi_0}(\mathbb{R}^n)\right\|^{1-\theta} \times \left\|T\colon\ F^{s_1}_{p_1q_1}(\mathbb{R}^n) \to F^{\sigma_1}_{\pi_1\chi_1}(\mathbb{R}^n)\right\|^{\theta}. \ \blacksquare$$

We omit the proof, as it is straightforward from [8, Lemma 2.4.6/3] and the definitions involved.

We recall the following fundamental result [8, Theorem 2.4.7].

Theorem 2.3.3. *Let* $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 < \infty$ *and* $0 < \theta < 1$. *If* $s = (1-\theta)\, s_0 + \theta s_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$, *then*

$$\Big(F^{s_0}_{p_0 q_0}(\mathbb{R}^n),\, F^{s_1}_{p_1 q_1}(\mathbb{R}^n)\Big)_\theta \;=\; F^s_{pq}(\mathbb{R}^n)$$

(equivalent quasi-norms). ∎

Remark 2.3.4. In the two-sided estimate corresponding to the equivalence of the quasi-norms in the above theorem, the constants can be taken independent of the particular θ considered, as was kindly pointed out to me by Prof. Triebel. □

2.3.2. Multipliers and embeddings

Proposition 2.3.5. *Given* $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 < \infty$ *and* $\rho > \max\Big\{0,\, s_0,\, s_1,\, n/\min\{p_0, q_0\} - s_0,\, n/\min\{p_1, q_1\} - s_1\Big\}$, *there exists* $c > 0$ *such that*

$$\|\psi\, f \,|\, F^s_{pq}(\mathbb{R}^n)\| \;\leq\; c\, \|\psi \,|\, \mathcal{C}^\rho(\mathbb{R}^n)\|\; \|f \,|\, F^s_{pq}(\mathbb{R}^n)\|\;,$$

for all $f \in F^s_{pq}(\mathbb{R}^n)$, *all* $\psi \in \mathcal{S}$ *and all* s, p, q *given by* $s = (1-\theta)\, s_0 + \theta\, s_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$ *for any* $\theta \in [0,1]$, *where* $\mathcal{C}^\rho(\mathbb{R}^n)$ *are the Zygmund spaces (cf. [8, p. 36]).*

Sketch of Proof: For $\theta = 0$ and $\theta = 1$ the result follows from [8, Theorem 2.8.2]. In particular, in these two cases of θ, it holds for $\sum_{j=0}^N (\varphi_j \hat{\psi})^\vee$ instead of ψ, for each $N \in \mathbb{N}$. Interpolation with this as multiplier is possible (that is, Proposition 2.3.2 can be applied), so that the proof finishes by taking care of Remark 2.3.4 and letting N tend to infinity. ∎

Proposition 2.3.6. *Given* $\psi \in \mathcal{D}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 < \infty$ *with* $p_0 \geq q_0$ *and* $p_1 \geq q_1$, $\theta \in [0,1]$, $1/p \equiv (1-\theta)/p_0 + \theta/p_1$ *and* $1/q \equiv (1-\theta)/q_0 + \theta/q_1$ $(\Rightarrow p \geq q)$, *the map*

$$\begin{aligned} F^s_{p2}(\mathbb{R}^n) &\to F^s_{q2}(\mathbb{R}^n)\\ f &\mapsto \psi\, f \end{aligned}$$

is a linear continuous operator, the quasi-norm of which can be bounded above independently of θ.

Sketch of Proof: Consider a bounded C^∞-domain Ω in $\mathbb{R}^n$ such that $\operatorname{supp}\psi \subset \Omega$. Then it is known that the restriction operator $R\colon \mathcal{S}' \to \mathcal{D}'(\Omega)$ is a retraction with a common coretraction S for the family of spaces involved in the proposition (cf. [8, p. 201]). Denote by P the continuous projection SR. Using the fact that the embeddings $F^s_{p_l 2}(\Omega) \hookrightarrow F^s_{q_l 2}(\Omega)$, for $l = 0, 1$, exist (cf. [8, Theorem 3.3.1(iii)]), we can conclude that the following composition

$$\begin{array}{ccccccccc} F^s_{p2}(\mathbb{R}^n) & \to & PF^s_{p2}(\mathbb{R}^n) & \to & PF^s_{q2}(\mathbb{R}^n) & \to & F^s_{q2}(\mathbb{R}^n) & \to & F^s_{q2}(\mathbb{R}^n) \\ g & \mapsto & Pg & \mapsto & Pg & \mapsto & Pg & \mapsto & \psi Pg \end{array}$$

makes sense. On the other hand, it is easily seen it gives the map in the proposition, so that all that remains is to control the quasi-norms of the various operators in this composition. We can do that by means of Proposition 2.3.5 and some interpolation more (cf. the arguments in [8, p. 204]). ■

Corollary 2.3.7. *Given a bounded domain* Ω *in* $\mathbb{R}^n$, $s \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 < \infty$ *with* $p_0 \geq q_0$ *and* $p_1 \geq q_1$, $\theta \in [0,1]$, $1/p \equiv (1-\theta)/p_0 + \theta/p_1$ *and* $1/q \equiv (1-\theta)/q_0 + \theta/q_1$ $(\Rightarrow p \geq q)$, *there exists the embedding*

$$F^s_{p2}(\Omega) \hookrightarrow F^s_{q2}(\Omega)$$

and its quasi-norm can be bounded above independently of θ.

Proof: Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ be such that $\psi \equiv 1$ on Ω. Given $g \in F^s_{p2}(\Omega)$, there exists $f \in F^s_{p2}(\mathbb{R}^n)$ such that $f|_\Omega = g$. By the previous proposition, $\psi f \in F^s_{q2}(\mathbb{R}^n)$ with $\|\psi f \mid F^s_{q2}(\mathbb{R}^n)\| \leq c\,\|f \mid F^s_{p2}(\mathbb{R}^n)\|$, where the constant c is independent of θ. Since $(\psi f)|_\Omega = f|_\Omega = g$, then $g \in F^s_{q2}(\Omega)$ and

$$\|g \mid F^s_{q2}(\Omega)\| \;\leq\; \|\psi f \mid F^s_{q2}(\mathbb{R}^n)\| \;\leq\; c\,\|f \mid F^s_{p2}(\mathbb{R}^n)\| \;.$$

Taking the infimum for all possible choices of $f \in F^s_{p2}(\mathbb{R}^n)$ with $f|_\Omega = g$, we get $\|g \mid F^s_{q2}(\Omega)\| \leq c\,\|g \mid F^s_{p2}(\Omega)\|$, finishing the proof. ■

Remark 2.3.8. The above corollary also holds for the spaces $H^s_p(\Omega)$ introduced in Remark 2.2.2, when the parameters p_0, p_1, q_0, q_1 are further restricted to be strictly greater than 1. This can easily be seen by complex interpolation of Banach spaces. □

2.4. Logarithmic Sobolev spaces

Let Ω be a bounded domain in $\mathbb{R}^n$.

We adopt the following convention for the rest of the paper:

If $p > 0$ is the parameter appearing in a F^s_{pq}- or a H^s_p- space and $r \in \mathbb{R}$ is given, by p^r we mean the positive number such that $1/p^r = 1/p + r/n$. In particular, in order this definition always makes sense, in the case $r < 0$ we are assuming that $r > -n/p$.

Definition 2.4.1. Given $s \in \mathbb{R}$, $0 < p < \infty$, $a < 0$ and $J \in \mathbb{N}$, the logarithmic Sobolev space $H^s_p(\log H)_a(\Omega)$ is defined as the set of all $f \in \mathcal{D}'(\Omega)$ such that

$$\left(\sum_{j=J}^{\infty} 2^{jap}\, \|f \mid F^s_{p^{\sigma(j)}2}(\Omega)\|^p\right)^{1/p} < \infty\,, \tag{5}$$

where $\sigma(j)$ stands for 2^{-j} for each j. □

Definition 2.4.2. Given $s \in \mathbb{R}$, $0 < p < \infty$, $a > 0$ and $J \in \mathbb{N}$ such that $2^J > p/n$, the logarithmic Sobolev space $H^s_p(\log H)_a(\Omega)$ is defined as the set of all $f \in \mathcal{D}'(\Omega)$ which can be represented as $f = \sum_{j=J}^{\infty} g_j$ in $\mathcal{D}'(\Omega)$, $g_j \in F^s_{p^{\lambda(j)}2}(\Omega)$, with

$$\left(\sum_{j=J}^{\infty} 2^{jap}\, \|g_j \mid F^s_{p^{\lambda(j)}2}(\Omega)\|^p\right)^{1/p} < \infty\,, \tag{6}$$

where $\lambda(j)$ stands for -2^{-j} for each j. □

Remark 2.4.3. The definition does not depend on the particular J considered (use Corollary 2.3.7) nor on the particular function φ fixed in accordance with 2.1 (use interpolation for the spaces in $\mathbb{R}^n$ and an argument of the type shown in Proposition 2.2.1 to reach the spaces in Ω). □

We have the following properties of $H^s_p(\log H)_a(\Omega)$:

(**i**) It is a quasi-Banach space for the quasi-norm given by (5) in the case $a < 0$ and for the quasi-norm given by "the infimum of (6) over all possibilities of $(g_j)_{j\geq J}$ according to the definition of the space" in the case $a > 0$ (use Corollary 2.3.7 and standard arguments; in the case $a > 0$ you might also need to prove before-hand the set-theoretic inclusion $H^s_p(\log H)_a(\Omega) \subset F^s_{p2}(\Omega)$ and the upper estimate "constant times (6)" for the quasi-norm in $F^s_{p2}(\Omega)$ of all functions f of $H^s_p(\log H)_a(\Omega)$).

(ii) Different choices for the fixed J or φ (see remark above) in the same space give rise to equivalent quasi-norms (this shows up in the course of proving the independence referred to in the above remark).

(iii) In the case $a<0$, it is easily seen that (5) is a $\widetilde{p_J}$-norm, where $\widetilde{p_J} \equiv \min\{1, p^{\sigma(J)}\}$. In particular, it is a $(p/2)$-norm if $0<p\leq 1$, no matter how J is chosen in accordance to the definition. It is a norm if $p>1$ and J is chosen in such a way that $2^J \geq p'/n$ (p' conjugate to p).

(iv) In the case $a>0$, it is easily seen that $H^s_p(\log H)_a(\Omega)$ is a p-normed space if $0<p<1$ and a normed space otherwise.

Remark 2.4.4. In the case $p>1$ (and, for $a<0$, with the further restriction $2^J > p'/n$) we could also have used H^s_p-spaces instead of F^s_{p2}-spaces in (5) and (6) in order to define $H^s_p(\log H)_a(\Omega)$. Using interpolation for the spaces in $\mathbb{R}^n$ and an argument of the type shown in Proposition 2.2.1 to reach the spaces in Ω, we get in fact that the two possible definitions for those spaces coincide. More than this, in the case $a<0$ the expression corresponding to (5) gives us an equivalent quasi-norm in $H^s_p(\log H)_a(\Omega)$, while in the case $a>0$ it is the infimum of the expression corresponding to (6) (taken over all possibilities of $(g_j)_{j\geq J}$ according to the definition of the space) which gives us also an equivalent quasi-norm. Actually, as now we are dealing with $p>1$, these quasi-norms are, in fact, norms. □

This remark explains the “Sobolev” in the name of the spaces (the Bessel-potential spaces are also known as fractional Sobolev spaces). The “logarithmic” comes from the fact that at least for bounded connected C^∞-domains and $p>1$ we have $H^0_p(\log H)_a(\Omega) = L_p(\log L)_a(\Omega)$, where the latter space can be defined with the help of logarithms (for more information, see [5, 2.6]).

Convention: From now on we will denote $F^s_{p2}(\mathbb{R}^n)$ and $F^s_{p2}(\Omega)$ respectively by $H^s_p(\mathbb{R}^n)$ and $H^s_p(\Omega)$, for all $s\in\mathbb{R}$ and $0<p<\infty$. In particular, the quasi-norm to be considered in an H^s_p-space is the quasi-norm of the corresponding F^s_{p2}-space, for some fixed φ in accordance with 2.1. □

We complete now the definitions given in the beginning of this subsection in the following way: for all $s\in\mathbb{R}$ and $0<p<\infty$,

$$H^s_p(\log H)_0(\Omega) \equiv H^s_p(\Omega)\ . \tag{7}$$

Recalling that Ω is any bounded domain in $\mathbb{R}^n$, using Corollary 2.3.7 it is easy to prove the following result.

Proposition 2.4.5. *Given any* $s \in \mathbb{R}$, $0 < p < \infty$, $\varepsilon > 0$ *and* $-\infty < a_2 \leq a_1 < \infty$, *there exist the embeddings*

$$H^s_{p+\varepsilon}(\Omega) \hookrightarrow H^s_p(\log H)_{a_1}(\Omega) \hookrightarrow H^s_p(\log H)_{a_2}(\Omega) \hookrightarrow H^s_{p-\varepsilon}(\Omega) ,$$

where in the last one we are also assuming that $p - \varepsilon > 0$. ∎

3 – The embedding $H^{s_1}_{p_1^{\lambda(j-1)}}(\Omega) \hookrightarrow H^{s_2}_{p_2^{\lambda(j)}}(\Omega)$

Let Ω be a bounded domain in $\mathbb{R}^n$ and

$$(8) \qquad -\infty < s_2 < s_1 < \infty, \quad 0 < p_1 < p_2 < \infty, \qquad \text{with } \; s_1 - n/p_1 = s_2 - n/p_2$$

(note that, in presence of this equality, each one of the two preceding conditions — $s_1 > s_2$ or $p_1 < p_2$ — implies the other).

3.1. The main result

In this section we are going to show the following result.

Proposition 3.1.1. *Given any* $\Lambda \in \Big]\max\{s_2 - s_1, \, -n/p_2, \, -n/(2p_1)\}, \, 0\Big[$, *there exists* $c > 0$ *such that for all* $k \in \mathbb{N}$ *and all* $\lambda \in [\Lambda, 0[$,

$$(9) \qquad e_k\Big(id_\lambda \colon \, H^{s_1}_{p_1^{2\lambda}}(\Omega) \hookrightarrow H^{s_2}_{p_2^{\lambda}}(\Omega)\Big) \;\leq\; c\,(-\lambda)^{-2(s_1-s_2)/n - 1/\tilde{p}_2}\, k^{-(s_1-s_2)/n} ,$$

where $\tilde{p}_2 = \min\{1, p_2\}$ *and* e_k *stands for the* k*-th entropy number.*

We recall that, for $k \in \mathbb{N}$, the k-th entropy number $e_k(S)$ of a bounded linear operator $S \colon E \to F$, where E and F are quasi-Banach spaces, is defined by

$$e_k(S) \equiv \inf\left\{\varepsilon > 0 \colon \;\; S(U_E) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon\, U_F) \;\; \text{for some } \; b_1, ..., b_{2^{k-1}} \in F\right\} .$$

Here U_E and U_F stand for the closed unit balls respectively in E and F.

For a brief introduction to these numbers and their properties, see [5, pp. 7–8].

3.2. The proof

First of all note that $H^{s_1}_{p_1^{2\lambda}}(\Omega) \hookrightarrow H^{s_2}_{p_2^{\lambda}}(\Omega)$ makes sense: indeed, the hypothesis on λ imply that $p_1^{2\lambda} < p_2^{\lambda}$ and $s_1 - n/p_1^{2\lambda} - (s_2 - n/p_2^{\lambda}) = -\lambda > 0$, so that the existence of the embedding follows from 2.2(iii).

3.2.1. Reductions

Note that the map $\Psi_\lambda : H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n) \to H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)$ given by $\Psi_\lambda(f) = \psi f$, where the fixed $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfies $\psi \equiv 1$ on Ω, is well-defined: it can be thought of as the composition

$$\begin{array}{ccccc} H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n) & \to & H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n) & \to & H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n) \\ f & \mapsto & f & \mapsto & \psi f \end{array}$$

(cf. 2.1(iii) and Proposition 2.3.5).

Lemma 3.2.1. *For all* $k \in \mathbb{N}$, $e_k(id_\lambda) \le 2\, e_k(\Psi_\lambda)$.

Proof: This is similar to the proof of Corollary 2.3.7. First note that $e_k(\Psi_\lambda) < \infty$ ($e_k(\Psi_\lambda) \le \|\Psi_\lambda\|$ and Ψ_λ is, clearly, a bounded linear operator). Consider now any $\varepsilon > e_k(\Psi_\lambda)$, so that there exist 2^{k-1} balls of radius ε in $H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)$ which together cover $\Psi_\lambda(U(\mathbb{R}^n))$, where $U(\mathbb{R}^n)$ is the closed unit ball of $H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$. Denote by b_l, $l = 1, ..., 2^{k-1}$, the centers of those balls. Given g in the closed unit ball $U(\Omega)$ of $H^{s_1}_{p_1^{2\lambda}}(\Omega)$, there exists $f \in H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$ such that $f|_\Omega = g$ and $\|f \,|\, H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)\| \le 2$. Then $(1/2)\, f \in U(\mathbb{R}^n)$ and therefore $\|\psi((1/2)\, f) - b_l \,|\, H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)\| \le \varepsilon$ for some b_l, that is, $\|\psi f - 2\, b_l \,|\, H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)\| \le 2\,\varepsilon$. Since the hypothesis $\psi \equiv 1$ on Ω implies that $(\psi f)|_\Omega = f|_\Omega = g$, we can also write $\|g - 2\, b_l|_\Omega \,|\, H^{s_2}_{p_2^{\lambda}}(\Omega)\| \le 2\,\varepsilon$. We have thus shown that $e_k(id_\lambda) \le 2\,\varepsilon$. Since ε was any number greater than $e_k(\Psi_\lambda)$, the lemma is proved. ∎

As a consequence of this lemma, in order to prove the main result in 3.1 it suffices to show that (9) holds with the operator Ψ_λ instead of id_λ, for some ψ chosen as indicated above.

Let now ψ_r, for $r \in \mathbb{Z}$, denote any $\mathcal{S}$-function with $\psi_r \equiv 1$ on $2^r B^n_\infty$ and $\operatorname{supp} \psi_r \subset 2^{r+1} B^n_\infty$, where B^n_∞ is the closed unit ball in the space ℓ^n_∞ of (complex) n-sequences with the norm $|\cdot|_\infty$. Let $\Psi_{r,\lambda}$ denote the corresponding bounded linear operator from $H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$ into $H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)$ defined by $\Psi_{r,\lambda}(f) = \psi_r f$.

It is clear that it is possible to find an $r \in \mathbb{Z}$ such that $\Omega \subset 2^r B_\infty^n$. Fix this r, fix a function ψ_0 and define $\psi_r \equiv \psi_0(2^{-r}\cdot)$. Note that, in particular, ψ_r is a $\mathcal{D}(\mathbb{R}^n)$-function with $\psi_r \equiv 1$ on Ω.

Lemma 3.2.2. *There exists $c > 0$ such that, for all $k \in \mathbb{N}$ and all $\lambda \in [\Lambda, 0[$,*

$$e_k(\Psi_{r,\lambda}) \leq c\, e_k(\Psi_{0,\lambda}) .$$

Proof: Note that $\Psi_{r,\lambda}$ is given by the composition

$$\begin{array}{ccccccc} H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n) & \xrightarrow{A_\lambda} & H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n) & \xrightarrow{\Psi_{0,\lambda}} & H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n) & \xrightarrow{B_\lambda} & H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n) \\ f & \longmapsto & f(2^r\cdot) & \longmapsto & \psi_0\, f(2^r\cdot) & \longmapsto & (\psi_0\, f(2^r\cdot))(2^{-r}\cdot) \end{array}$$

where $\langle f(m\cdot), \varphi\rangle = m^{-n}\langle f, \varphi(m^{-1}\cdot)\rangle$, for all $\varphi \in \mathcal{S}$ and all $m \in \mathbb{R}\backslash\{0\}$ (cf. also [9, p. 209]), so that the multiplicativity of the entropy numbers yields

$$e_k(\Psi_{r,\lambda}) \leq \|A_\lambda\|\, \|B_\lambda\|\, e_k(\Psi_{0,\lambda})$$

for all $k \in \mathbb{N}$. It only remains to show that $\|A_\lambda\|\, \|B_\lambda\|$ can be bounded above by a positive constant independent of λ. Consider $R \in \mathbb{R}\backslash\{0\}$ and the linear operator $T\colon \mathcal{S}' \to \mathcal{S}'$ given by $Tf = f(2^R\cdot)$. It is a straightforward exercise to show that the set of all $\mathcal{S}'$-analytic functions in $A = \{z \in \mathbb{C}\colon 0 < \Re z < 1\}$ is invariant under T (in the sense explained in Definition 2.3.1). The interpolation theory explained in 2.3.1 applies then to give upper estimates for $\|A_\lambda\|$ and $\|B_\lambda\|$ which are independent of λ (for example, in the case of B_λ we make $R = -r$, so that B_λ is obtained by interpolating between the parameters p_2 and p_2^Λ with $\theta = \lambda/\Lambda$). ∎

As a consequence of the two preceding lemmas, in order to prove the main result in 3.1 it suffices to show that (9) holds with the operator $\Psi_{0,\lambda}$ instead of id_λ, for some ψ_0 chosen as indicated above.

3.2.2. Main estimate

To simplify notation, let us write ψ and Ψ_λ for ψ_0 and $\Psi_{0,\lambda}$ respectively. So, in this subsubsection ψ is some fixed $\mathcal{S}$-function with $\psi \equiv 1$ on B_∞^n and $\operatorname{supp}\psi \subset 2\,B_\infty^n$ (but see below for the specialization assumed from Step 2 on).

Note that any $f \in \mathcal{S}'$ can be written as

$$f = \sum_{j=0}^{\infty} (\varphi_j \hat{f})^\vee$$

where $(\varphi_j)_{j\in\mathbb{N}_0}$ is a sequence fixed according to 2.1, and that the action of

$$\begin{aligned} \Psi_\lambda\colon\ H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n) &\to H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n) \\ f &\mapsto \psi f \end{aligned} \tag{10}$$

on $f\in H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$ can be decomposed by means of

$$\psi f \,=\, \psi\sum_{j=0}^{\infty}(\varphi_j\hat{f})\check{}\,=\,\psi\sum_{j=0}^{N}(\varphi_j\hat{f})\check{}+\psi\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\,\equiv\, f_N+f^N\ , \tag{11}$$

for any $N\in\mathbb{N}$.

Step 1: *The estimate for f^N*

Using Proposition 2.3.5, we can write

$$\left\|\psi\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\mid H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)\right\| \,\le\, c_1\left\|\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\mid H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)\right\|\ , \tag{12}$$

where $c_1>0$ is independent of f, N and λ ($\in[\Lambda,0[$).

Note that, by 2.1(iii) and the Fourier multiplier assertion from [8, 1.6.3],

$$\begin{aligned} \left\|\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\mid H^{s_2}_{p_2}(\mathbb{R}^n)\right\| &\le c_2\left\|\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\mid H^{s_1}_{p_1}(\mathbb{R}^n)\right\| \\ &\le c_3\,\|f\mid H^{s_1}_{p_1}(\mathbb{R}^n)\|\ , \end{aligned} \tag{13}$$

for all $f\in H^{s_1}_{p_1}(\mathbb{R}^n)$, and where $c_2,c_3>0$ are independent of f and N.

On the other hand, taking advantage of the fact that $(\operatorname{supp}\sum_{j=N+1}^{\infty}\varphi_j\hat{f})\cap 2^N\mathring{B}^n_2=\emptyset$ (where $\mathring{B}^n_2$ is the open unit ball in the Euclidean $\mathbb{R}^n$) and using again 2.1(iii) and [8, 1.6.3],

$$\begin{aligned} \left\|\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\mid H^{s_2}_{p_2^{\Lambda}}(\mathbb{R}^n)\right\| &\le c_4\,2^{-N(n/p_2^{\Lambda}-s_2)}\left\|\Big(\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\Big)(2^{-N+1}\cdot)\mid H^{s_2}_{p_2^{\Lambda}}(\mathbb{R}^n)\right\| \\ &\le c_5\,2^{-N(n/p_2^{\Lambda}-s_2)}\left\|\Big(\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\Big)(2^{-N+1}\cdot)\mid H^{s_1}_{p_1^{2\Lambda}}(\mathbb{R}^n)\right\| \\ &\le c_6\,2^{-N(n/p_2^{\Lambda}-s_2)+N(n/p_1^{2\Lambda}-s_1)}\left\|\sum_{j=N+1}^{\infty}(\varphi_j\hat{f})\check{}\mid H^{s_1}_{p_1^{2\Lambda}}(\mathbb{R}^n)\right\| \\ &\le c_7\,2^{N\Lambda}\,\|f\mid H^{s_1}_{p_1^{2\Lambda}}(\mathbb{R}^n)\|\ , \end{aligned} \tag{14}$$

for all $f\in H^{s_1}_{p_1^{2\Lambda}}(\mathbb{R}^n)$, with $c_4,c_5,c_6,c_7>0$ independent of f and N.

Consider now the linear operator $T\colon \mathcal{S}' \to \mathcal{S}'$ given by $Tf = \sum_{j=N+1}^{\infty}(\varphi_j \hat{f})^\vee$. It is easy to see that the set of all $\mathcal{S}'$-analytic functions in $A = \{z \in \mathbb{C}\colon 0 < \Re z < 1\}$ is invariant under T (in the sense explained in Definition 2.3.1). Interpolating then between p_1 and $p_1^{2\Lambda}$ (for the source space) and between p_2 and p_2^{Λ} (for the target space), in both cases with $\theta = \lambda/\Lambda$, we get, with the help of (13) and (14) above and the interpolation theory explained in 2.3.1,

$$\left\| \sum_{j=N+1}^{\infty} (\varphi_j \hat{f})^\vee \mid H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n) \right\| \le c_8\, 2^{N\lambda}\, \|f \mid H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)\| ,$$

for all $f \in H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$ and with $c_8 > 0$ independent of f, N and λ. Putting this estimate in (12), we finally get

$$\|f^N \mid H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)\| \le c_1\, c_8\, 2^{N\lambda}\, \|f \mid H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)\| . \tag{15}$$

Step 2: *The estimate for $f_{N,2}$*

As in [5, p. 108], we use now the decomposition

$$f_N = \sum_{j=0}^{N} f_N^j + f_{N,2} \equiv f_{N,1} + f_{N,2} , \tag{16}$$

where, for each $j \in \mathbb{N}_0$,

$$f_N^j = C\,\psi \sum_{m\in\mathbb{Z}^n,\, |m|\le N_j(\lambda)} (\varphi_j \hat{f})^\vee (2^{-j}m)\, (\psi - \psi_\ell)^\vee (2^{j+1}\cdot - 2\,m) \tag{17}$$

(with the convention that ψ_ℓ doesn't show up when $j = 0$) and

$$f_{N,2} = C\,\psi \sum_{j=0}^{N} \sum_{m\in\mathbb{Z}^n,\, |m|> N_j(\lambda)} (\varphi_j \hat{f})^\vee (2^{-j}m)\, (\psi - \psi_\ell)^\vee (2^{j+1}\cdot - 2\,m) \tag{18}$$

(with the same convention as before for the case $j = 0$), where $C = 2^n(2\pi)^{-n/2}$, ℓ can be taken to be $\ell = \log_2(8\sqrt{n})$, $\psi_\ell = \psi(2^\ell \cdot)$ and

$$N_j(\lambda) = \max\left\{N^2\lambda^2,\, 2^{j+2}\sqrt{n}\right\}, \qquad j \in \mathbb{N}_0 . \tag{19}$$

To go further in our estimates, we need to specialize a little bit our function ψ fixed in the beginning of 3.2.2. We assume from now on that ψ also has the following property: for any $a > 0$ and any $\gamma \in \mathbb{N}_0^n$ there exists $c_{\gamma,a} > 0$ such that for all x in $\mathbb{R}^n$ with $|x| \geq 1$,

$$|D^\gamma \check{\psi}(x)| \;\leq\; c_{\gamma,a}\, 2^{-\sqrt{|x|}}\, |x|^{-a}$$

(for the existence of such functions ψ, see [7, 1.2.1 and 1.2.2]).

Reasoning much in the same way as in [5, pp. 108–109] for the $f_{N,2}$, but controlling the dependence on λ (to the effect of which we can take advantage of Proposition 2.3.5 and the arguments in [5, p. 143]), we get

$$\|f_{N,2} \,|\, H^{s_2}_{p_2^\lambda}(\mathbb{R}^n)\| \;\leq\; c\, 2^{N\lambda}\, \|f \,|\, H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)\| \tag{20}$$

for all $f \in H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$, with $c > 0$ independent of f, N and λ ($\in [\Lambda, 0[$).

Step 3: *The estimate for S_j and T_j*

For each $j \in \mathbb{N}$, let $F_j\colon H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n) \to H^{s_2}_{p_2^\lambda}(\mathbb{R}^n)$ be given by $F_j f = f_N^j$ (cf. (16) and (17) above).

If $N^2\lambda^2 \leq 2^{j+2}\sqrt{n}$, F_j can be obtained as the composition

$$H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n) \xrightarrow{\;S_j\;} \ell^{M_j}_{p_1^{2\lambda}} \xrightarrow{\;emb_j\;} \ell^{M_j}_{p_2^\lambda} \xrightarrow{\;T_j\;} H^{s_2}_{p_2^\lambda}(\mathbb{R}^n)\;, \tag{21}$$

where M_j is the number of n-tuples $m \in \mathbb{Z}^n$ such that $|m| \leq 2^{j+2}\sqrt{n}$, emb_j is the natural embedding,

$$S_j f \;=\; \Big((\varphi_j \hat{f})\check{}\,(2^{-j}m)\Big)_{|m|\leq 2^{j+2}\sqrt{n}}$$

and

$$T_j(a_m)_{|m|\leq 2^{j+2}\sqrt{n}} \;=\; C\,\psi \sum_{|m|\leq 2^{j+2}\sqrt{n}} a_m (\psi - \psi_\ell)\check{}\,(2^{j+1}\cdot - 2\,m)\;.$$

Now, following the same type of arguments as in [5, p. 110], however paying attention to the possible dependence on λ, one gets

$$\|S_j\| \;\leq\; c_1\, 2^{j(n/p_1^{2\lambda} - s_1)} \tag{22}$$

and

$$\|T_j\| \;\leq\; c_2\, 2^{j(s_2 - n/p_2^\lambda)}\;, \tag{23}$$

with $c_1, c_2 > 0$ independent of $j \in \mathbb{N}$ and $\lambda \in [\Lambda, 0[$.

The multiplicativity of the entropy numbers applied to (21) then gives

$$e_{k_j}(F_j) \;\leq\; c_1\, c_2\, 2^{j\lambda}\, e_{k_j}(emb_j) \tag{24}$$

for all $k_j \in \mathbb{N}$ and all $j \in \mathbb{N}$, where

$$e_{k_j}(emb_j) \le c_3 \begin{cases} 1 & \text{if } 1 \le k_j \le \log_2(2M_j) \\ \Big(k_j^{-1} \log_2\Big(1+(2M_j)/k_j\Big)\Big)^{1/p_1^{2\lambda}-1/p_2^{\lambda}} & \text{if } \log_2(2M_j) \le k_j \le 2M_j \\ 2^{-k_j/(2M_j)}\,(2M_j)^{1/p_2^{\lambda}-1/p_1^{2\lambda}} & \text{if } k_j \ge 2M_j \end{cases}, \tag{25}$$

with $c_3 > 0$ independent of M_j, k_j and λ. Here we followed the proof of [5, Proposition 3.2.2], taking care on the possible dependence on λ.

We remark that $M_j \le c_4\, 2^{nj}$, for some positive constant c_4, and, to the effect of using formula (24) to estimate $e_{k_j}(F_j)$, there is no loss of generality in assuming that $M_j = c_4\, 2^{nj}$. We shall take advantage of this in the sequel.

Step 4: *The estimate for the terms $f_{N,3}$, $f_{N,4}$ and $f_{N,5}$*

For each $k \in \mathbb{N}$ and $\lambda \in [\Lambda, 0[$ we define now

$$N \equiv \Big[(s_1 - s_2)\,(-\lambda)^{-1} \log_2(-k\,\lambda)/n\Big]$$

and also

$$L \equiv \Big[\log_2(-k\,\lambda)/n\Big] \quad \text{and} \quad H \equiv \Big[\log_2\Big((N^2\lambda^2)/\sqrt{n}\Big) - 1\Big]\,. \tag{26}$$

It is clear that if we assume that $k \ge c\,(-\lambda)^{-1}$, for a suitable choice of the positive constant c, then $N, L, H \in \mathbb{N}$, $N \ge L$ and

$$2^{L+2}\sqrt{n} > 2\sqrt{n}\; 2^{-\lambda N/(s_1-s_2)} > 2\,N^2\,\lambda^2 \ge 2^{H+2}\sqrt{n}\,,$$

from which follows, in particular, that

$$N \ge L > H > 0\,.$$

We are thus going to make that assumption $k \ge c\,(-\lambda)^{-1}$ and split $f_{N,1}$ (cf. (16)) as follows:

$$f_{N,1} = \sum_{j=0}^{N} f_N^j = \sum_{j=0}^{H} f_N^j + \sum_{j=H+1}^{L} f_N^j + \sum_{j=L+1}^{N} f_N^j \equiv f_{N,3} + f_{N,4} + f_{N,5}\,.$$

Note that these three terms define operators — which we denote, respectively, by $F_{N,3}$, $F_{N,4}$ and $F_{N,5}$ — from $H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$ into $H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)$, each of them being a sum of some of the F_j's defined in Step 3. Note also that in the case of $F_{N,4}$ and

$F_{N,5}$ the j's involved satisfy the condition $N^2\lambda^2 \le 2^{j+2}\sqrt{n}$, so that we can take advantage of formula (24) (and (25)). We obtain then, by a reasoning similar to that used to estimate the terms $f_{N,4}$ and $f_{N,5}$ in [5, pp. 111–112] (and, in the case of the latter term, also an argument as in [5, (3.4.2/28)]) that

$$e_{c_1[-\lambda k]}(F_{N,4}) \le c_2(-\lambda)^{-(s_1-s_2)/n}\, k^{-(s_1-s_2)/n} \tag{27}$$

and

$$e_{c_3 k}(F_{N,5}) \le c_4(-\lambda)^{-2(s_1-s_2)/n-1/\tilde{p}_2}\, k^{-(s_1-s_2)/n}\;, \tag{28}$$

with $\tilde{p}_2 \equiv \min\{1, p_2\}$, where $c_1, c_2, c_3, c_4 > 0$ are independent of k and λ and, moreover, $c_1, c_3 \in \mathbb{N}$.

As to $F_{N,3}$, first we write, for $f \in H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$,

$$\|f_{N,3} \mid H^{s_2}_{p_2^\lambda}(\mathbb{R}^n)\|^{\tilde{p}_2} = \left\|\sum_{j=0}^{H} f_N^j \mid H^{s_2}_{p_2^\lambda}(\mathbb{R}^n)\right\|^{\tilde{p}_2} \le$$

$$\le c_5\, \|f \mid H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)\|^{\tilde{p}_2} + \left\|\sum_{j=0}^{H} C\,\psi \sum_{m\in\mathbb{Z}^n} (\varphi_j \hat{f})\check{}\,(2^{-j}m)\,(\psi-\psi_\ell)\check{}\,(2^{j+1}\cdot\, - 2\,m) \mid H^{s_2}_{p_2^\lambda}(\mathbb{R}^n)\right\|^{\tilde{p}_2}, \tag{29}$$

with $c_5 > 0$ independent of f, N, H and λ, where we have used (17) and (20) (slightly adapted). This is similar to [5, (3.3.2/45)]. Next we note (cf. (11), (16), (17) and (18)) that the second term on the right-hand side of (29) is

$$\left\|\psi \sum_{j=0}^{H} (\varphi_j \hat{f})\check{} \mid H^{s_2}_{p_2^\lambda}(\mathbb{R}^n)\right\|^{\tilde{p}_2},$$

which, by Proposition 2.3.5, is bounded above by

$$c_6 \left\|\sum_{j=0}^{H} (\varphi_j \hat{f})\check{} \mid H^{s_2}_{p_2^\lambda}(\mathbb{R}^n)\right\|^{\tilde{p}_2},$$

with $c_6 > 0$ independent of f, H and λ.

Consider now momentarily that H is any natural number (so not necessarily defined as in (26)). By the same type of arguments as used in (13), we can write, for all $f \in H^{s_1}_{p_1}(\mathbb{R}^n)$,

$$\left\|\sum_{j=0}^{H} (\varphi_j \hat{f})\check{} \mid H^{s_2}_{p_2}(\mathbb{R}^n)\right\| \le c_7\, \|f \mid H^{s_1}_{p_1}(\mathbb{R}^n)\|$$

and also, for all $f \in H^{s_1}_{p_1^{2\Lambda}}(\mathbb{R}^n)$,

$$\left\| \sum_{j=0}^{H} (\varphi_j \hat{f})\check{} \mid H^{s_2}_{p_2^{\Lambda}}(\mathbb{R}^n) \right\| \leq c_8 \, \|f \mid H^{s_1}_{p_1^{2\Lambda}}(\mathbb{R}^n)\| ,$$

where $c_7, c_8 > 0$ are independent of f and H. An interpolation argument, as in the last part of Step 1 above, gives us then

$$\left\| \sum_{j=0}^{H} (\varphi_j \hat{f})\check{} \mid H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n) \right\| \leq c_9 \, \|f \mid H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)\|$$

for all $f \in H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n)$ and with $c_9 > 0$ independent of f, H and λ.

Returning now to (29), we get

$$\|F_{N,3}\| \leq c_{10} , \tag{30}$$

with c_{10} independent of N, H and λ.

Now we decompose $F_{N,3}$ as

$$\begin{array}{ccccccc} H^{s_1}_{p_1^{2\lambda}}(\mathbb{R}^n) & \to & H(\lambda, d) & \xrightarrow{I_\lambda} & H(\lambda, d) & \to & H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n) \\ f & \mapsto & f_{N,3} & \mapsto & f_{N,3} & \mapsto & f_{N,3} \end{array}$$

where $H(\lambda, d)$ is the range of $F_{N,3}$, which is a finite-dimensional subspace of $H^{s_2}_{p_2^{\lambda}}(\mathbb{R}^n)$ with dimension $d \leq 5^n L(N\lambda)^{2n}$.

By the multiplicativity of the entropy numbers and (30), we can write, for all $r \in \mathbb{N}$,

$$e_r(F_{N,3}) \leq c_{10} \, e_r(I_\lambda) . \tag{31}$$

Adapting the proof of [1, pp. 73–74] — cf. also, for real spaces, [6, Theorem 12.1.10 and Proposition 12.1.13] — to the case of the identity operator in a $\tilde{p}_2$-normed space of dimension d, we obtain, for all $r \in \mathbb{N}$,

$$e_r(I_\lambda) \leq 4^{1/\tilde{p}_2} \, 2^{-(r-1)/(2d)} .$$

Putting this estimate in (31) and choosing $r = [-N\lambda \, 2 \, d] + 2$, we get, for $k \geq c'(-\lambda)^{-1}$ (for a suitable choice of the positive constant c'),

$$e_{c_{11}k}(F_{N,3}) \leq c_{12} \, (-\lambda)^{-(s_1-s_2)/n} \; k^{-(s_1-s_2)/n} , \tag{32}$$

with $c_{11}, c_{12} > 0$ independent of k and λ and, moreover, with $c_{11} \in \mathbb{N}$.

Step 5: *Assembling things together*

Using the $\tilde{p}_2$-additivity of the entropy numbers (see [5, Lemma 1.3.1/1(iii)]) — valid when the target space is a $\tilde{p}_2$-Banach space —, we can put together the estimates that have been obtained so far (i.e., (15), (20), (27), (28) and (32)) and write

$$e_{c_1 k}(\Psi_\lambda) \le c_2\,(-\lambda)^{-2(s_1-s_2)/n-1/\tilde{p}_2}\; k^{-(s_1-s_2)/n}\;, \tag{33}$$

for all natural $k \ge c_3(-\lambda)^{-1}$ and $\lambda \in [\Lambda, 0[$, where $c_1, c_2, c_3 > 0$ are independent of k and λ and, moreover, $c_1 \in \mathbb{N}$.

By standard arguments, we can write k instead of $c_1 k$ in (33) and lift the restriction $k \ge c_3(-\lambda)^{-1}$.

3.2.3. Conclusion

Combining the result just obtained with the reductions established in 3.2.1, we finally get (9). ∎

3.3. A corollary

Under the same conditions of the main result in 3.1, (9) also holds with id_λ replaced by the natural embedding $H^{s_1}_{p_1^{2\lambda}}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\Omega)$. This is an immediate consequence of (9), Corollary 2.3.7 and the multiplicativity of the entropy numbers applied to the composition

$$H^{s_1}_{p_1^{2\lambda}}(\Omega) \hookrightarrow H^{s_2}_{p_2^{\lambda}}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\Omega)\;.$$

4 – The limiting embedding $H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)$

Let Ω be a bounded domain in $\mathbb{R}^n$ and let the parameters s_1, s_2, p_1 and p_2 satisfy (8). Let $a_1, a_2 \in \mathbb{R}$ be such that $a_2 \le a_1$ and $a_1 > 0$ (we are ruling out the case $a_1 \le 0$ because it was already considered in [5, p. 149], at least when Ω is a smooth domain).

First of all let us make it clear that, under these conditions, there exists the embedding of the title of this section. In fact,

- in the case $a_2 \le 0$ it follows immediately from the composition

$$H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \hookrightarrow H^{s_1}_{p_1}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)$$

(cf. 2.2(iii), (7) and Proposition 2.4.5);

- in the case $a_2 > 0$, we proceed as follows: given $f \in H^{s_1}_{p_1}(\log H)_{a_1}(\Omega)$, we have $f = \sum_{j=J}^{\infty} g_j$ in $\mathcal{D}'(\Omega)$, with $g_j \in H^{s_1}_{p_1^{\lambda(j)}}(\Omega)$ and $(\sum_{j=J}^{\infty} 2^{ja_1p_1} \|g_j | H^{s_1}_{p_1^{\lambda(j)}}(\Omega)\|^{p_1})^{1/p_1} < \infty$ (cf. Definition 2.4.2); due to the relation between the parameters and 2.2(iii), then each g_j belongs also to $H^{s_2}_{p_2^{\lambda(j)}}(\Omega)$ and $\|g_j \,|\, H^{s_2}_{p_2^{\lambda(j)}}(\Omega)\| \le c\,\|g_j \,|\, H^{s_1}_{p_1^{\lambda(j)}}(\Omega)\|$, with $c > 0$ independent of j (this can be seen by interpolation for the spaces in $\mathbb{R}^n$ and Proposition 2.2.1); finally, using $a_2 \le a_1$ and $p_1 < p_2$, we get

$$\left(\sum_{j=J}^{\infty} 2^{ja_2p_2}\, \|g_j \,|\, H^{s_2}_{p_2^{\lambda(j)}}(\Omega)\|^{p_2}\right)^{1/p_2} \le c\left(\sum_{j=J}^{\infty} 2^{ja_1p_1}\, \|g_j \,|\, H^{s_1}_{p_1^{\lambda(j)}}(\Omega)\|^{p_1}\right)^{1/p_1},$$

and the result follows.

4.1. Lower estimate

Under the hypothesis considered above, here we want to show that there exists $c > 0$ such that, for all $k \in \mathbb{N}$,

$$e_k(id) \ge c\,k^{-(s_1-s_2)/n}, \tag{34}$$

where

$$id\colon\ H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\log H)_{a_2}(\Omega). \tag{35}$$

First note that, for $\varepsilon > 0$ such that $p_2 - \varepsilon > 0$,

$$H^{s_1}_{p_1+\varepsilon}(\Omega) \hookrightarrow H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\log H)_{a_2}(\Omega) \hookrightarrow H^{s_2}_{p_2-\varepsilon}(\Omega)$$

(cf. Proposition 2.4.5), so that the multiplicativity property of the entropy numbers yields, for all $k \in \mathbb{N}$,

$$\begin{aligned} e_k(id) \ \ge\ & \left\|H^{s_1}_{p_1+\varepsilon}(\Omega) \hookrightarrow H^{s_1}_{p_1}(\log H)_{a_1}(\Omega)\right\|^{-1} \\ & \times \left\|H^{s_2}_{p_2}(\log H)_{a_2}(\Omega) \hookrightarrow H^{s_2}_{p_2-\varepsilon}(\Omega)\right\|^{-1} \\ & \times e_k\Big(H^{s_1}_{p_1+\varepsilon}(\Omega) \hookrightarrow H^{s_2}_{p_2-\varepsilon}(\Omega)\Big). \end{aligned} \tag{36}$$

It is well-known (cf., for example, [5, Theorem 3.3.3/1]) that the entropy numbers on the right-hand side of (36) can be estimated from below by "positive constant times $k^{-(s_1-s_2)/n}$", at least in the case of smooth domains Ω. This restriction about Ω can, however, be lifted using the same arguments as in [5, proof of Theorem 3.5] (cf. also the proof of Lemma 3.2.1 above), and this finishes the proof of (34).

4.2. Upper estimate

Under the hypothesis above we have the following result (where $\tilde{p}_2 \equiv \min\{1, p_2\}$ and id is given by (35)).

Proposition 4.2.1. *If* $0 \le a_2 < a_1 - 2(s_1 - s_2)/n - 1/\tilde{p}_2$ *there exists* $c > 0$ *such that, for all* $k \in \mathbb{N}$,

$$e_k(id) \le c\, k^{-(s_1-s_2)/n} .$$

Proof: Consider first the case $a_2 > 0$.

Choose a natural $J > 1$ such that $\lambda(J) \equiv -2^{-J} \ge \Lambda$, where Λ is a negative number chosen in accordance with the assertion of Proposition 3.1.1. Apply this proposition to write that there exists $c_1 > 0$ such that for all $k, j \in \mathbb{N}$ with $j \ge J$,

$$e_k(id_{\lambda(j)}) \le c_1\, 2^{ja}\, k^{-(s_1-s_2)/n} , \tag{37}$$

where $a \equiv 2(s_1 - s_2)/n + 1/\tilde{p}_2$ and $\lambda(j) = -2^{-j}$, as in (6).

Consider any natural $L > J$ and, for each $j = J, ..., L$, define

$$k(j) \equiv \left[2^{n(L-j)\gamma/(s_1-s_2)} + 1\right] ,$$

where $\gamma \equiv a_1 - a - a_2 - \varepsilon > 0$ (for some fixed $\varepsilon > 0$), so that (37) applied to these $k(j)$ (and the definition of the entropy numbers — see 3.1) allows us to conclude that, for each $j = J, ..., L$, there exist $2^{k(j)-1}$ balls of radius

$$2\, c_1\, 2^{ja}\, k(j)^{-(s_1-s_2)/n}$$

in $H^{s_2}_{p_2^{\lambda(j)}}(\Omega)$ which together cover the closed unit ball of $H^{s_1}_{p_1^{2\lambda(j)}}(\Omega)$. Denote by h_{jm} $(m = 1, ..., 2^{k(j)-1})$ the centers of those balls and note that $h_{jm} \in H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)$ (cf. Proposition 2.4.5).

Given now any g in the closed unit ball of $H^{s_1}_{p_1}(\log H)_{a_1}(\Omega)$, let $g_j \in H^{s_1}_{p_1^{\lambda(j)}}(\Omega)$, $j \ge J-1$, be such that $g = \sum_{j=J-1}^{\infty} g_j$ in $\mathcal{D}'(\Omega)$ and

$(\sum_{j=J-1}^{\infty} 2^{ja_1p_1} \|g_j \,|\, H^{s_1}_{p_1^{\lambda(j)}}(\Omega)\|^{p_1})^{1/p_1} \leq 2$ (cf. Definition 2.4.2). This clearly implies that, for each $j \geq J$, $2^{(j-1)a_1-1}g_{j-1}$ belongs to the closed unit ball of $H^{s_1}_{p_1^{2\lambda(j)}}(\Omega)$.

For each $j = J, ..., L$ we choose h_{jm} such that

$$\left\|2^{(j-1)a_1-1}\, g_{j-1} - \, h_{jm} \,|\, H^{s_2}_{p_2^{\lambda(j)}}(\Omega)\right\| \leq 2\ c_1\, 2^{j(a_1-a_2)}\, 2^{-L\gamma}\ ,$$

which is equivalent to

$$\left\|g_{j-1} - \, 2^{-(j-1)a_1+1}\, h_{jm} \,|\, H^{s_2}_{p_2^{\lambda(j)}}(\Omega)\right\| \leq 4\ c_1\, 2^{a_1}\, 2^{-ja_2}\, 2^{-L\gamma} \tag{38}$$

$(j = J, ..., L)$.

For $j > L$ we just consider the following crude estimate taken from (37) when $k = 1$:

$$\|g_{j-1} \,|\, H^{s_2}_{p_2^{\lambda(j)}}(\Omega)\| \leq 2^{1/\tilde{p}_2}\, c_1\, 2^{a_1}\, 2^{-j(a_1-a)}\ .$$

This, (38) and Definition 2.4.2 imply that

$$\begin{aligned}
&\left\|g - \sum_{j=J}^{L} 2^{-(j-1)a_1+1}\, h_{jm} \,|\, H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)\right\| = \\
&\quad = \left\|\sum_{j=J}^{L} (g_{j-1} - 2^{-(j-1)a_1+1}\, h_{jm}) + \sum_{j=L+1}^{\infty} g_{j-1} \,|\, H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)\right\| \\
&\quad \leq \left(\sum_{j=J}^{L} 2^{ja_2p_2} \left\|g_{j-1} - 2^{-(j-1)a_1+1}\, h_{jm} \,|\, H^{s_2}_{p_2^{\lambda(j)}}(\Omega)\right\|^{p_2}\right. \\
&\qquad \left. + \sum_{j=L+1}^{\infty} 2^{ja_2p_2}\, \|g_{j-1} \,|\, H^{s_2}_{p_2^{\lambda(j)}}(\Omega)\|^{p_2}\right)^{1/p_2} \\
&\quad \leq c_2\, 2^{-L\gamma}\ ,
\end{aligned} \tag{39}$$

where $c_2 > 0$ does not depend on g nor on L.

Since the number of possible choices of the elements of $H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)$ of the form $\sum_{j=J}^{L} 2^{-(j-1)a_1+1}h_{jm}$ clearly does not exceed $\prod_{j=J}^{L} 2^{k(j)-1}$, which is bounded above by $2^{c_3 2^{n\gamma L/(s_1-s_2)}}$ (where $c_3 > 0$ does not depend on L), we have just shown that, for each natural $L > J$,

$$e_{[c_3 2^{n\gamma L/(s_1-s_2)}]+2}(id) \leq c_2\, 2^{-L\gamma}\ ,$$

where c_2 and c_3 are as above.

The result for $a_2 > 0$ then follows by standard arguments.

In the case $a_2 = 0$ the proof is similar, the main differences being that then we are using the result in 3.3 instead of Proposition 3.1.1 and in the deductions corresponding to (39) we take advantage of the $\tilde{p}_2$-triangle inequality. ∎

Corollary 4.2.2. *The result of Proposition 4.2.1 holds true in the two following cases:*

(**i**) $a_2 < 0 < a_1 - 2\,(s_1 - s_2)/n - 1/\tilde{p}_2\,;$

(**ii**) $a_2 < a_1 - 2\,(s_1 - s_2)/n - 1/\tilde{p}_2 \le 0$ *and* $a_2 < -2\,(s_1 - s_2)/n\,.$

Proof: (**i**) In this case, *id* can be written as the composition

$$H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\log H)_{a_2}(\Omega) ,$$

so that the result follows by the multiplicativity of the entropy numbers and Proposition 4.2.1.

(**ii**) In this case *id* can be decomposed as

$$H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \hookrightarrow H^{s_1}_{p_1}(\Omega) \hookrightarrow H^{s_2}_{p_2}(\log H)_{a_2}(\Omega) ,$$

so that the result follows from [5, p. 149], again with the help of the multiplicativity of the entropy numbers. ∎

4.3. The estimate

Putting together what we have shown so far, we have the following result.

Theorem 4.3.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* id *be as in (35). Let the parameters* s_1, s_2, p_1 *and* p_2 *satisfy (8). If the real numbers* a_1 *and* a_2 *satisfy the conditions* $a_1 > 0$, $a_1 \notin\,]1/\tilde{p}_2,\, 2\,(s_1 - s_2)/n + 1/\tilde{p}_2]$ *and* $a_2 < a_1 - 2\,(s_1 - s_2)/n - 1/\tilde{p}_2$ *then there exist* $c_1, c_2 > 0$ *such that, for all* $k \in \mathbb{N}$,

$$c_1\, k^{-(s_1-s_2)/n} \le e_k(id) \le c_2\, k^{-(s_1-s_2)/n} \,. \blacksquare$$

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