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MEASURE-VALUED SOLUTIONS AND WELL-POSEDNESS OF MULTI-DIMENSIONAL CONSERVATION LAWS IN A BOUNDED DOMAIN *

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Abstract: We propose a general framework to establish the strong convergence of approximate solutions to multi-dimensional conservation laws in a bounded domain, provided uniform bounds on their L^p norm and their entropy dissipation measures are available. To this end, existence, uniqueness, and compactness results are proven in a class of entropy measure-valued solutions, following DiPerna and Szepessy. The new features lie in the treatment of the boundary condition, which we are able to formulate by relying only on an L^p uniform bound. This framework is applied here to prove the strong convergence of diffusive approximations of hyperbolic conservation laws.

Introduction

In this paper, we are interested in the boundary and initial value problem for a hyperbolic conservation law in several space dimensions:

$$\partial_t u + \operatorname{div} f(u) = 0\,, \qquad u(x,t) \in \mathbb{R}, \quad x \in \Omega, \quad t > 0\,, \tag{1.1}$$

$$u(x,0) = u_0(x)\,, \qquad x \in \Omega\,, \tag{1.2}$$

where Ω is an open and bounded subset of $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$. Here we have set $x = (x_1, x_2, ..., x_d)$, $\operatorname{div} f(u) := \sum_{j=1}^{d} \partial_j f_j(u)$, and $\partial_j := \partial_{x_j}$ for

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$j = 1, ..., d$. The flux-function $f = (f_1, f_2, ..., f_d)\colon \mathbb{R} \to \mathbb{R}^d$ is a given continuous mapping and the initial datum u_0 belongs to the space $L^p(\Omega)$ for some $p \in (1, \infty]$. Furthermore along the boundary we impose a boundary datum $u_B\colon \partial\Omega \times \mathbb{R}_+ \to \mathbb{R}^d$,

$$u(x,t) = u_B(x,t)\,, \qquad x \in \partial\Omega, \quad t > 0\ , \tag{1.3}$$

however expressed in the *weak* sense of Bardos, Leroux, and Nedelec [1]. Concerning the boundary conditions for hyperbolic conservation laws, we refer the reader to LeFloch [10], Dubois and LeFloch [6], Szepessy [16], Cockburn, Coquel, and LeFloch [3], Joseph and LeFloch [8], Otto (see [12]), Chen and Frid [2], and the references therein.

In the present paper, in Sections 2 and 3, we develop a general framework aimed at proving the convergence of a sequence of approximate solutions u^ε toward a solution of (1.1)–(1.3). The notion of entropy measure-valued solution introduced by DiPerna [5] plays here a central role. These are Young measures (Tartar [18]) satisfying the equation and the entropy inequalities in a weak sense. Under some natural assumption, a sequence of approximate solutions always generates an entropy measure-valued solution.

The key is given by a suitable generalization of Kruzkov's L^1 contraction property [9] discovered by DiPerna [5] and extended by Szepessy [15, 16], to include on one hand the boundary conditions and, on the other hand, measure-valued solutions in L^p. Our approach in the present paper covers both approximate solutions in L^p and a bounded domain. Recall that L^p Young measures associated with hyperbolic conservation laws were first studied by Schonbek [14]. Recall also that DiPerna's strategy was applied to proving the convergence of numerical schemes by Szepessy [17], Coquel and LeFloch [4], and Cockburn, Coquel, and LeFloch [3].

Section 4 contains the application of the above compactness framework to the vanishing diffusion problem with boundary condition.

2 – Measure-valued solutions

We aim at developing a general framework to study multi-dimensional conservation laws in a bounded domain, encompassing all of the fundamental issues of existence, uniqueness, regularity and compactness of entropy solutions. First we observe that, for the problem (1.1)–(1.3), it is easy to construct sequences of *approximate* solutions $u^h : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ satisfying certain natural uniform bounds; see (2.3)–(2.5) below. In particular, u^h satisfies "approximate" entropy inequalities of the form

$$\partial_t U(u^h) + \operatorname{div} F(u^h) \;\leq\; R_U^h \;\longrightarrow\; 0$$

in the sense of distributions. Such approximate solutions indeed will be constructed explicitly in Section 4.

The main difficulty is proving that these approximations converge in a strong topology and that the limit is an entropy solution of (1.1)–(1.2), satisfying a relaxed version of (1.3). To this end, following DiPerna [5], we introduce the notion of *entropy measure-valued solution*, designed to handle weak limits of the sequence u^h. The key result to be proven concerns the *regularity* and *uniqueness* of the measure-valued solution which, in fact, will coincide with a weak solution in the standard sense. This approach relies heavily on the entropy inequalities and on the L^1 contraction property of the solution-operator. The classical approach uses a compactness embedding (Helly's theorem) instead. Another characteristic of the present strategy is that it provides at once the strong convergence of the sequence u^h *and* a characterization of its limit.

Consider the problem (1.1)–(1.3) where the initial data u_0 belongs to $L^p(\Omega)$ with $1 < p \leq \infty$ and the boundary datum $u_B\colon\ \partial\Omega \times \mathbb{R}_+ \mapsto \mathbb{R}$ is a smooth and bounded function. We make the following assumptions on the flux-function f:

(**1**) f is *continuous* on $\mathbb{R}$.

(**2**) When $p < \infty$, f satisfies the *growth condition* at infinity

$$f(u) = O\Big(1 + |u|^r\Big) \tag{2.1}$$

for some $r \in [1, p)$.

We also define $q := p/r$. The following terminology will be used.

Definition 2.1. A continuous function satisfying (2.1) (when $p < \infty$) will be called *admissible*. When $p = \infty$, no growth condition is imposed.

More generally, a function $g = g(u, x, t)$ is called *admissible* if it continuous in $u \in \mathbb{R}$ and Lebesgue measurable in $x \in \Omega$ and $t \in \mathbb{R}_+$ with

$$|g(u, x, t)| \;\leq\; g_1(u) + g_2(x, t)\ ,$$

where the function $g_1 \geq 0$ satisfies (2.1) and $g_2 \geq 0$ belongs to $L^\infty(\mathbb{R}_+, L^q(\Omega))$.

A pair of smooth functions $(U, F)\colon\ \mathbb{R} \to \mathbb{R} \times \mathbb{R}^d$ is called a *tame entropy pair* if $\nabla F = U'\,\nabla f$ and the function U is *affine* outside a compact set. It is said to be convex if U is convex. ▫

Note that, when $p = \infty$, the behavior at infinity is irrelevant. A truly remarkable property of the scalar conservation laws is the existence of a special family of one-parameter, symmetric and convex entropies, the so-called Kruzkov entropies [9],

$$\tilde{U}(u,v) := |u - v|\,, \qquad \tilde{F}(u,v) := \operatorname{sgn}(u-v)\Big(f(u) - f(v)\Big)\,, \tag{2.2}$$

which play an important role in the theory of conservation laws.

For $h \in (0,1)$, let $u^h\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be a sequence of piecewise smooth functions with the following properties:

(**i**) The *uniform bound*

$$\|u^h(t)\|_{L^p(\Omega)} \le C(T)\,, \qquad t \in (0,T)\,, \tag{2.3}$$

holds for a constant $C(T) > 0$ independent of h and for each time $T > 0$.

(**ii**) The *entropy inequalities* ($\mathcal{H}_{d-1}$ being the $(d-1)$-dimensional Haussdorf measure)

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}_+} \Big(U(u^h)\,\partial_t\theta + F(u^h)\cdot \operatorname{grad}\theta\Big)\,dx\,dt \;+ \int_\Omega U(u^h(0))\,\theta(x,0)\,dx \\ - \iint_{\partial\Omega\times\mathbb{R}_+} B_U^h(x,t)\,\theta(x,t)\,d\mathcal{H}_{d-1}(x)\,dt \;\ge \iint_{\Omega\times\mathbb{R}_+} R_U^h(\theta)\,dx\,dt\end{aligned} \tag{2.4a}$$

hold for every convex and tame entropy-pair (U,F) and every test-function $\theta = \theta(x,t) \ge 0$ in $\mathcal{C}_c^1(\overline{\Omega}\times(0,\infty))$. We are assuming here that there exists a (smooth) *approximate boundary flux* b^h and an element $b \in W^{-1,\infty}([0,T), W^{-1/q,q}(\partial\Omega))$ (for all $T > 0$ with $1/q + 1/q' = 1$) such that

$$b^h \to b \quad \text{in the sense of distributions}\,. \tag{2.4b}$$

We have set also

$$B_U^h \;=\; F(u_B)\cdot N \;+\; U'(u_B)\Big(b^h - f(u_B)\cdot N\Big)\,,$$

where N is the outside unit normal along $\partial\Omega$. Moreover, in (2.4a), $R_U^h:\ \Omega\times\mathbb{R}_+ \to \mathbb{R}$ are piecewise smooth functions, possibly depending on U and converging to zero

$$\iint_{\Omega\times\mathbb{R}_+} R_U^h(\theta)\,dx\,dt \;\longrightarrow\; 0\,, \qquad h \to 0\,. \tag{2.4c}$$

(**iii**) The initial traces $u^h(0)$ approach the initial datum u_0 when $h \to 0$, in the following weak sense

$$\limsup_{h\to 0} \int_\Omega U(u^h(0))\,\theta\; dx \;\leq\; \int_\Omega U(u_0)\,\theta\; dx \tag{2.5}$$

for all arbitrary $\theta = \theta(x) \geq 0$ in $\mathcal{C}(\Omega)$ and for all convex tame entropy U. In particular (choosing $U(u) = \pm u$), (2.5) implies the weak convergence property

$$\lim_{h\to 0} \int_\Omega u^h(0)\,\theta\; dx \;=\; \int_\Omega u_0\,\theta\; dx\ .$$

For instance, (2.5) holds whenever $u^h(0)$ tends to u_0 in L^1 strongly.

It is well-known that, from a sequence u^h satisfying the uniform bound (2.3), one can extract a subsequence converging in the weak topology, but not necessarily converging in the strong topology. More generally, for any *admissible* function $g = g(u, x, t)$ we have

$$\|g(u^h, \cdot, \cdot)\|_{L^\infty(\mathbb{R}_+, L^q(\Omega))} \;\leq\; C(T)$$

for some uniform constant $C(T) > 0$ depending on g. By a weak compactness theorem, there exists a limit $\bar g \in L^\infty((0, T), L^q)$ (for each $T > 0$) and a subsequence still labelled u^h and possibly depending on g such that $g(u^h) \rightharpoonup \bar g$ weakly, i.e. for every $\theta = \theta(x, t)$ in $\mathcal{C}_c(\overline{\Omega} \times [0, \infty))$

$$\iint_{\Omega\times\mathbb{R}_+} g\Big(u^h(x,t), x, t\Big)\,\theta(x,t)\; dx\, dt \;\longrightarrow\; \iint_{\Omega\times\mathbb{R}_+} \bar g(x,t)\,\theta(x,t)\; dx\, dt\ . \tag{2.6}$$

A fundamental difficulty must be overcome. Given a nonlinear function $g = g(u)$, the weak limit of the composite function $g(u^h)$ of u^h need not coincide with the composite function of the weak limit (say $\bar u$) of u^h. In other words $\bar g \neq g(\bar u)$. In this juncture, the Young measure provides us with a powerful tool to express the weak limit $\bar g$ in (2.6) from the function g. To each $(x, t) \in \Omega \times \mathbb{R}_+$, it associates a probability measure $\nu_{x,t}$ on $\mathbb{R}$, i.e., an element of the space of all positive measures with unit mass, such that $\bar g(x, t)$ be the expected value of g with respect to the measure $\nu_{x,t}$. This means that

$$\bar g(x,t) = \langle \nu_{x,t}, g\rangle := \int_{\mathbb{R}} g(\bar u)\, d\nu_{x,t}(\bar u) \qquad \text{for a.e. } \ (x,t)\ .$$

Recall that a measure is a linear mapping μ from the linear space $\mathcal{C}(\mathbb{R})$ of all continuous functions g (or a subset of them) such that for some $C > 0$

$$|\langle \mu, g\rangle| := \left|\int_{\mathbb{R}} g(\bar u)\, d\mu(\bar u)\right| \;\leq\; C\,\|g\|_{C(\mathbb{R})}\ ,$$

the mass of μ being then

$$|\mu| := \sup_{\substack{g \in \mathcal{C}(\mathbb{R}) \\ \|g\|_{C(\mathbb{R})}=1}} \langle \mu, g \rangle \ .$$

Recall also that μ is said to be positive iff $\langle \mu, g\rangle \geq 0$ for all functions $g \geq 0$.

Theorem 2.2 (Tartar [18], Schonbek [14]). *Given a sequence u^h satisfying (2.3) for some $p \in (1, \infty]$, there exists a subsequence of u^h and a family of probability measures $\{\nu_{x,t}\}_{(x,t)\in\Omega\times\mathbb{R}_+}$ with the following property. For all admissible functions $g = g(u, x, t)$, the function $(x, t) \mapsto \langle \nu_{x,t}, g(\cdot, x, t)\rangle$ belongs to $L^\infty(\mathbb{R}_+, L^q(\Omega))$ and we have*

$$g(u^h, x, t) \rightharpoonup \langle \nu_{x,t},\ g(\cdot, x, t)\rangle \quad \text{in the weak sense}\ . \tag{2.7}$$

Let $\bar{u}$ be the weak limit of u^h. Then

$$u^h \to \bar{u} \quad \text{strongly}$$

iff

$$\nu_{x,t} = \delta_{\bar{u}(x,t)} \quad \text{for a.e. } (x,t) \in \Omega \times \mathbb{R}_+ \ ,$$

where $\delta_{\bar{u}(x,t)}$ denotes the Dirac measure at the point $\bar{u}(x, t)$.

The mapping ν constructed in Theorem 2.2 is called a Young measure associated with the sequence $\{u^h\}_{h>0}$. The concept of a Young measure is now applied to the conservation law (1.1).

Definition 2.3. A Young measure $\nu_{x,t}$ is an *entropy measure-valued solution* of the problem (1.1)–(1.3) iff there exists an element $b \in W^{-1,\infty}([0,T), W^{-1/q,q}(\partial\Omega))$ (for all $T > 0$ with $1/q + 1/q' = 1$) such that for every convex and tame entropy pair (U, F), every smooth function $\theta = \theta(x, t) \geq 0$ and every $T > 0$ we have

$$\begin{aligned}\int_0^T\!\!\int_\Omega \Big(\langle \nu, U\rangle\, \partial_t\theta + \langle \nu, F\rangle \cdot \operatorname{grad}\theta\Big)\, dx\, dt \ + \int_\Omega U(u_0)\, \theta(x,0)\, dx \\ - \int_0^T\!\!\int_{\partial\Omega} \Big(F(u_B)\cdot N + U'(u_B)\,\big(b - f(u_B)\cdot N\big)\Big)\, \theta\ d\mathcal{H}_{d-1}\, dt \ \geq\ 0\ .\end{aligned} \tag{2.8}$$

A function $u \in L^\infty((0,T), L^p(\Omega))$ for all $T > 0$ is called an *entropy weak solution* of the problem (1.1)–(1.3) iff the Young measure $(x, t) \mapsto \delta_{u(x,t)}$ is an entropy measure-valued solution. □

In particular, (2.8) implies that the inequality

$$\partial_t\langle \nu, U\rangle + \operatorname{div}\langle \nu, F\rangle \ \leq\ 0 \tag{2.9}$$

holds in the sense of distributions.

Definition 2.3 is directly motivated by the following observation, which is easily deduced from the property (2.7) of the Young measure and the assumptions (2.4) and (2.5): A Young measure associated with a sequence satisfying (2.3)–(2.5) is an entropy measure-valued solution of (1.1)–(1.2).

We discuss now the regularity of the measure-valued solutions. We show that the initial data u_0 is assumed in a strong sense, and we investigate in what sense ν satisfies the boundary condition (1.3) along $\partial\Omega$.

Theorem 2.4. *Let $\nu = \nu_{x,t}$ be an entropy measure-valued solution of* (1.1)–(1.3).

(**a**) *For every convex and tame entropy $U = U(u)$ and for every smooth function $\theta = \theta(x)$ with compact support in Ω, the function*

$$t \longmapsto \int_\Omega \langle \nu_{x,t}, U\rangle \, \theta \, dx \tag{2.10}$$

has locally bounded total variation and admits a trace as $t \to 0+$.

(**b**) *For every function $U = U(u,x)$, that is convex in u and measurable in x and such that $|U(u,x)| \le c\,|u| + |\tilde{U}(x)|$ where $\tilde{U} \in L^1(\Omega)$ and $c \ge 0$, we have*

$$\limsup_{t\to 0+} \int_\Omega \langle \nu_{x,t}, U(\cdot,x)\rangle \, dx \le \int_\Omega U(u_0(x),x)\, dx \ . \tag{2.11}$$

(**c**) *In particular, the Young measure assumes its initial datum u_0 in the following strong sense:*

$$\limsup_{t\to 0+} \int_\Omega \Big\langle \nu_{x,t},\ |id - u_0(x)| \Big\rangle \, dx = 0 \ . \tag{2.12}$$

Proof: Using in the weak formulation (2.8) a function $\theta(x,t) = \theta_1(x)\,\theta_2(t)$, compactly supported in $\Omega \times [0,\infty)$ and having $\theta_1, \theta_2 \ge 0$, we obtain

$$\begin{aligned}\int_0^\infty \frac{d\theta_2}{dt} \int_\Omega \langle \nu, U\rangle \, \theta_1 \, dx\, dt + \theta_2(0) \int_\Omega U(u_0)\, \theta_1 \, dx &\ge -\int_0^\infty \theta_2 \int_\Omega \operatorname{grad}\theta_1 \cdot \langle \nu, F\rangle \, dx\, dt \\ &\ge -C_1 \int_0^\infty \theta_2 \, dt \ ,\end{aligned}$$

for some constant $C_1 > 0$ depending on θ_1. Thus the function

$$V_1(t) := -C_1\, t + \int_\Omega \langle \nu_{x,t}, U\rangle \, \theta_1 \, dx$$

satisfies the inequality

$$(2.13)\qquad -\int_0^\infty V_1(t)\,\frac{d\theta_2}{dt}\,dt \;\le\; \theta_2(0)\int_\Omega U(u_0)\,\theta_1\,dx\ .$$

Using in (2.13) a test-function $\theta_2 \ge 0$ compactly supported in $(0,\infty)$, we find

$$-\int_0^\infty V_1(t)\,\frac{d\theta_2}{dt}\,dt \;\le\; 0$$

that is (in the sense of distributions) the function $V_1(t)$ is decreasing and, therefore, has locally bounded total variation. Since it is uniformly bounded, $V_1(t)$ has a limit as $t \to 0+$. This proves (a).

To establish the item (b), we fix a time $t_0 > 0$ and consider the sequence of continuous functions

$$\theta_2^\varepsilon(t) \;=\; \begin{cases} 1 & \text{for } \ t\in[0,t_0], \\ (t_0+\varepsilon-t)/\varepsilon & \text{for } \ t\in[t_0,t_0+\varepsilon], \\ 0 & \text{for } \ t\ge t_0+\varepsilon\ . \end{cases}$$

Relying on the regularity property (a) above, we see easily that

$$-\int_0^\infty V_1(t)\,\frac{d\theta_2^\varepsilon}{dt}\,dt \;\longrightarrow\; V_1(t_0+)\ .$$

Since $\theta_2^\varepsilon(0) = 1$ and t_0 is arbitrary, (2.13) yields

$$V_1(t_0) \;=\; -C_1\,t_0 + \int_\Omega \langle \nu_{x,t_0}, U\rangle\,\theta_1\,dx \;\le \int_\Omega U(u_0)\,\theta_1\,dx$$

for all $t_0 > 0$ and, in particular,

$$(2.14)\qquad \lim_{t\to 0+}\int_\Omega \langle \nu_{x,t}, U\rangle\,\theta_1\,dx \le \int_\Omega U(u_0)\,\theta_1\,dx \qquad \text{for all } \ \theta_1 = \theta_1(x) \ge 0\ .$$

Note that the left-hand limit exists, in view of (a).

Consider the set of all linear, convex and finite combinations of the form $\sum_j \alpha_j\,\theta_{1,j}(x)\,U_j(u)$, where $\alpha_j \ge 0$, $\sum_j \alpha_j = 1$, the functions U_j are smooth and convex in u and the functions $\theta_{1,j}(x) \ge 0$ are smooth and compactly support, with moreover

$$|U_j(u)\,\theta_{1,j}(x)| \;\le\; c\,|u| + |\tilde U_j(x)|$$

with $c \ge 0$ and $\tilde U_j \in L^1(\Omega)$. This set is dense (for the uniform topology in u and the L^1 topology in x) in the set of all functions $U = U(u,x)$ that are convex in u and measurable in x and satisfy

$$|U(u,x)| \;\le\; c\,|u| + |\tilde U(x)|$$

for some $c > 0$ and $\tilde{U} \in L^1(\Omega)$. Therefore by density we can deduce the statement (b) from (2.14).

The statement (c) follows from (b) by choosing $U(u,x) = |u - u_0(x)|$. ∎

To identify some properties of the Young measure along the boundary, we will need the following:

Lemma 2.5. *Let $V\colon \Omega \to \mathbb{R}^d$ be a function in $L^q(\Omega)$ $(q > 1)$, satisfying*

$$\operatorname{div} V \le 0 \quad \text{in the sense of distributions .}$$

Then the function V admits a normal trace along the boundary of Ω, in the following sense. Consider the change of coordinate $x = \chi(\bar{x}, y) = \bar{x} + y\,N(\bar{x})$, where $(\bar{x}, y) \in \partial\Omega \times (0, \varepsilon)$ for some $\varepsilon > 0$ sufficiently small. Call $J = J(\bar{x}, y) = |\frac{\partial\chi}{\partial(\bar{x},y)}|$ is the Jacobian of this transformation. Then for each test-function θ of the single variable $y \in (0, \varepsilon)$ given by

$$A(y) = \int_{\partial\Omega} V(\bar{x}, y)\cdot N(\bar{x})\,\theta(\bar{x}) J(\bar{x}, y)\; d\mathcal{H}_{d-1}(\bar{x})$$

is a monotone increasing function and so admits a limit, say A_0, in $[-\infty, \infty)$ as $y \to 0^+$ with

$$A_0 \le C\,\|V\|_{L^q(\Omega)}\,\|\theta\|_{W^{1,q'}(\partial\Omega)}\ , \tag{2.15}$$

Proof: The change of coordinates $x \mapsto (\bar{x}, y)$ is well defined in a neighborhood of $\partial\Omega$, for $(\bar{x}, y) \in \partial\Omega \times (0, \varepsilon)$. Given $\theta \in \mathcal{C}(\partial\Omega)$, consider the associated function A. For any test function $\psi(x) = \theta(\bar{x})\,\varphi(y)$ with $\theta \in \mathcal{C}^\infty(\partial\Omega)$, and $\varphi \in \mathcal{C}_c^\infty((0, \varepsilon))$, we have

$$\nabla_x(\theta\,\varphi) = \varphi(y)\,\nabla_x\theta + \theta\,\varphi'(y)\,\partial_x y\ ,$$

with $\partial_x y = -N(\bar{x})$. On the other hand from the inequality satisfied by V we deduce that

$$\int_\Omega \nabla_x(\theta\,\varphi)\cdot V\,dx \ge 0\ ,$$

i.e. using the change of variables $x \mapsto (\bar{x}, y)$

$$\int_0^\varepsilon \varphi(y) \int_{\partial\Omega} V(\bar{x}, y)\cdot\nabla_x\theta(\bar{x})\,J(\bar{x}, y)\; d\mathcal{H}_{d-1}(\bar{x})\,dy \\ - \int_0^\varepsilon \varphi'(y) \int_{\partial\Omega} V(\bar{x}, y)\cdot N(\bar{x})\,\theta(\bar{x})\,J(\bar{x}, y)\; d\mathcal{H}_{d-1}(\bar{x})\,dy \ \ge\ 0\ .$$

Setting

$$B(y) := \int_{\partial\Omega} V(\bar{x}, y) \cdot \nabla_x \theta(\bar{x})\, J(\bar{x}, y)\; d\mathcal{H}_{d-1}(\bar{x}) \;,$$

we obtain

$$A'(y) + B(y) \;\geq\; 0 \tag{2.16}$$

in the sense of distributions on $(0, \varepsilon)$. Observe that both A and B are Lebesgue measurable functions. Now (2.16) implies that $A' + B$ is a non-negative locally bounded Borel measure on $(0, \varepsilon)$. Hence $A(y) + \int_0^y B(y')\, dy'$ has a pointwise limit as $y \to 0+$, which belongs to $[-\infty, \infty)$. By assumption $b \in L^q(\partial\Omega)$ so $\int_0^y B(y')\, dy' \to 0$ as $y \to 0+$. This establishes that $A(y)$ admits a limit A_0 when $y \to 0+$.

On the other hand, from (2.16) we deduce that for

$$A(y) - A_0 + \int_0^y B(y')\, dy' \;\geq\; 0 \;,$$

so

$$\begin{aligned} A_0 \;&\leq\; \frac{1}{\varepsilon} \int_0^\varepsilon |A(y)|\, dy \;+\; \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^y |B(y')|\; dy'\, dy \\ &\leq\; \frac{C}{\varepsilon}\, \|V\|_{L^q(\Omega)}\, \|\theta\|_{L^{q'}(\partial\Omega)} \;+\; \frac{C}{\varepsilon}\, \|V\|_{L^q(\Omega)}\, \|\nabla\theta\|_{L^{q'}(\partial\Omega)} \;, \end{aligned}$$

which gives (2.15). This completes the proof of Lemma 2.5. ∎

Theorem 2.6. *Let $\nu = \nu_{x,t}$ an entropy measure-valued solution. There exists a Young measure $\nu^B_{x,t}$ defined along the boundary, for $(x,t) \in \partial\Omega \times \mathbb{R}_+$ such that for each continuous function $F\colon \mathbb{R} \to \mathbb{R}^d$ satisfying the growth condition (2.1), $\langle \nu^B_{x,t},\, F \cdot N(x) \rangle$ belongs to the distribution space $W^{-1,\infty}([0,T), W^{-1/q,q}(\partial\Omega))$ for all $T > 0$ (with $1/q + 1/q' = 1$). This Young measure represents the trace of $\langle \nu_{x,t},\, F \cdot N(x) \rangle$ along the boundary, in the following sense (using the notation of Lemma 2.5), as $y \to 0$,*

$$\begin{aligned} A(y) \;&:=\; \int_{\mathbb{R}_+} \int_{\partial\Omega} \langle \nu_{\bar{x},y,t},\; F \cdot N(x) \rangle\, \theta_1(\bar{x})\, \theta_2(t)\, J(\bar{x}, y)\; d\mathcal{H}_{d-1}(\bar{x})\, dt \\ &\longrightarrow\; \int_{\mathbb{R}_+} \int_{\partial\Omega} \langle \nu^B_{\bar{x},t}, F \cdot N(\bar{x}) \rangle\, \theta_1(\bar{x})\, \theta_2(t)\; d\mathcal{H}_{d-1}(\bar{x})\, dt \end{aligned}$$

for every test-functions θ_1 and θ_2.

Moreover, the following boundary entropy inequality

$$\Big\langle \nu^B,\; N \cdot \Big(F(\cdot) - F(u_B) - U'(u_B) \Big(f(\cdot) - f(u_B) \Big) \Big) \Big\rangle \;\geq\; 0 \tag{2.17}$$

on the boundary $\partial\Omega \times \mathbb{R}_+$

in the sense of distributions, and

$$b = \langle \nu^B, f\cdot N\rangle \quad \text{on the boundary } \partial\Omega\times\mathbb{R}_+ \ . \tag{2.18}$$

Proof: Use the weak formulation (2.8) with a function $\theta(x,t)=\theta_1(x)\,\theta_2(t)$, compactly supported in $\overline{\Omega}\times[0,\infty)$ and having $\theta_1,\theta_2\geq 0$:

$$\begin{aligned}
&\int_\Omega \operatorname{grad}\theta_1 \cdot \int_0^\infty \langle\nu,F\rangle\,\theta_2\;dx\,dt \;+\; \theta_2(0)\int_\Omega U(u_0)\,\theta_1\,dx \\
&\quad - \int_{\partial\Omega}\int_0^\infty \Big(F(u_B)\cdot N + U'(u_B)\,\Big(b - f(u_B)\cdot N\Big)\Big)\,\theta_1\theta_2\;dt\,d\mathcal{H}_{d-1} \\
&\geq\; -\int_\Omega \theta_1 \int_0^\infty \langle\nu,U\rangle\,\frac{d\theta_2}{dt}\;dt\,dx \\
&\geq\; -\,C_2(\theta_2)\int_\Omega \theta_1 \int_0^T |\langle\nu,U\rangle|\;dt\,dx \\
&=\; C_2(\theta_2)\int_\Omega \nabla\theta_1\cdot X\,dx \ ,
\end{aligned}$$

where
$$C_2(\theta_2) \;=\; \Big\|\frac{d\theta_2}{dt}\Big\|_{L^\infty(\mathbb{R}_+)}$$
and X is a solution of (see [7])

$$\operatorname{div} X = \int_0^T |\langle\nu,U\rangle|\,dt\,, \qquad X\in W_0^{1,p}(\Omega) \ .$$

Thus the vector-valued function

$$V_2(x) \;=\; \int_0^\infty \langle\nu_{x,t},F\rangle\,\theta_2(t)\;dt \;-\; C_2(\theta_2)\,X$$

satisfies the inequality

$$\begin{aligned}
&-\int_\Omega V_2\cdot\operatorname{grad}\theta_1\;dx \\
&\leq\; -\int_{\partial\Omega}\theta_1 \int_0^\infty \Big(F(u_B)\cdot N + U'(u_B)\,\Big(b - f(u_B)\cdot N\Big)\Big)\,\theta_2\;dt\,d\mathcal{H}_{d-1} \\
&\quad +\; \theta_2(0)\int_\Omega U(u_0)\,\theta_1\,dx \ .
\end{aligned} \tag{2.19}$$

Using in (2.19) a test-function $\theta_1\geq 0$ compactly supported in Ω, and θ_2 such that $\theta_2(0)=0$ we find

$$-\int_\Omega V_2\cdot\operatorname{grad}\theta_1\,dx \;\leq\; 0 \ ,$$

that is $\operatorname{div} V_2 \le 0$ in the sense of distributions, with $V_2 \in L^q(\Omega)$. Applying Lemma 2.5 we see that the normal trace V_2 exists along the boundary. We thus define

$$G(F\cdot N;\, \theta_1, \theta_2) := \lim_{y\to 0+} \int_{\partial\Omega} V_2(\bar x, y)\cdot N(\bar x)\, \theta_1(\bar x)\, J(\bar x, y)\; d\mathcal{H}_{d-1}(\bar x) .$$

We now return to the general inequality (2.19), and we let the test-function θ_1 tend to a zero in the interior of Ω. We find for all $\theta_2 \ge 0$ and $\theta_1 \colon \partial\Omega \to \mathbb{R}_+$

$$\begin{aligned} (2.19) \quad & -G(F\cdot N;\, \theta_1, \theta_2) \\ & \le -\int_{\mathbb{R}_+} \int_{\partial\Omega} \Big(F(u_B)\cdot N + U'(u_B)\, \big(b - f(u_B)\cdot N\big)\Big)\, \theta_1(\bar x)\, \theta_2(t)\; dt\, d\mathcal{H}_{d-1}(\bar x) . \end{aligned}$$

In view of (2.15) and with the regularity available on b and u_B, we see that the mapping G satisfies the following estimate:

$$|G(F\cdot N;\, \theta_1, \theta_2)| \le C\, \|\theta_1\, \theta_2\|_{W^{1,1}\left([0,T), W^{1,q'}(\partial\Omega)\right)} .$$

This gives us the desired regularity of G.

Finally, rewritting G via a Young measure ν^B we obtain

$$G(F\cdot N;\, \theta_1, \theta_2) = \int_{\mathbb{R}_+} \int_{\partial\Omega} \langle \nu^B_{\bar x,t}, F\cdot N(\bar x)\rangle\, \theta_1(\bar x)\, \theta_2(t)\; d\mathcal{H}_{d-1}(\bar x)\, dt$$

Using similar estimates as above but with $U(u) = \pm u$ we actually have the equality

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\partial\Omega} \langle \nu^B_{\bar x,t}, f\rangle\, \theta_1(\bar x)\, \theta_2(t)\; d\mathcal{H}_{d-1}(\bar x)\, dt \\ & \qquad = \int_{\mathbb{R}_+} \int_{\partial\Omega} \Big(f(u_B)\cdot N + b - f(u_B)\cdot N\Big)\, \theta_1(\bar x)\, \theta_2(t)\; d\mathcal{H}_{d-1}(\bar x)\, dt , \end{aligned}$$

so that (2.18) holds. This completes the proof of Theorem 2.6. ∎

3 – Existence, uniqueness, and compactness

Continuing the investigation of the properties of the measure-valued solutions, we now arrive at a general theory of existence and uniqueness for the problem (1.1)–(1.3), based on extending to measure-valued solutions the standard Kruzkov's contraction property [9].

Theorem 3.1. *Let ν_1 and ν_2 be two entropy measure-valued solutions. Then, in the sense of distributions we have*

$$\partial_t \langle \nu_1 \otimes \nu_2, \tilde{U} \rangle + \operatorname{div} \langle \nu_1 \otimes \nu_2, \tilde{F} \rangle \;\le\; 0 \;, \tag{3.1}$$

where $(\tilde{U}, \tilde{F})$ is the Kruzkov parametrized entropy pair (see (2.2)), and the tensor product of measures is defined by

$$\langle \nu_1 \otimes \nu_2, \tilde{U} \rangle \;:=\; \iint \tilde{U}(u_1, u_2)\; d\nu_1(u_1)\, d\nu_2(u_2) \;.$$

Proof: Formally we have

$$\begin{aligned} &\partial_t \langle \nu_1 \otimes \nu_2, \tilde{U} \rangle + \operatorname{div} \langle \nu_1 \otimes \nu_2, \tilde{F} \rangle \\ &= \Big\langle \nu_1 \Big(\partial_t \langle \nu_2, \tilde{U} \rangle + \operatorname{div} \langle \nu_2, \tilde{F} \rangle \Big) \Big\rangle \;+\; \Big\langle \nu_2 \Big(\partial_t \langle \nu_1, \tilde{U} \rangle + \operatorname{div} \langle \nu_1, \tilde{F} \rangle \Big) \Big\rangle \\ &\le 0 \;, \end{aligned}$$

where we used (2.9) and the positivity of the measures ν_1 and ν_2. The proof can be made rigorous by regularization in the (x,t)-variable. ∎

Theorem 3.2. *Let ν_1 and ν_2 be two entropy measure-valued solutions satisfying the same initial datum $u_0 \in L^p(\Omega)$. Then there exists a function $u \in L^\infty(\mathbb{R}_+, L^p(\Omega))$ such that*

$$\nu_{1,(x,t)} = \nu_{2,(x,t)} = \delta_{u(x,t)} \qquad \text{for almost every} \;\; (x,t) \;. \tag{3.4}$$

In particular, the problem (1.1)–(1.3) has exactly one entropy solution in $L^\infty(\mathbb{R}_+, L^p(\Omega))$, which moreover satisfies its initial data in the sense

$$\limsup_{t \to 0+} \int_\Omega |u(x,t) - u_0(x)|\, dx \;=\; 0 \;. \tag{3.5}$$

Furthermore, given two such solutions u_1 and u_2 associated with the boundary data u_B, we have for all $t \ge s \ge 0$

$$\int_\Omega |u_1(x,t) - u_2(x,t)|\; dx \;\le\; \int_\Omega |u_1(x,s) - u_2(x,s)|\; dx \;. \tag{3.6}$$

The existence part in Theorem 3.2 is based on the assumption that a family of approximate solutions satisfying (2.3)–(2.5) does exist, in order to generate at least one entropy measure-valued solution. Recall that Section 4 below will indeed provide such approximate solutions. In the applications, in order to establish the strong convergence of a sequence of approximate solutions, we will appeal to the following immediate consequence of Theorem 3.2 and Theorem 2.2.

Corollary 3.3. *Let u^h be a sequence of approximate solutions satisfying the conditions* (2.3)–(2.5). *Then there exists a function $u \in L^\infty(\mathbb{R}_+, L^p(\Omega))$ such that*

$$u^h \to u \quad \textit{strongly} \ ,$$

and u is the unique entropy solution to the problem (1.1)–(1.3). *The entropy solution $u \in L^\infty(\mathbb{R}_+, L^p(\Omega))$ of* (1.1)–(1.3) *satisfies the additional regularity:*

$$t \mapsto \int_\Omega U(u)\,\theta \, dx \quad \textit{has locally bounded variation} \tag{3.7}$$

for all tame entropy U and all smooth $\theta \geq 0$ having compact support in $\Omega \times \mathbb{R}_+$.

To prove Theorem 3.2, we shall need:

Lemma 3.4. *Let u_B be given in $\mathbb{R}$, together with some unit vector $N \in \mathbb{R}^d$. Let ν_1 and ν_2 be probability measures on $\mathbb{R}$ (acting on all admissible functions and) satisfying the boundary entropy inequalities*

$$\Big\langle \nu_1,\ N \cdot \Big(F(\cdot) - F(u_B) - U'(u_B)\,\big(f(\cdot) - f(u_B)\big)\Big)\Big\rangle \ \geq\ 0 \tag{3.8a}$$

and

$$\Big\langle \nu_2,\ N \cdot \Big(F(\cdot) - F(u_B) - U'(u_B)\,\big(f(\cdot) - f(u_B)\big)\Big)\Big\rangle \ \geq\ 0 \tag{3.8b}$$

for all convex and tame entropy pairs (U, F). Then we have

$$\langle \nu_1 \otimes \nu_2,\, N \cdot \tilde{F} \rangle \ \leq\ 0 \ . \tag{3.9}$$

Proof: Using Kruzkov's entropies, the conditions (3.8) and (3.9) are found to be equivalent to: for each $v_2, v_1 \in \mathbb{R}$

$$\Big\langle \nu_1,\ \big(\mathrm{sgn}(u_1 - v_2) - \mathrm{sgn}(u_B - v_2)\big)\,\big(f(u_1) - f(v_2)\big) \cdot N \Big\rangle \ \geq\ 0 \ , \tag{3.10a}$$

$$\Big\langle \nu_2,\ \big(\mathrm{sgn}(u_2 - v_1) - \mathrm{sgn}(u_B - v_1)\big)\,\big(f(u_2) - f(v_1)\big) \cdot N \Big\rangle \ \geq\ 0 \ . \tag{3.10b}$$

Taking succesively $v_2 < u_B$, then $v_2 = u_B$, and finally $v_2 > u_B$, we obtain

$$\int_{u_1 < u_B} N \cdot \tilde{F}(u_1, u_2)\, d\nu_1(u_1) \ \geq\ 0 \,, \qquad v_2 < u_B \ , \tag{3.11i}$$

$$\int_{u_1\in\mathbb{R}} N\cdot\tilde{F}(u_1,u_B)\,d\nu_1(u_1)\ \geq\ 0\,, \tag{3.11ii}$$

and

$$\int_{u_1>u_B} N\cdot\tilde{F}(u_1,u_2)\,d\nu_1(u_1)\ \geq\ 0\,,\qquad v_2>u_B\ . \tag{3.11iii}$$

Similarly we get

$$\int_{u_2<u_B} N\cdot\tilde{F}(u_1,u_2)\,d\nu_2(u_2)\ \geq\ 0\,,\qquad v_1<u_B\ , \tag{3.12i}$$

$$\int_{u_2\in\mathbb{R}} N\cdot\tilde{F}(u_2,u_B)\,d\nu_2(u_2)\ \geq\ 0\,, \tag{3.12ii}$$

and

$$\int_{u_2>u_B} N\cdot\tilde{F}(u_1,u_2)\,d\nu_2(u_2)\ \geq\ 0\,,\qquad v_1>u_B\ . \tag{3.12iii}$$

These conditions (3.11)–(3.12) imply immediately that

$$\iint_{Q_1\cup Q_3} N\cdot\tilde{F}(u_2,u_B)\,d\nu_1(u_1)\,d\nu_2(u_2)\ \geq\ 0\,, \tag{3.13}$$

where $Q_1 := \{u_1\geq u_B,\ u_2\geq u_B\}$ and $Q_3 := \{u_1\leq u_B,\ u_2\leq u_B\}$.

To estimate the sign in the region $Q_4 := \{u_1>u_B,\ u_2<u_B\}$, we use (3.11iii) which gives us

$$\int_{u_1>v_2} N\cdot f(u_1)\,d\nu_1(u_1)\ \geq\ \int_{u_1>v_2} N\cdot f(v_2)\,d\nu_1(u_1)\,,\qquad v_2>u_B\ .$$

We use also (3.12i) which gives

$$\int_{u_2<v_1} N\cdot f(u_2)\,d\nu_2(u_2)\ \leq\ \int_{u_2<v_1} N\cdot f(v_1)\,d\nu_2(u_2)\,,\qquad v_1<u_B\ .$$

Combining these two inequalities we arrive at

$$\begin{aligned}\int_{u_2<u_B}\int_{u_1>v_2} N\cdot f(u_1)\,d\nu_1(u_1)\,d\nu_2(u_2) &\geq \int_{u_2<u_B}\int_{u_1>v_2} N\cdot f(v_2)\,d\nu_1(u_1)\,d\nu_2(u_2)\\ &\longrightarrow \int_{u_2<u_B}\int_{u_1>u_B} N\cdot f(u_B)\,d\nu_1(u_1)\,d\nu_2(u_2)\ ,\end{aligned}$$

as $v_2\to u_B$. We also have

$$\begin{aligned}\int_{u_1>u_B}\int_{u_2<v_1} N\cdot f(u_2)\,d\nu_1(u_1)\,d\nu_2(u_2) &\geq \int_{u_1>u_B}\int_{u_2<v_1} N\cdot f(v_1)\,d\nu_1(u_1)\,d\nu_2(u_2)\\ &\longrightarrow \int_{u_1>u_B}\int_{u_2<u_B} N\cdot f(u_B)\,d\nu_1(u_1)\,d\nu_2(u_2)\ ,\end{aligned}$$

as $v_1 \to u_B$. This implies exactly

$$\iint_{Q_4} N \cdot \tilde{F}(u_2, u_B)\; d\nu_1(u_1)\, d\nu_2(u_2) \;\geq\; 0\;. \tag{3.14}$$

A similar argument applies on $Q_2 := \{u_1 < u_B,\; u_2 > u_B\}$

$$\iint_{Q_2} N \cdot \tilde{F}(u_2, u_B)\; d\nu_1(u_1)\, d\nu_2(u_2) \;\geq\; 0\;. \tag{3.15}$$

This completes the proof of Lemma 3.4. ∎

Proof of Theorem 3.2: Consider two solutions ν_1 and ν_2 associated with a pair of data u_{01}, u_B and u_{02}, u_B, respectively. With the Green formula and Theorem 3.1, together with the existence of the normal trace (Theorem 2.6), we obtain immediately for test-functions $\theta_1, \theta_2 \geq 0$

$$\begin{aligned}
&-\int_{\mathbb{R}_+}\int_\Omega \langle \nu_1 \otimes \nu_2,\, \tilde{U}\rangle\, \theta_1(x)\, \theta_2'(t)\; dx\, dt \\
&-\int_{\mathbb{R}_+}\int_\Omega \langle \nu_1 \otimes \nu_2,\, N\cdot\tilde{F}\rangle \cdot \nabla\theta_1(x)\, \theta_2'(t)\; dx\, dt \;\leq\; 0\;,
\end{aligned}$$

and so

$$\begin{aligned}
&-\int_{\mathbb{R}_+}\int_\Omega \langle \nu_1 \otimes \nu_2,\, \tilde{U}\rangle\, \theta_1(x)\, \theta_2'(t)\; dx\, dt \\
&+\int_{\mathbb{R}_+}\int_{\partial\Omega} \langle \nu_1^B \otimes \nu_2^B,\, N\cdot\tilde{F}\rangle\, \theta_1(x)\, \theta_2'(t)\; d\mathcal{H}_{d-1}(x)\, dt \;\leq\; 0\;.
\end{aligned} \tag{3.17}$$

In view of Lemma 3.4, we have

$$-B(t) \;\leq\; 0\;.$$

therefore we arrive at

$$\frac{dA}{dt}(t) + B(t) \;\leq\; 0\;, \tag{3.18}$$

where

$$A(t) := \int_\Omega \langle \nu_1 \otimes \nu_2,\, \tilde{U}\rangle\, \theta_1(x)\; dx\;.$$

We now turn to evaluate of A. Since $\tilde{U}(\lambda_1, \lambda_2) = |\lambda_2 - \lambda_1|$, the term $A(t)$ is regarded as the L^1 norm between the two solutions. On the other hand, from (3.17)–(3.18) we deduce

$$A(t) - A(s) \;\leq\; 0, \qquad 0 < s \leq t\;. \tag{3.19}$$

First of all, suppose that ν_1 and ν_2 assume the same boundary and initial data u_B and u_0. Since the Young measures satisfy the initial condition in the strong sense (2.12), we obtain for all $t > 0$

$$\begin{aligned} A(t) \;&\le\; \int_\Omega \Big\langle \nu_1 \otimes \nu_2,\; |\bar u_1 - u_0| + |\bar u_2 - u_0| \Big\rangle \, dx \\ &\le\; \int_\Omega \langle \nu_1,\, |\bar u_1 - u_0| \rangle \, dx \;+\; \int_\Omega \langle \nu_2,\, |\bar u_2 - u_0| \rangle \, dx \;, \end{aligned}$$

thus

$$\limsup_{t\to 0+} A(t) = 0 \;.$$

Therefore letting $s \to 0$ in (3.19),

$$A(t) \equiv 0\,, \qquad t \ge 0$$

and thus

$$\int_\Omega \langle \nu_1 \otimes \nu_2,\, \tilde U \rangle \, dx \;=\; 0 \;.$$

Thus, for almost every (x,t), the measures $\nu_1 = \nu_{1,(x,t)}$ and $\nu_2 = \nu_{2,(x,t)}$ satisfy

$$\iint |\bar u_2 - \bar u_1| \; d\nu_1(\bar u_1)\, d\nu_2(\bar u_2) \;=\; \langle \nu_1 \otimes \nu_2,\, \tilde U \rangle \;=\; 0 \;. \tag{3.20}$$

Fix (x,t) such that (3.20) holds. We claim that there exists some $w \in \mathbb{R}$ such that

$$\nu_1 = \nu_2 = \delta_w \;.$$

Otherwise there would exist $w_1 \in \operatorname{supp} \nu_1$ and $w_2 \in \operatorname{supp} \nu_2$ with $w_1 \ne w_2$. By definition of the support of a measure, there exist continuous functions $\varphi_j \ge 0$ such that $\operatorname{supp} \varphi_j \subset B(w_j, \varepsilon) \subset \Omega$ (the ball with center w_j and radius ε) and $\langle \nu_j, \varphi_j \rangle \ne 0$. One can always assume that ε is so small that $B(w_1,\varepsilon) \cap B(w_2,\varepsilon) = \emptyset$. To conclude, we observe that

$$0 \;<\; \iint \varphi_1 \varphi_2 \; d\nu_1 \otimes d\nu_2 \;\le\; \left\| \frac{\varphi_1\, \varphi_2}{\bar u_2 - \bar u_1} \right\|_\infty \iint |\bar u_2 - \bar u_1| \; d\nu_1(\bar u_1)\, d\nu_2(\bar u_2) \;=\; 0 \;,$$

which is a contradiction. The proof of Theorem 3.2 is completed. ∎

4 – Zero diffusion limit

The theory developed in Sections 2 and 3 is now applied to analyze a singular limit problem. We treat a class of multi-dimensional conservation laws containing vanishing diffusion. Precisely, we consider the problem (1.1)–(1.3), where the flux-function satisfies a growth condition like (2.1) with $r = 1$. Given a diffusion parameter $\varepsilon > 0$ and a (uniformly in x) positive-definite matrix $(b_{ij}(x))_{1\le i,j\le d}$ depending smoothly on x, we study the equation

$$u_t^\varepsilon + \operatorname{div} f(u^\varepsilon) = \varepsilon \sum_{i,j=1}^{d} \partial_i(b_{ij}\,\partial_j u^\varepsilon) \,, \tag{4.1}$$

together with

$$u(x,0) = u_0^\varepsilon(x)\,, \qquad x \in \Omega \,, \tag{4.2}$$

$$u(x,t) = u_B^\varepsilon(x,t)\,, \qquad x \in \partial\Omega, \quad t > 0 \,. \tag{4.3}$$

Here u_0^ε and u_B^ε are sufficiently smooth initial and boundary data. Standard existence results show that for all $\varepsilon > 0$, the problem (4.1)–(4.3) admits a unique smooth solution u^ε defined globally in time. The aim of this section is to prove the convergence of u^ε toward the entropy solution of (1.1)–(1.3).

Theorem 4.1. *Suppose that the flux f satisfies the growth condition*

$$f'(u) = O(1) \tag{4.4}$$

and consider an initial datum u_0 in $L^2(\Omega)$ and a sequence of smooth data u_0^ε satisfying the uniform bound

$$\|u_0^\varepsilon\|_{L^2(\Omega)} \le C \tag{4.5}$$

and a boundary data u_B such that

$$u_B \in L^\infty_{\mathrm{loc}}\Big([0,\infty), H^{1/2}(\partial\Omega)\Big) \quad \text{and} \quad \partial_t u_B \in L^2_{\mathrm{loc}}\Big([0,\infty), L^2(\partial\Omega)\Big) \,. \tag{4.6}$$

Then for each $T > 0$ the solutions u^ε of (4.1)–(4.3) satisfy for

$$\|u^\varepsilon(t)\|_{L^2(\Omega)} + \varepsilon^{1/2}\,\|\nabla u^\varepsilon\|_{L^2\left((0,T),L^2(\Omega)\right)} \le C(T)\,, \qquad t \in (0,T) \,, \tag{4.7}$$

for some constant $C(T) > 0$, and converge strongly to the unique entropy solution $u \in L^\infty_{\mathrm{loc}}([0,\infty), L^2(\Omega))$ of the hyperbolic problem (1.1)–(1.3).

Proof: Let $\tilde u_B \colon \overline{\Omega} \times [0,\infty) \mapsto \mathbb{R}$ be an extension of the function u_B to the whole domain. In view of (4.6) we can assume that $\tilde u_B \in L^\infty_{\mathrm{loc}}([0,\infty), H^1(\Omega))$ and $\partial_t \tilde u_B \in L^2_{\mathrm{loc}}([0,\infty), H^{1/2}(\Omega))$. Define

$$F\Big(u^\varepsilon(x,t), \tilde u_B(x,t)\Big) = \int_{\tilde u_B(x,t)}^{u^\varepsilon(x,t)} \Big(v - \tilde u_B(x,t)\Big)\, f'(v)\; dv \,.$$

We have

$$\operatorname{div}\Big(F\Big(u^\varepsilon(x,t),\tilde u_B(x,t)\Big)\Big) = (u^\varepsilon-\tilde u_B)\,f'(u^\varepsilon)\,\nabla u^\varepsilon - \int_{\tilde u_B}^{u^\varepsilon}\nabla\tilde u_B f'(v)\,dv$$
$$= (u^\varepsilon-\tilde u_B)\,f'(u^\varepsilon)\,\nabla u^\varepsilon + \nabla\tilde u_B\Big(f(\tilde u_B)-f(u^\varepsilon)\Big)\,.$$

Using this and (4.1) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac12 (u^\varepsilon-\tilde u_B)^2\,dx\\
&= \int_\Omega (u^\varepsilon-\tilde u_B)\Big(-\operatorname{div} f(u^\varepsilon) + \varepsilon\sum_{i,j=1}^d \partial_i(b_{ij}\,\partial_j u^\varepsilon)\Big)\,dx - \int_\Omega (u^\varepsilon-\tilde u_B)\,\partial_t\tilde u_B\,dx\\
&= \int_\Omega\Big(\nabla\tilde u_B\cdot\Big(f(\tilde u_B)-f(u^\varepsilon)\Big) - \operatorname{div} F(u^\varepsilon,\tilde u_B)\Big)\,dx\\
&\quad + \varepsilon\sum_{i,j=1}^d\int_\Omega (u^\varepsilon-\tilde u_B)\,\partial_i\Big(b_{ij}\,\partial_j(u^\varepsilon-\tilde u_B)\Big)\,dx\\
&\quad + \varepsilon\sum_{i,j=1}^d\int_\Omega (u^\varepsilon-\tilde u_B)\,\partial_i\Big(b_{ij}\,\partial_j(\tilde u_B)\Big)\,dx - \int_\Omega (u^\varepsilon-\tilde u_B)\,\partial_t\tilde u_B\,dx\,.
\end{aligned}$$

By integration by parts using that $F(\tilde u_B,\tilde u_B)=0$ we find

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac12 (u^\varepsilon-\tilde u_B)^2\,dx\\
&\quad= \int_\Omega \nabla\tilde u_B\cdot\Big(f(\tilde u_B)-f(u^\varepsilon)\Big)\,dx - \varepsilon\sum_{i,j=1}^d\int_\Omega \partial_i(u^\varepsilon-\tilde u_B)\,b_{ij}\,\partial_j(u^\varepsilon-\tilde u_B)\,dx\\
&\qquad + \int_\Omega (u^\varepsilon-\tilde u_B)\Big(-\partial_t\tilde u_B + \varepsilon\sum_{i,j=1}^d \partial_i(b_{ij}\,\partial_j\tilde u_B)\Big)\,dx\,.
\end{aligned}$$

Since $\sum_{i,j=1}^d \alpha_i\,b_{ij}\,\alpha_j \ge c\sum_{i,j=1}^d \alpha_i^2$ for some $c>0$, and since f is Lipschitz continuous by (4.4), we find with Cauchy–Schwarz inequality

$$\begin{aligned}
&\frac{d}{dt}\frac12\,\|u^\varepsilon(t)-\tilde u_B(t)\|^2_{L^2(\Omega)}\\
&\qquad\le \|\nabla\tilde u_B(t)\|_{L^2(\Omega)}\ \mathrm{Lip}(f)\,\|u^\varepsilon(t)-\tilde u_B(t)\|_{L^2(\Omega)}\\
&\qquad\quad - \varepsilon\,c\,\|\nabla u^\varepsilon(t)-\tilde u_B(t)\|^2_{L^2(\Omega)} + C(\tilde u_B)\,\|u^\varepsilon(t)-\tilde u_B(t)\|_{L^2(\Omega)}\,,
\end{aligned}$$

where $C(\tilde u_B)$ is bounded by the conditions (4.6).

We find that for some constant $C>0$ depending on u_B and f

$$\frac{d}{dt}\|u^\varepsilon(t)-\tilde u_B(t)\|^2_{L^2(\Omega)} + \varepsilon\,c\,\|\nabla u^\varepsilon(t)-\nabla\tilde u_B(t)\|^2_{L^2(\Omega)} \le C + \|u^\varepsilon(t)-\tilde u_B(t)\|^2_{L^2(\Omega)}$$

Then, by Gronwall inequality, for any $T > 0$, there exists $C(T) > 0$ such that for all $t \in [0,T]$,

$$\begin{aligned}\|u^\varepsilon(t) - \tilde u_B(t)\|^2_{L^2(\Omega)} + \varepsilon \int_0^T \|\nabla u(t) - \nabla \tilde u_B(t)\|^2_{L^2(\Omega)}\, dt \\ \le C(T) \Big(1 + \|u_0 - \tilde u_B(0)\|^2_{L^2(\Omega)}\Big) \\ \le C'(T)\, .\end{aligned}$$

In view of (4.6) we thus have proved (4.7).

To apply the framework of Sections 3 and 4, we need to check several assumptions. First of all (for a subsequence at least) we claim that

$$\sum_{i,j=1,\dots,d} \varepsilon\, b_{ij}\, \partial_j u^\varepsilon_{|\partial\Omega}\, N_i \quad \text{converges in the sense of distributions to some } q \in H^{-1}\Big((0,T), H^{-1/2}(\partial\Omega)\Big)\, .$$

Namely, multiplying the equation (4.1) by a test function $\varphi = \varphi(x,t)$ and integrating on $\Omega \times (0,T)$, we obtain

$$\begin{aligned}(4.8)\quad & \sum_{i,j=1}^d \int_0^T\!\!\int_{\partial\Omega} \varepsilon\, b_{ij}\, \varphi\, \partial_j u^\varepsilon\, N_i\ d\mathcal{H}_{d-1}\, dt \\ &= \int_\Omega u^\varepsilon(x,T)\, \varphi(x,T)\, dx - \int_\Omega u^\varepsilon(x,0)\, \varphi(x,0)\, dx \\ &\quad - \int_0^T\!\!\int_\Omega \Big(u^\varepsilon\, \partial_t\varphi + f(u^\varepsilon)\cdot\nabla\varphi\Big)\, dx\, dt + \int_0^T\!\!\int_{\partial\Omega} \varphi\, f(u_B)\cdot N\ d\mathcal{H}_{d-1}\, dt \\ &\quad + \varepsilon \sum_{i,j=1}^d \int_0^T\!\!\int_{\partial\Omega} b_{ij}\, \partial_i\varphi\, \partial_j u^\varepsilon\, dx\, dt\, .\end{aligned}$$

Using the bounds (4.7) we estimate the boundary flux in the following way:

$$\begin{aligned}(4.9)\quad & \left| \sum_{i,j=1}^d \int_0^T\!\!\int_{\partial\Omega} \varepsilon\, b_{ij}\, \varphi\, \partial_j u^\varepsilon N_i\ d\mathcal{H}_{d-1}\, dt \right| \\ &\le C\, \|\varphi(T)\|_{L^2(\Omega)} + C\, \|\varphi(0)\|_{L^2(\Omega)} \\ &\quad + C\, \|\partial_t\varphi\|_{L^1((0,T),L^2(\Omega))} + C\, \|\nabla\varphi\|_{L^1((0,T),L^2(\Omega))} \\ &\quad + C\, \|\varphi\|_{L^1((0,T),L^1(\partial\Omega))} + \varepsilon^{1/2}\, C\, \|\nabla\varphi\|_{L^2((0,T),L^2(\Omega))}\, .\end{aligned}$$

It follows that $\sum_{i,j=1}^d \varepsilon\, b_{ij}\, \partial_j u^\varepsilon_{\partial\Omega} N_j$ is uniformly bounded in some distribution space and, for a subsequence at least, admits a limit, say q, in the sense of distributions. Furthermore this limit satisfies the inequality

$$(4.10)\qquad \left| \int_0^T\!\!\int_{\partial\Omega} q\,\varphi\; d\mathcal{H}_{d-1}\,dt \right| \\ \le\; C\,\|\varphi(T)\|_{L^2(\Omega)} + C\,\|\varphi(0)\|_{L^2(\Omega)} + C\,\|\partial_t\varphi\|_{L^1\left((0,T),L^2(\Omega)\right)} \\ + C\,\|\nabla\varphi\|_{L^1\left((0,T),L^2(\Omega)\right)} + C\,\|\varphi\|_{L^1\left((0,T),L^1(\partial\Omega)\right)}\ .$$

Restricting attention to test-functions compactly supported in time in $[0,T)$ we have $\varphi(T)\equiv 0$ and

$$\|\varphi(0)\|_{L^2(\Omega)}\;\le\;\|\partial_t\varphi\|_{L^1\left((0,T),L^2(\Omega)\right)}\ .$$

Therefore we arrive at

$$(4.11)\qquad \left| \int_0^T\!\!\int_{\partial\Omega} q\,\varphi\; d\mathcal{H}_{d-1}\,dt \right| \\ \le\; C\,\|\partial_t\varphi\|_{L^1\left((0,T),L^2(\Omega)\right)} + C\,\|\nabla\varphi\|_{L^1\left((0,T),L^2(\Omega)\right)} + C\,\|\varphi\|_{L^1\left((0,T),L^1(\partial\Omega)\right)}\ .$$

On the other hand, as we are interested in the trace along the boundary only, we can always pick up any φ on $\partial\Omega$ and extend it to the whole of Ω so that

$$\|\partial_t\varphi\|_{L^1\left((0,T),L^2(\Omega)\right)} + \|\nabla\varphi\|_{L^1\left((0,T),L^2(\Omega)\right)} \\ \le\; C\,\|\partial_t\varphi\|_{L^1\left((0,T),H^{-1/2}(\partial\Omega)\right)} + C\,\|\varphi\|_{L^1\left((0,T),H^{1/2}(\partial\Omega)\right)}\ .$$

Finally we obtain

$$(4.12)\qquad \left| \int_0^T\!\!\int_{\partial\Omega} q\,\varphi\; d\mathcal{H}_{d-1}\,dt \right| \\ \le\; C\,\|\partial_t\varphi\|_{L^1\left((0,T),H^{-1/2}(\partial\Omega)\right)} + C\,\|\varphi\|_{L^1\left((0,T),H^{1/2}(\partial\Omega)\right)} + C\,\|\varphi\|_{L^1\left((0,T),L^1(\partial\Omega)\right)} \\ \le\; C\,\|\varphi\|_{W^{1,1}\left((0,T),H^{1/2}(\partial\Omega)\right)}\ .$$

This proves that the limiting trace q satisfies at least

$$(4.13)\qquad q\;\in\;W^{-1,\infty}\Big((0,T),H^{-1/2}(\partial\Omega)\Big)\ .$$

It remains to check the conditions (2.4) of Section 2.

Multiplying the equation (4.1) by $U'(u^\varepsilon)\,\theta$ with $\theta=\theta(x,t)\ge 0$ in $\mathcal{C}^1_c(\overline{\Omega}\times[0,\infty))$ and integrating over $\Omega\times[0,\infty)$, we obtain

$$(4.14)\qquad \iint_{\Omega\times\mathbb{R}_+}\Big(U(u^\varepsilon)\,\partial_t\theta + F(u^\varepsilon)\cdot\operatorname{grad}\theta\Big)\,dx\,dt\;+\int_\Omega U(u_0^\varepsilon(0))\,\theta(x,0)\,dx \\ -\iint_{\partial\Omega\times\mathbb{R}_+} B_U^\varepsilon(x,t)\,\theta(x,t)\;d\mathcal{H}_{d-1}(x)\,dt\;=\;\iint_{\Omega\times\mathbb{R}_+}\tilde{R}_U^\varepsilon\,dx\,dt\ ,$$

where (U,F) is an arbitrary convex and tame entropy-pair, and

$$B^\varepsilon_U \;=\; F(u^\varepsilon_B)\cdot N + U'(u^\varepsilon_B)\Big(b^\varepsilon - f(u^\varepsilon_B)\cdot N\Big)\;,$$

$$b^\varepsilon \;=\; f(u^\varepsilon_B)\cdot N \;-\; \sum_{i,j=1}^{d} \varepsilon\, b_{ij}\, \partial_j u^\varepsilon N_i\;,$$

and

$$\tilde R^\varepsilon_U \;=\; \sum_{i,j=1}^{d} b_{ij}\,\varepsilon\, \partial_i(U'(u^\varepsilon)\,\theta)\,\partial_j u^\varepsilon\;.$$

Using (4.7) and $U'' \ge 0$ we obtain

$$\tilde R^\varepsilon_U \;=\; \hat R^\varepsilon_U + R^\varepsilon_U$$

where $\hat R^\varepsilon_U \ge 0$ and

$$R^\varepsilon_U \;:=\; \sum_{i,j=1}^{d} b_{ij}\,\varepsilon\, U'(u^\varepsilon)\,\partial_i\theta\,\partial_j u^\varepsilon \;\longrightarrow\; 0\;.$$

On the other hand b admits a limit in the sense of distributions

$$b^\varepsilon \;\longrightarrow\; f(u_B) - q\;.$$

Therefore (4.14) leads us to the conditions (2.4). The framework developed in Sections 2 and 3 applies and we conclude that the approximate solutions constructed by (4.1)–(4.3) converge strongly to the unique entropy solution. ∎

REFERENCES

[1] Bardos, C.W.; Leroux, A.-Y. and Nedelec, J.-C. – First order quasilinear equations with boundary conditions, *Comm. Part. Diff. Eqns.,* 4 (1979), 75–78.

[2] Chen, Q.-G. and Frid, H. – Divergence measure fields and hyperbolic conservation laws, *Arch. Rational Mech. Anal.,* 147 (1999), 89–118.

[3] Cockburn, B.; Coquel, F. and LeFloch, P.G. – Convergence of finite volume methods for multi-dimensional conservation laws, *SIAM J. Numer. Anal.,* 32 (1995), 687–705.

[4] Coquel, F. and LeFloch, P.G. – Convergence of finite difference schemes for scalar conservation laws in several space variables: the corrected antidiffusive-flux approach, *Math. of Comput.,* 57 (1991), 169–210.

[5] DiPerna, R.J. – Measure-valued solutions to conservation laws, *Arch. Rational Mech. Anal.,* 88 (1985), 223–270.

[6] Dubois, F. and LeFloch, P.G. – Boundary conditions for nonlinear hyperbolic systems of conservation laws, *J. Differential Equations,* 31 (1988), 93–122.

[7] Gilbarg D. and Trudinger N. – *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, New York, 1983.

[8] Joseph, K.T. and LeFloch, P.G. – Boundary layers in weak solutions of hyperbolic conservation laws, *Arch. Rational Mech. Anal.,* 147 (1999), 47–88.

[9] Kružkov, S.N. – First order quasilinear equations in several independent variables, *Mat. Sbornik,* 81 (1970), 285–355; English translation in *Math. USSR Sb.,* 10 (1970), 217–243.

[10] LeFloch, P.G. – Explicit formula for scalar conservation laws with boundary condition, *Math. Meth. Appl. Sc.,* 10 (1988), 265–287.

[11] LeFloch, P.G. – *Hyperbolic Systems of Conservation Laws: The theory of classical and nonclassical shock waves*, E.T.H. Lecture Notes Series, Birkhäuser, 2001.

[12] Malek, J.; Necas, Rokyta M. and Ruzicka, M. – *Weak and measure-valued solutions to evolutionnary PDE's*, Applied Math. and Math. Comput., 13, Chapmann and Hall, 1996.

[14] Schonbek M.E. – Convergence of solutions to nonlinear dispersive equations, *Comm. Part. Diff. Equa.,* 7 (1982), 959–1000.

[15] Szepessy, A. – An existence result for scalar conservation laws using measure-valued solutions, *Comm. Part. Diff. Eqns.,* 14 (1989), 1329–1350.

[16] Szepessy A. – Measure-valued solutions of scalar conservation laws with boundary conditions, *Arch. Rational Mech. Anal.,* 107 (1989), 181–193.

[17] Szepessy A. – Convergence of a streamline diffusion finite element method for a conservation law with boundary conditions, *RAIRO Model. Math. Anal. Numer.,* 25 (1991), 749–783.

[18] Tartar, L. – *The compensated compactness method applied to systems of conservation laws*, in "Systems of Nonlinear Partial Differential Equations" (J.M. Ball, Ed.), NATO ASI Series, C. Reidel publishing Col., 1983, pp. 263–285.

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