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ON DIVERGENT DIAGRAMS OF FINITE CODIMENSION

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Abstract: We obtain the formal classification of finite codimension singular points of smooth divergent diagrams of the type $(f,g)\colon (\mathbb{R},0)\longleftarrow(\mathbb{R}^n,0)\longrightarrow(\mathbb{R},0)$, $n\geq 2$. We also define a complete set of topological invariants for this classification.

Introduction

Divergent diagrams of smooth mappings on smooth manifolds

$$(f,g)\colon\ P\longleftarrow N\longrightarrow Q$$

appear in various contexts of applications of singularity theory such as envelope theory, web geometry, singularities of first order differential equations, vision theory. Two divergent diagrams $(f_i,g_i)\colon\ P_i\longleftarrow N_i\longrightarrow Q_i$ $(i=1,2)$ are equivalent if there exist diffeomorphisms $h\colon\ N_1\to N_2$, $k\colon\ P_1\to P_2$ and $l\colon\ Q_1\to Q_2$ such that $f_2\circ h=k\circ f_1$, $g_2\circ h=l\circ g_1$.

The stability of such diagrams with respect to this equivalence relation was extensively studied by J.P. Dufour in [4], [5], [6]. Applications of this theory to geometry are given in [7], [8], [9], [10] and [14].

In the present paper, we obtain the formal classification of finite codimension singular points of divergent diagrams of the type $(f,g)\colon(\mathbb{R},0)\longleftarrow(\mathbb{R}^n,0)\longrightarrow(\mathbb{R},0)$, $n\geq 2$. The following theorem summarises our main result.

Theorem 0.1. *Let $(f,g)\colon\ (\mathbb{R}^n,0)\to(\mathbb{R}^2,0)$ be a divergent diagram, where $n\geq 2$. Then (f,g) has formal finite codimension if and only if it is formally equivalent to one of the following normal forms given in the Table 1 below.*

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Type $f=(f_1,f_2)$	Formal Normal Form	formal $\mathrm{cod}(f)$
1 submersion	(x,y)	0
2 transverse fold	$\big(x^2+y+q(z),\, y\big)$	0
$2'$ tangent fold with contact of order $k+2$; $\pm$ agree for odd k	$\big(x^2 \pm y^{k+2}+q(z),\, y\big)$ or $\big(x,\, \pm x^{k+2}+y^2+q(z)\big)$	k
3 transverse cusp	$\big(x^3+xy+y+q(z),\, y\big)$	0
4 transverse k-lips/beak to beak; k even	$\big(x^3 \pm xy^k + s\, y^{3k/2}+y+q(z)\,,\; y\big)$	k
5 transverse k-lips/beak to beak; k odd, $k>1$	$\big(x^3+xy^k+y+q(z)\,,\; y\big)$	$k-1$

Table 1

where $z=(z_1,...,z_{n-2})$ *and* $q(z)=\sum_{i=1}^{n-2} \pm {z_i}^2$. ∎

In the final section, we define a complete set of topological invariants for this classification.

1 – Notations and basic definitions

A divergent diagram $(f_1,f_2):\ (\mathbb{R}^p,0) \longleftarrow (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^q,0)$ is a pair of map-germs $f_1\colon (\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ and $f_2\colon (\mathbb{R}^n,0)\to(\mathbb{R}^q,0)$.

Definition 1.1. Two divergent diagrams $(f_1,f_2),(g_1,g_2)\colon (\mathbb{R}^p,0)\longleftarrow(\mathbb{R}^n,0) \longrightarrow(\mathbb{R}^q,0)$ are equivalent if there exist germs of diffeomorphisms h of $(\mathbb{R}^n,0)$, k_1 of $(\mathbb{R}^p,0)$ and k_2 of $(\mathbb{R}^q,0)$ commuting the following diagram:

$$\begin{array}{ccccc} (\mathbb{R}^p,0) & \xleftarrow{f_1} & (\mathbb{R}^n,0) & \xrightarrow{f_2} & (\mathbb{R}^q,0) \\ k_1\downarrow & & \downarrow h & & \downarrow k_2 \\ (\mathbb{R}^p,0) & \xleftarrow{g_1} & (\mathbb{R}^n,0) & \xrightarrow{g_2} & (\mathbb{R}^q,0) \end{array}$$ □

We identify a divergent diagram $(f_1,f_2):\ (\mathbb{R}^p,0)\longleftarrow(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^q,0)$ with the map-germ $f:\ (\mathbb{R}^n,0)\to(\mathbb{R}^p{\times}\mathbb{R}^q,(0,0))$, $f(x)=(f_1(x),f_2(x))$. With this identification, the equivalence of divergent diagrams corresponds to the action of a subgroup of the group $\mathcal{A}$, consisting of elements such that the germ of diffeomorphism in the target is of product type, i.e., preserves the product structure of $\mathbb{R}^p \times \mathbb{R}^q$.

We need some notation to describe the tangent spaces to the orbits associated with these groups.

For any non-negative integer m, let C_m be the local ring of smooth function-germs at the origin in $\mathbb{R}^m$, and $\mathcal{M}_m$ the corresponding maximal ideal. Let $C(m,s)$ be the space of smooth map-germs $f\colon (\mathbb{R}^m,0) \to (\mathbb{R}^s,0)$. Given a map-germ $f \in C(m,s)$, let $f^*\colon C_s \to C_m$ be the ring homomorphism defined by $f^*(\phi) = \phi \circ f$. Let θ_f denote the C_m-module of vector fields along f, and set $\theta_m = \theta_{I(\mathbb{R}^m,0)}$. The C_m-homomorphism $tf\colon \theta_m \to \theta_f$ is defined by $tf(\xi) = df(\xi)$, and the morphism over f^*, $wf\colon \theta_s \to \theta_f$ by $wf(\eta) = \eta \circ f$. The tangent space and the extended tangent space associated with the group $\mathcal{A}$ acting in $C(m,s)$ are defined by $T\mathcal{A}(f) = tf(\mathcal{M}_m\theta_m) + wf(\mathcal{M}_s\theta_s)$ and $T\mathcal{A}_e(f) = tf(\theta_m) + wf(\theta_s)$. The $\mathcal{A}_e$-codimension of f is defined by $\operatorname{cod}(\mathcal{A}_e, f) = \dim_{\mathbb{R}} \theta_f / T\mathcal{A}_e(f)$.

Let $f = (f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, (0,0))$ be a divergent diagram. We write $\theta_f = \theta_{f_1} \oplus \theta_{f_2}$ and define the tangent space and the extended tangent space associated with the equivalence of divergent diagrams by

$$T(f) = tf(\mathcal{M}_n\theta_n) + \Big[wf_1(\mathcal{M}_p\theta_p) \oplus wf_2(\mathcal{M}_q\theta_q)\Big]$$

and

$$T_e(f) = tf(\theta_n) + \Big[wf_1(\theta_p) \oplus wf_2(\theta_q)\Big] .$$

Then, $T_e(f)$ (resp. $T(f)$) is the set of all $\sigma = (\sigma_1, \sigma_2) \in \theta_f$ (resp. $\sigma \in \mathcal{M}_n\theta_f$) such that there exist $\xi \in \theta_n$ (resp. $\xi \in \mathcal{M}_n\theta_n$), $\eta_1 \in \theta_p$ (resp. $\eta_1 \in \mathcal{M}_p\theta_p$) and $\eta_2 \in \theta_q$ (resp. $\eta_2 \in \mathcal{M}_q\theta_q$) satisfying

$$\begin{cases} \sigma_1 = df_1(\xi) + \eta_1 \circ f_1 , \\ \sigma_2 = df_2(\xi) + \eta_2 \circ f_2 . \end{cases}$$

The codimension of the diagram f is defined by

$$\operatorname{cod}(f) = \dim_{\mathbb{R}} \theta_f / T_e(f) .$$

In the calculations of the codimension, we shall use the following conventions:

1) If $(x_1, ..., x_m)$ indicates the coordinate system in $(\mathbb{R}^m, 0)$, C_m will be written $C_{(x_1,...,x_m)}$, and analogously for $\mathcal{M}_m$.

2) For a map-germ $f: (\mathbb{R}^m, 0) \to (\mathbb{R}^s, 0)$, θ_f will be identified with $(C_m)^s$ via its free basis $\left\{\frac{\partial}{\partial y_1} \circ f, ..., \frac{\partial}{\partial y_s} \circ f\right\}$ for the given coordinated system $(y_1, ..., y_s)$ in $(\mathbb{R}^s, 0)$.

2 – Auxiliary results

Let $f = (f_1, f_2)\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, (0,0))$ be a divergent diagram and $T_e(f_1)$ the set of all vector fields $\sigma_1 \in \theta_{f_1}$ such that $\sigma = (\sigma_1, 0) \in T_e(f)$ $(0 \in \theta_{f_2})$.

Proposition 2.1. *If f_2 is $\mathcal{A}$-finitely determined, then* $\mathrm{cod}(f)$ *is finite if and only if* $\dim_{\mathbb{R}} \theta_{f_1}/T_e(f_1)$ *is finite, and in this case*

$$\mathrm{cod}(f) = \dim_{\mathbb{R}} \theta_{f_1}/T_e(f_1) + \mathrm{cod}(\mathcal{A}_e, f_2) \ .$$

Proof: It is enough to observe that the following sequence is exact:

$$0 \longrightarrow \frac{\theta_{f_1}}{T_e(f_1)} \xrightarrow{i^*} \frac{\theta_f}{T_e(f)} \xrightarrow{\pi^*} \frac{\theta_{f_2}}{T\mathcal{A}_e(f_2)} \longrightarrow 0$$

where i^* and π^* are defined by

$$i^*([\sigma]) = [(\sigma, 0)] \ ,$$
$$\pi^*([(\sigma_1, \sigma_2)]) = [\sigma_2] \ . \ \blacksquare$$

Corollary 2.2. *If f_2 is $\mathcal{A}$-infinitesimally stable, then* $\mathrm{cod}(f) = \dim_{\mathbb{R}} \theta_{f_1}/T_e(f_1)$.

Proposition 2.3. *Let f be the divergent diagram*

$$f = (g, \pi)\colon\ (\mathbb{R}^p, 0) \longleftarrow (\mathbb{R}^n \times \mathbb{R}^q, 0) \longrightarrow (\mathbb{R}^q, 0) \ ,$$

where π is the canonical projection $\pi(x,y) = y$. Consider the map-germ $g_0\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ given by $g_0(x) = g(x, 0)$. Then,

$$\mathrm{cod}(\mathcal{A}_e, g_0) \ \le\ \mathrm{cod}(f) + q \ .$$

Proof: Let $i^*: \theta_g \to \theta_{g_0}$ be the surjective $\mathbb{R}$-linear transformation defined by $i^*(\sigma) = \sigma \circ i$, where $i\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \mathbb{R}^q, 0)$ is the canonical inclusion. Then i^* induces a $\mathbb{R}$-linear isomorphism

$$\widetilde{i}\,^*\colon\ \frac{\theta_g}{T_e(g) + \mathcal{M}_q\theta_g} \ \to\ \frac{\theta_{g_0}}{T\mathcal{A}_e(g_0) + \left\langle \frac{\partial g}{\partial y_1}\Big|_{y=0}, ..., \frac{\partial g}{\partial y_q}\Big|_{y=0}\right\rangle_{\mathbb{R}}} .$$

The result follows now from Corollary 2.2. ∎

3 – Classification of divergent diagrams of finite codimension

In this section, we prove Theorem 0.1, in which we obtain formal normal forms for all divergent diagrams $f = (f_1, f_2)\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$, $n \geq 2$, of finite codimension. For corank one diagrams f, f receives the adjective transverse if the image of $df(0)$ is transversal to both subspaces $\mathbb{R}\times 0$ and $0\times\mathbb{R}$ of $\mathbb{R}^2$, and tangent, if the image of $df(0)$ coincides with one of them.

The classification of the stable diagrams was obtained by Dufour in [4], [5]. In Proposition 3.1, the finite codimension divergent diagrams of fold type are classified. Diagrams of cusp type or more degenerate are treated in the Propositions 3.2, 3.3, 3.4 and 3.6 below. For corank one diagrams f, we shall assume that the germ f_2 is nonsingular and, hence, $\mathcal{A}$-infinitesimally stable; for the calculation of the codimension of f, we use Corollary 2.2.

Proposition 3.1. *Let $f = (f_1, f_2)\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$, $n \geq 2$, be a divergent diagram of fold type and let $k \geq 1$. Then, $\operatorname{cod}(f) = k$ if and only if f is a tangent fold with contact of order $k + 2$. In this case, the normal form is:*

$$(x, y, z) \longmapsto \Big(x^2 \pm y^{k+2} + q(z),\, y\Big) ,$$

where $z = (z_1, z_2, ..., z_{n-2})$ *and* $q(z) = \sum_{i=1}^{n-2} \pm {z_i}^2$.

Proof: We can choose coordinates in the source such that f is of the following form (see[19])

$$(x, y, z) \longmapsto \Big(\pm x^2 + \lambda(y) + q(z),\, y\Big) ,$$

with $\lambda \in \mathcal{M}_y$.

For this divergent diagram, $T_e(f_1)$ is the set of all vector fields $\sigma \in \theta_{f_1}$ satisfying the equation:

$$\begin{aligned}\sigma(x,y,z) \;=\;& \pm 2\,x\,\xi(x,y,z) + \lambda'(y)\,\nu(y)\\ &+ \sum_{i=1}^{n-2} \pm 2\,z_i\,\xi_i(x,y,z) + \eta\Big(\pm x^2 + \lambda(y) + q(z)\Big)\end{aligned}$$

where $\xi,\ \xi_i \in C_{(x,y,z)}$ $(i=1,...,n-2)$, $\eta \in C_u,\ \nu \in C_v$, $((u,v)$ denotes the target coordinates).

To solve this equation, it is equivalent to solve the following:

$$\sigma(y) \;=\; \lambda'(y)\,\nu(y) + \eta(\lambda(y)) \ .$$

Then,

$$\dim_{\mathbb{R}} \theta_{f_1}/T_e(f_1) = k$$

if and only if $\mathrm{ord}(\lambda) = k+2$. Under this condition, it follows that the divergent diagram f is equivalent to the divergent diagram given by:

$$(x,y,z) \longmapsto \Big(x^2 \pm y^{k+2} + q(z),\ y\Big) \ . \ \blacksquare$$

Next, we analyse the divergent diagrams arising from germs $f=(f_1,f_2)$ of the following types: tangent cusp, lips, beak to beak and swallowtail. By choosing coordinates, we can assume that the diagram has the following form:

$$(x,y,z) \longmapsto \Big(g(x,y) + q(z),\ y\Big) \ ,$$

where z and $q(z)$ are as before, $g_0 \in \mathcal{M}_x^3,\ g_0(x) = g(x,0)$.

As in the proof of Proposition 3.1, this classification problem reduces to the classification of the following divergent diagram:

$$f\colon\ (x,y) \longmapsto (g(x,y),y) \ ,$$

and to compute the codimension of the diagram, we consider the space $T_e(g)$, given by the set of all vector fields $\sigma \in \theta_g$ satisfying the equation:

$$\sigma(x,y) \;=\; \frac{\partial g}{\partial x}(x,y)\,\xi_1(x,y) + \frac{\partial g}{\partial y}(x,y)\,\xi_2(y) + \eta(g(x,y)) \ ,$$

where $\xi_1 \in C_{(x,y)},\ \eta \in C_u$ and $\xi_2 \in C_v$.

We denote by $\sum(f)$ the singular set of f. Then, $\sum(f)$ is given by the equation $\frac{\partial g}{\partial x} = 0$. When $\frac{\partial g}{\partial x}$ is a submersion defining the germ of a regular curve, transversal to the y-axis, we shall assume that $\frac{\partial g}{\partial x} = 0 = y - \alpha(x)$, and denote by γ a parametrization of its image by f, $f(\sum(f))$.

Lemma 3.1. *With the above conditions, we have that* $\mathrm{cod}(f)$ *is finite if and only if* $\mathrm{cod}(\gamma)$ *is finite, and in this case,*

$$\mathrm{cod}(f) \leq \mathrm{cod}(\gamma) \leq\ 2\ \mathrm{cod}(f) + 1\ ,$$

and the codimensions are taken with respect to the equivalence of divergent diagrams.

Proof: **Step 1.** Let $\delta\colon (\mathbb{R},0) \to (\mathbb{R}^2,0)$ be the parametrization of $\Sigma(f)$ given by $\delta(x) = (x,\alpha(x))$. Let $\delta^*\colon\ C_{(x,y)} \to C_x$ be the $\mathbb{R}$-linear transformation given by $\delta^*(\sigma) = \sigma \circ \delta$. We have that $\ker \delta^* = \langle \frac{\partial g}{\partial x} \rangle$, where $\langle \frac{\partial g}{\partial x} \rangle$ is the ideal of $C_{(x,y)}$ generated by $\frac{\partial g}{\partial x}$. Moreover, δ^* is surjective since given $\mu \in C_x$, and any $h \in \ker \delta^*$, the germ $\sigma \in C_{(x,y)}$, defined by $\sigma(x,y) = \mu(x) + h(x,y)$, is such that $\delta^*(\sigma) = \mu$. Since $\ker \delta^* \subset T_e(g)$, δ^* induces a $\mathbb{R}$-linear isomorphism from $\frac{\theta g}{T_e(g)}$ onto $\frac{C_x}{\delta^*(T_e(g))}$.

Thus, for the calculation of $\mathrm{cod}(f)$, it is sufficient to consider the equation:

$$\mu(x)\ =\ \frac{\partial g}{\partial y}(x,\alpha(x))\,\xi(\alpha(x)) +\ \eta(g(x,\alpha(x)))\ ,$$

where $\eta \in C_u$ and $\xi \in C_v$.

Step 2. We consider the divergent diagram

$$\gamma\ =\ (\gamma_1,\gamma_2)\colon\ (\mathbb{R},0) \longrightarrow (\mathbb{R}^2,0)\ ,$$

with $\gamma = f \circ \delta$. In this case, $T_e(\gamma)$ is the set of all vector fields $\sigma = (\sigma_1,\sigma_2) \in \theta_\gamma$ satisfying to the system:

$$\begin{cases} \sigma_1(x)\ =\ \dfrac{\partial g}{\partial y}(x,\alpha(x))\,\alpha'(x)\,\xi(x) + \eta_1(g(x,\alpha(x)))\ , \\ \sigma_2(x)\ =\ \alpha'(x)\,\xi(x) + \eta_2(\alpha(x))\ , \end{cases}$$

where $\xi \in C_x$, $\eta_1 \in C_u$ and $\eta_2 \in C_v$.

Assume that $\mathrm{cod}(\gamma) = l < \infty$. Let us choose $p_1 = (p_{11},p_{21}),...,p_l = (p_{1l},p_{2l}) \in \theta_\gamma$, such that

$$\theta_\gamma\ =\ T_e(\gamma) \oplus \langle p_1,...,p_l \rangle_{\mathbb{R}}\ .$$

Then, given any $\sigma = (\sigma_1, \sigma_2) \in \theta_\gamma$, there exist $\xi \in C_x$, $\eta_1 \in C_u$, $\eta_2 \in C_v$, $a_1, ..., a_l \in \mathbb{R}$ such that

$$\begin{cases} \sigma_1(x) \;=\; \dfrac{\partial g}{\partial y}(x, \alpha(x))\, \alpha'(x)\, \xi(x) + \eta_1(g(x, \alpha(x))) + \displaystyle\sum_{i=1}^{l} a_i\, p_{1i}(x) \;, \\ \sigma_2(x) \;=\; \alpha'(x)\, \xi(x) + \eta_2(\alpha(x)) + \displaystyle\sum_{i=1}^{l} a_i\, p_{2i}(x) \;. \end{cases}$$

Taking $\sigma_2 = 0$ in the above system, we conclude that any $\mu \in C_x$ can be written in the form

$$(*) \qquad \mu(x) \;=\; \frac{\partial g}{\partial y}(x, \alpha(x))\, \xi(\alpha(x)) + \eta(g(x, \alpha(x))) + \sum_{i=1}^{l} a_i\, q_i(x) \;,$$

where $q_i(x) = p_{1i}(x) - \frac{\partial g}{\partial y}(x, \alpha(x))\, p_{2i}(x)$ $(i = 1, ..., l)$, $\eta \in C_u$ and $\xi \in C_v$. It follows from step 1 that $\operatorname{cod}(f) \le l = \operatorname{cod}(\gamma)$.

Now, assume that $\operatorname{cod}(f) = r < \infty$. Then, from step 1, there exist $q_1, q_2, ..., q_r \in C_x$, such that any $\mu \in C_x$ can be written as in $(*)$.

Moreover, setting $s = \operatorname{ord}(\alpha)$, we obtain $2 \le s = \operatorname{ord}(g_0) - 1$.

Hence, $\alpha(x) = x^s \phi(x)$, where $\phi \in C_x$ is invertible. Let us write $\alpha'(x) = x^{s-1}\psi(x)$, ψ invertible. Then, $\alpha(x) = x\, \alpha'(x)\, \frac{\phi(x)}{\psi(x)}$. Given $\xi \in C_v$, we have

$$\xi(\alpha(x)) \;=\; a + \alpha'(x) \left(x\, \frac{\phi(x)}{\psi(x)}\, \lambda(\alpha(x)) \right) \;=\; a + \alpha'(x)\, \bar\xi(x) \;,$$

where $\bar\xi(x) = x\, \frac{\phi(x)}{\psi(x)}\, \lambda(\alpha(x))$, $\bar\xi \in \mathcal{M}_x$. Substituting in $(*)$, we obtain

$$\begin{cases} \mu(x) \;=\; \dfrac{\partial g}{\partial y}(x, \alpha(x))\, \alpha'(x)\, \bar\xi(x) + \eta_1(g(x, \alpha(x))) + a\, \dfrac{\partial g}{\partial y}(x, \alpha(x)) + \displaystyle\sum_{i=1}^{r} a_i\, q_i(x) \;, \\ 0 \;=\; \alpha'(x)\, \bar\xi(x) + \eta_2(\alpha(x)) \;, \end{cases}$$

where $\eta_2 = a - \xi$.

Hence,

$$\theta_{\gamma_1} \;=\; T_e(\gamma_1) + \left\langle \frac{\partial g}{\partial y} \circ \delta,\, q_1, ..., q_r \right\rangle_{\mathbb{R}} \;.$$

Since $\dim_{\mathbb{R}} \frac{\theta_{\gamma_1}}{T_e(\gamma_1)} \le r + 1 = \operatorname{cod}(f) + 1$ and $\operatorname{cod}(\mathcal{A}_e, \alpha) = s - 2$, it follows from Proposition 2.1 that

$$\operatorname{cod}(\gamma) \;\le\; \operatorname{cod}(f) + s - 1 \;.$$

Now, we have that $\mathrm{cod}(\mathcal{A}_e, g_0) = s-1$, hence it follows from Proposition 2.3 that

$$\mathrm{cod}(\gamma) \leq 2\,\mathrm{cod}(f) + 1\ . \ \blacksquare$$

Lemma 3.2 ([3], Part III, Lemma I.1)**.** *Let* $\gamma = (\gamma_1, \gamma_2)\colon (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *be a divergent diagram such that* $\mathrm{ord}(\gamma_1) \geq 3$ *and* $\mathrm{ord}(\gamma_2) \geq 2$*. Then,* $\mathrm{cod}(\gamma) = \infty$. $\blacksquare$

Remark 3.2. One can get more precise information on divergent diagrams γ as above. In fact, using the Method of Complete Transversals ([1]), we can show that any γ such that $j^3\gamma = (x^3, x^2)$, where $j^3\gamma$ denotes the Taylor polynomial of degree 3 of γ, is formally equivalent to a diagram of the following type:

$$\left(x^3 + \varepsilon\, x^4 + \sum_{i=1}^{\infty} a_i\, x^{6i+2} + \sum_{j=1}^{\infty} b_j\, x^{6j+4},\ x^2\right), \qquad \varepsilon = 0, 1, \quad a_i, b_j \in \mathbb{R}\ .$$

The least degenerate diagram in this family not only has infinite codimension but also infinite modality, but the codimension of the modular stratum is 2. This example shows that divergent diagrams of infinite codimension form a bigger set than some readers might expect. $\square$

Proposition 3.3. *Let* $f = (f_1, f_2)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a divergent diagram of tangent cusp type. Then,* f *has infinite codimension.*

Proof: We can assume that f is of the following form

$$(x, y) \longmapsto \Big(x^3 + x\,y + \lambda(y),\ y\Big)\ ,$$

with $\lambda'(0) = 0$.

In this case, $\Sigma(f)$ is the curve defined by $y + 3\,x^2 = 0$. Hence, it follows from Lemma 3.2 that the divergent diagram $\gamma = (\gamma_1, \gamma_2)$ obtained from the parametrization of $f(\Sigma(f))$ has infinite codimension. Now, the result follows from Lemma 3.1. $\blacksquare$

Proposition 3.4. *Let* $f(x, y) = (g(x, y), y)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a divergent diagram with* $g_0 \in \mathcal{M}_x^4$*. Then,* f *has infinite codimension.*

Proof: From the upper semicontinuity of the codimension, it suffices to show that any divergent diagram $f(x, y) = (g(x, y), y)$ with $g(x, y) = x^4 + \phi(x)\, y + \eta(x, y)\, y^2 + \psi(x) + \lambda(y)$, where $\phi(0) = 0$, $\phi'(0) \neq 0$, $\eta(0, 0) = 0$, $\psi \in \mathcal{M}_x^5$, has infinite codimension.

The conditions imply that the singular set $\Sigma(f)$ is a regular curve given by $y=\alpha(x)=a\,x^3+h.o.t$, and as in the above proposition, the result follows from Lemmas 3.1 and 3.2. ∎

It is a consequence of our previous results and of the upper semicontinuity of the codimension that, if $f(x,y)=(g(x,y),y)$ is a finite codimension divergent diagram, then $\mathrm{ord}(g_0)=3$, and the origin is a singularity of transverse type.

Proposition 3.5. *Let $f(x,y)=(g(x,y),y)\colon(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ be a divergent diagram $\mathcal{A}$-equivalent to $(x^3\pm x\,y^2,y)$, the origin being a singularity of transverse type. Then $f(x,y)$ is equivalent to*

$$(x,y)\longmapsto\Big(y+x^3\pm x\,y^2+s\,y^3+\lambda(x,y),\;y\Big)\;,$$

with $\lambda\in\mathcal{M}^\infty_{(x,y)}$. Moreover, $\mathrm{cod}(f)\geq 2$. ∎

Lemma 3.3. *Let $g(x,y)=y+x^3\pm x\,y^2+\lambda(x,y),\;\lambda\in{\mathcal{M}^4}_{(x,y)}$.*

(i) $\Big\langle x^{r-s}y^s;\;s=0,1,...,r\Big\rangle_{\mathbb{R}}\subset T_e(g)+\mathcal{M}^{r+1}_{(x,y)}$, *for every $r\geq 2$, $r\neq 5$.*

(ii) $\mathcal{M}^6_{(x,y)}\subset T_e(g)+\mathcal{M}^\infty_{(x,y)}$.

(iii) $T_e(g)+\mathcal{M}^\infty_{(x,y)}\supset\theta_g-\Big\langle x,x^2y,y^3,x^5,x^3y^2,xy^4\Big\rangle_{\mathbb{R}}+\Big\langle 3x^2y\pm y^3,3x^5\pm x^3y^2,3x^3y^2\pm xy^4,\pm 2xy^4+y^3\Big\rangle_{\mathbb{R}}$.

Proof: We recall that $T_e(g)$ is the vector space of all vector fields $\sigma\in\theta_g$ given by

$$\begin{aligned}\sigma(x,y)\;=\;&\Big(3\,x^2\pm y^2+\frac{\partial\lambda}{\partial x}(x,y)\Big)\,\xi_1(x,y)+\Big(1\pm 2\,x\,y+\frac{\partial\lambda}{\partial y}(x,y)\Big)\,\xi_2(y)\\&+\;\eta\Big(y+x^3\pm x\,y^2+\lambda(x,y)\Big)\;,\end{aligned}$$

where $\xi_1\in C_{(x,y)},\;\xi_2\in C_v$ and $\eta\in C_u$.

(**i**) **Case 1**: r even, $r\geq 2$.

Taking in the above equation

(**a**) $\xi_1(x,y)=x^{r-2j-2}\,y^{2j},\;j=0,...,(r-2)/2,\;\xi_2=0,\;\eta=0$,

(**b**) $\xi_1=0,\;\xi_2(y)=y^r,\;\eta=0$,

we obtain

(**1**) $3\,x^{r-2j}\,y^{2j}\pm x^{r-2j-2}\,y^{2j+2}\in T_e(g)+\mathcal{M}^{r+1}_{(x,y)},\quad j=0,...,(r-2)/2$,

(**2**) $y^r\in T_e(g)+\mathcal{M}^{r+1}_{(x,y)}$.

From (1) and (2), it follows that

$$x^{r-2k}\,y^{2k} \;\in\; T_e(g)+\mathcal{M}^{r+1}_{(x,y)}\,, \qquad k=0,...,r/2\;.$$

If $r=2$, taking $\xi_1=0$, $\xi_2=\pm 1/2$ and $\eta=\mp 1/2$, we obtain that $xy\in T_e(g)+\mathcal{M}^3_{(x,y)}$, and this completes the analysis in this case.

If $r\geq 4$, we take

(**c**) $\xi_1(x,y)=x^{r-(2j+1)-2}\,y^{2j+1}$, $j=0,...,(r-4)/2$, $\xi_2=0$, $\eta=0$.

(**d**) $\xi_1=0$, $\xi_2(y)=y^{r-2}$, $\eta=0$,

(**e**) $\xi_1=0$, $\xi_2=0$, $\eta(u)=u^{r-2}$,

we get, respectively,

(**3**) $3\,x^{r-(2j+1)}\,y^{2j+1}\pm x^{r-(2j+1)-2}\,y^{(2j+1)+2}\;\in\;T_e(g)+\mathcal{M}^{r+1}_{(x,y)}$, $j=0,...,(r-4)/2$,

(**4**) $y^{r-2}\pm 2\,x\,y^{r-1}\;\in\;T_e(g)+\mathcal{M}^{r+1}_{(x,y)}$,

(**5**) $y^{r-2}+(r-2)\,(x^3\,y^{r-3}\pm x\,y^{r-1})\;\in\;T_e(g)+\mathcal{M}^{r+1}_{(x,y)}$.

The relations (3) (with $j=(r-4)/2$), (4) and (5) give the following system:

$$\begin{aligned}
3\,x^3\,y^{r-3}\pm x\,y^{r-1} &= 0 \pmod{T_e(g)+\mathcal{M}^{r+1}_{(x,y)}}\;,\\
\pm 2\,x\,y^{r-1}+y^{r-2} &= 0 \pmod{T_e(g)+\mathcal{M}^{r+1}_{(x,y)}}\;,\\
(r-2)\,x^3\,y^{r-3}\pm(r-2)\,x\,y^{r-1}+y^{r-2} &= 0 \pmod{T_e(g)+\mathcal{M}^{r+1}_{(x,y)}}\;,
\end{aligned}$$

with the determinant of the matrix of the coefficients equals to $\mp(2\,r-10)$, and non-zero in this case.

Hence, it follows that $x^3\,y^{r-3}$, $x\,y^{r-1}\in T_e(g)+\mathcal{M}^{r+1}_{(x,y)}$, and these together with (3) will give:

$$x^{r-(2k+1)y^{2k+1}}\in T_e(g)+\mathcal{M}^{r+1}_{(x,y)}\,, \qquad k=0,...,(r-2)/2\;,$$

concluding this case.

Case 2: r odd, $r\geq 3$, and $r\neq 5$.

We follow the same method as above, but changing through $2\,j$ by $2\,j+1$ and vice-versa.

(ii) and (iii) follow easily from (i). Notice that the Malgrange Preparation Theorem does not hold for divergent diagrams. Hence, in (ii), we only get the formal relation. ∎

Proof of Proposition 3.4: With the hypothesis, we can write $g(x,y) = y + x^3 \pm x\,y^2 + \lambda(x,y)$, $\lambda \in \mathcal{M}^4_{(x,y)}$.

We can easily change coordinates in source and target to reduce $g(x,y)$ to the form:

$$g(x,y) \;=\; y + x^3 \pm x\,y^2 + t\,x\,y^4 + \lambda(x,y)\,, \qquad \lambda \in \mathcal{M}^6_{(x,y)}\;.$$

Applying Lemma 3.3 and Mather's Lemma ([15]), we get that, formally, the diagram $(x,y) \longmapsto f(x,y) = (g(x,y), y)$ is equivalent to

$$(x,y) \;\longmapsto\; \Big(y + x^3 \pm x\,y^2 + t\,x\,y^4,\;\, y\Big)\;.$$

Now, making the following change of coordinates in source

$$\begin{cases} x = X \\ y = Y \mp (t/2)\,Y^3 \end{cases}$$

we obtain the desired normal form. ∎

Remark 3.6. When $g(x,y) = y + x^3 + x\,y^2 + s\,y^3$, the space $T_e(g)$ contains the ideal $\langle 3\,x^2 + y^2\rangle$, which is an elliptic ideal of $C_{(x,y)}$. Hence, $\langle 3\,x^2 + y^2\rangle \supset \mathcal{M}^\infty_{(x,y)}$, (see [18]), and this implies that $\mathrm{cod}(f) = 2$. □

Proposition 3.7. *Let $f(x,y) = (g(x,y), y)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a divergent diagram $\mathcal{A}$-equivalent to $(x^3 \pm x\,y^k,\, y)$, $k \geq 3$, the origin being a singularity of transverse type. Then $f(x,y)$ is equivalent to*

(i) $$\Big(y + x^3 \pm x\,y^k + s\,y^{3k/2} + \lambda(x,y),\;\, y\Big)\,,$$

with $\lambda \in \mathcal{M}^\infty_{(x,y)}$, k even;

(ii) $$\Big(y + x^3 + x\,y^k + \lambda(x,y),\;\, y\Big)\,,$$

with $\lambda \in \mathcal{M}^\infty_{(x,y)}$, k odd.

Moreover, $\mathrm{cod}(f) \geq k$ *in case (i), and* $\mathrm{cod}(f) \geq k-1$ *in case (ii).* ∎

As in the previous case, the proof of this proposition will follow directly by computing the corresponding tangent spaces. The calculus are straightforward and we summarize the results in the next lemma.

Lemma 3.4. *Let $g(x,y) = y + x^3 \pm x\,y^k + \lambda(x,y)$, $k \geq 3$, $\lambda \in \mathcal{M}^{k+2}_{(x,y)}$.*

(i) $\Big\langle x^{r-s}\,y^s;\; s = 0, 1, ..., r\Big\rangle_{\mathbb{R}} \subset T_e(g) + \mathcal{M}^{r+1}_{(x,y)}$, *for every $r \geq k$, for all values of k, except when k is even and $r = 5\,k/2$.*

(ii) $\mathcal{M}^l_{(x,y)} \subset T_e(g) + \mathcal{M}^\infty_{(x,y)}$, *for* $l = k$, *if* k *is odd and* $l = 5\,k/2 + 1$, *if* k *is even.*

(iii) *If* k *is odd, then* $T_e(g) + \mathcal{M}^\infty_{(x,y)} \supset \theta_g - \left\langle x\,y^s,\ s = 0, 1, ..., k-2 \right\rangle_{\mathbb{R}}$.
If k *is even, then* $T_e(g) + \mathcal{M}^\infty_{(x,y)} \supset \theta_g - \left\langle x\,y^s,\ s = 0, 1, ..., k-2 \right\rangle_{\mathbb{R}} - \left\langle x^2\,y^{k/2},\ y^{3\,k/2},\ x^5\,y^{(k-2)/2},\ x^3\,y^{(3\,k-2)/2},\ x\,y^{(5\,k-2)/2} \right\rangle_{\mathbb{R}} + \left\langle 3\,x^2\,y^{k/2} \pm y^{3k/2},\ 3\,x^5\,y^{(k-2)/2} \pm x^3\,y^{(3k-2)/2},\ 3\,x^3\,y^{(3k-2)/2} \pm x\,y^{(5k-2)/2},\ \pm k\,x\,y^{(5k-2)/2} + y^{3k/2} \right\rangle_{\mathbb{R}}$. ∎

To complete the classification, we observe that divergent diagrams of corank 2, as map-germs to the plane and with respect to $\mathcal{A}$-classification, are adjacent to the cusp singularity (see [17]). Applying this to our case, it is easy to see that every divergent diagram of corank 2 has infinite codimension, since it is adjacent to the tangent cusp.

Remark 3.8. Dufour proved in [6] that there are no stable singular multigerms of divergent diagrams $f = (f_1, f_2)\colon (\mathbb{R}^2, S) \to (\mathbb{R}^2, 0)$, $S \geq 2$. One can see that there are no finite codimension singular multigerms of divergent diagrams, either. In fact, direct computations show that, even the case of transverse intersection of two folds, has infinite codimension. □

4 – Invariants for divergent diagrams

Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a finitely $\mathcal{A}$-determined map-germ and f_t be a stable perturbation of f. Then f_t has a finite number of cusps and double-folds, denoted by $c(f)$ and $d(f)$. Formulas to compute these numbers were determined in [12], [16] and [13]. When f has corank one, $f(x,y) = (g(x,y), y)$, we have

$$c(f) \;=\; \dim_{\mathbb{C}} \frac{\mathcal{O}_{(x,y)}}{\langle g_x, g_{xx} \rangle}$$

and

$$d(f) \;=\; \frac{1}{2}\,\dim_{\mathbb{C}} \frac{\mathcal{O}_{(x,\,x',\,y)}}{\left\langle h,\ h_x,\ (h_x - h_{x'})/(x - x') \right\rangle}\;,$$

where $\mathcal{O}_{(x,y)}$ and $\mathcal{O}_{(x,\,x',\,y)}$ denote the local ring of germs of holomorphic functions at the origin, in $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively, and $h(x, x', y) = \frac{g(x,y) - g(x',y)}{x - x'}$.

Now, given a finitely $\mathcal{A}$-determined real map-germ $f(x,y) = (g(x,y),y)$, and denoting by $f_{\mathbb{C}}$ its complexification, it follows that the numbers $c(f_{\mathbb{C}})$ and $d(f_{\mathbb{C}})$ are also invariants for the $\mathcal{A}$-classification of f.

Since every invariant for $\mathcal{A}$-classification is also an invariant for divergent diagrams, $c(f)$ and $d(f)$ are the first invariants we shall consider. However, according to [16], $d(f) = 0$ for all $\mathcal{A}$-finitely determined map-germs of the plane of multiplicity ≤ 3, and all our normal forms belong to one of the $\mathcal{K}$-orbits A_1, A_2 or A_3.

The classification of the stable divergent diagrams suggests new invariants: the numbers $b_1(f)$ and $b_2(f)$ of tangent folds with quadratic contact, appearing in a stable perturbation of the complexification of f, with respect to the horizontal and vertical axis, respectively. When $f(x,y) = (g(x,y),y)$, $b_2(f)$ is always zero, and $b_1(f)$ is equal to $\mu(g)$, the Milnor number of g.

We see in Table 2 that the invariants $\mathrm{cod}(f)$, $c(f)$, $b_1(f)$ distinguish all the germs in Table 1, except for the family 4. In the next proposition, we define an invariant for the family $(x^3 \pm x\,y^k + s\,y^{3k/2} + y,\, y)$, k even.

Type $f=(f_1,f_2)$	Formal Normal Form	formal cod(f)	c(f)	$b_1(f)$
2	$\big(x^2 + y + q(z),\, y\big)$	0	0	0
$2'$	$\big(x^2 \pm y^{k+2} + q(z),\, y\big)$ or $\big(x,\, \pm x^{k+2} + y^2 + q(z)\big)$	k	0	$k+1$
3	$\big(x^3 + x\,y + y + q(z),\, y\big)$	0	1	0
4	$\big(x^3 \pm x\,y^k + s\,y^{3k/2} + y + q(z),\, y\big)$ (k even)	k	k	0
5	$\big(x^3 \pm x\,y^k + y + q(z),\, y\big)$ (k odd, $k > 1$)	$k-1$	k	0

Table 2

Proposition 4.1. *Let f_s be as above.*

(**a**) *When $k = 0$ (mod 4), f_s and $f_{\bar{s}}$ are equivalent if and only if $s = \pm\bar{s}$.*

(**b**) *When $k \neq 0$ (mod 4), f_s and $f_{\bar{s}}$ are equivalent if and only if $s = \bar{s}$.*

Proof: Suppose that H and K are germs of diffeomorphisms in source and target such that $K \circ f_s = f_{\bar{s}} \circ H$, with the target diffeomorphism K of product type, that is, $K(u,v) = (K_1(u), K_2(v))$. Then, a direct calculation shows that these diffeomorphisms are in fact linear, and, more precisely: if $k = 0$ (mod 4), $(H,K) = (I_{\mathbb{R}^2}, I_{\mathbb{R}^2})$ or $(H,K) = (-I_{\mathbb{R}^2}, -I_{\mathbb{R}^2})$, where $I_{\mathbb{R}^2}$ denotes the identity map in $\mathbb{R}^2$; if $k \neq 0$ (mod 4), the only possibility is $(H,K) = (I_{\mathbb{R}^2}, I_{\mathbb{R}^2})$. ∎

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