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# ON MODULI OF REGULAR SURFACES WITH $K^2 = 8$ AND $p_g = 4$

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**Abstract:** Let $S$ be a surface of general type with not birational bicanonical map and that does not contain a pencil of genus 2 curves. If $K_S^2 = 8$, $p_g(S) = 4$ and $q(S) = 0$ then $S$ can be given as double cover of a quadric surface. We show that its moduli space is generically smooth of dimension 38, and single out an open subset. Note that for these surfaces $h^2(S, T_S)$ is not zero.

## 1 – Introduction

It is known that if $X$ is a surface of general type with a pencil of genus 2 curves then its bicanonical map is non birational (see [1]). On the other hand, there are also surfaces with non birational bicanonical map, which have no pencil of curves of genus 2. According with [2], these surfaces are said to be *special*. Under the assumption that $p_g \geq 4$, the classification of all special surfaces has been completed in [2]. There are three main types of such surfaces, while others of them can be obtained by specialization. Two of these types are classically known (see [1] and [3]), and also their moduli space has been studied (see [4]). Here we concern with the third type, discovered in [2]. These surfaces have the following invariants: $K^2 = 8$, $p_g = 4$ and $q = 0$.

We recall theorem (3.1) in [2].

**Theorem 1.1.** *If $S$ is a minimal regular surface of general type with $K^2 = 8$ and $p_g = 4$ which is special, then $S$ is one of the following types:*

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**(i)** *The canonical system* $K$ *has four distinct simple base points* $p_1, p_2, q_1, q_2$*. The canonical map* $\phi_{K_S}$ *is of degree* 2 *onto a smooth quadric* $Q$ *of* $\mathbb{P}^3$*. If* $p\colon \widetilde{S} \to S$ *is the blow-up of the points* $p_1, p_2, q_1, q_2$*, then there exists a morphism* $\varphi\colon \widetilde{S} \to Q \subset \mathbb{P}^3$ *such that* $\varphi = \phi_{K_S} \circ p$*. The morphism* $\varphi$ *is generically finite of degree 2, with branch curve* $B$ *on* $Q$ *of type* $B = \eta_1 + \eta_2 + \eta_1' + \eta_2' + B'$*, where* $\eta_1, \eta_1'$ *are two distinct lines of the same ruling of* $Q$*,* $\eta_2, \eta_2'$ *are two distinct lines of the other ruling,* $B'$ *is a curve of type* $(8, 8)$ *not containing* $\eta_i, \eta_i'$*, having 4-uple points at the intersection of the four lines, and no further essential singularity.*

**(ii)** *The canonical system* $|K_S|$ *has a fixed component which is an irreducible (-2)-curve* $Z$*. The linear system* $|K_S - Z|$ *has no fixed component but has two distinct simple base points. The canonical map* $\phi_{K_S}$ *has degree 2 onto a smooth quadric* $Q$ *of* $\mathbb{P}^3$*. If* $p\colon \widetilde{S} \to S$ *is the blow-up of the base points of* $|K_S - Z|$*, then there exists a morphism* $\varphi\colon \widetilde{S} \to Q$ *such that* $\varphi = \phi_{K_S} \circ p$*. The morphism* $\varphi$ *is generically finite of degree 2, with branch curve* $B$ *on* $Q$ *of type* $B = \eta + \eta' + B'$*, where* $\eta, \eta'$ *are two distinct lines of the same ruling of* $Q$*,* $B'$ *is a curve of type* $(8, 8)$ *not containing* $\eta, \eta'$*, having two* $[4, 4]$*-points at the intersection of the* $\eta, \eta'$ *with a line of the other ruling, and tangent lines* $\eta, \eta'$*, and no further essential singularity.*

**Remark 1.2.** Here, the essential singularities are the ones that affect the invariants of $S$.

The surfaces in theorem 1.1(ii) are specialization of the surfaces in theorem 1.1(i). We call *general* the latter surfaces and *particular* the former ones (see remark (3.10) in [2]). □

For the tangent bundle one has

$$\chi(T_S) \;=\; -10\,\chi(\mathcal{O}_S) + 2\,K_S^2 \;=\; -34\;. \tag{1}$$

We will prove the following:

**Theorem 1.3.** *The family* $\mathcal{F}$ *of regular surfaces with* $K^2 = 8$*,* $p_g = 4$ *with non trivial torsion and without a pencil of genus* 2 *curves described in theorem 1.1(i) corresponds to an open subset of its moduli space, which is irreducible, smooth of dimension* 38*.*

The prove is based on the geometric description of $S$ by means of the double map on the quadric surface $Q$.

### 1.1. Notations and set up

We recall the notations used in [2]: we consider $n = \eta_1 \cap \eta_2$, $n' = \eta_1' \cap \eta_2'$, $m = \eta_2 \cap \eta_1'$, $m' = \eta_1 \cap \eta_2'$, points on the quadric $Q$. We denote by $E_i$, $E_i'$ for $i = 1, 2$ the exceptional curves in $\widetilde{S}$ corresponding to the points $p_1$, $p_2$, $q_1$, $q_2$ of $S$ by the blow up $p$.

We introduce further notations. We denote by $\Gamma_1$ and $\Gamma_2$ the two pencils of lines on $Q$ to which $\eta_1$ and $\eta_2$ belong respectively. Let $bl : Y \to Q$ be the blow up of $Q$ on $n, n'$, $m, m'$, and denote by $E_n$, $E_{n'}$,$E_m$, $E_{m'}$ the exceptional curves corresponding to the points $n$, $n'$, $m$, $m'$. We write

$$E \,=\, E_n + E_{n'} + E_m + E_{m'} \ .$$

We mark with a bar the strict transforms of the divisors of $Q$ on $Y$.

We have the following commutative diagram (cf. the proof of theorem (3.1)($i$) in [2]):

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\ p\ } & S \\ \downarrow \psi & & \downarrow \phi_K \\ Y & \xrightarrow{\ bl\ } & Q \end{array} \tag{2}$$

The curves $E_1, E_2, E_1', E_2'$ are sent to the lines $\eta_1, \eta_2, \eta_1', \eta_2'$ respectively, by $\phi_K \circ p$.

Note that $\psi\colon \widetilde{S} \to Y$ is a $2\!:\!1$ morphism branched along a divisor $B_Y$ of $Y$. In fact, there are curves on $\widetilde{S}$, denoted by $\bar{N}, \bar{N}'$, $\bar{M}, \bar{M}'$ in [2], which are sent on $E_n$, $E_{n'}$,$E_m$, $E_{m'}$ respectively. By theorem 1.1 $B_Y$ belongs to the linear system

$$|10\,\bar{\Gamma}_1 + 10\,\bar{\Gamma}_2 - 6\,E| \,=\, \bar{\eta}_1 + \bar{\eta}_2 + \bar{\eta}_1' + \bar{\eta}_2' + |B_Y'| \ ,$$

where

$$B_Y' \,\in\, |8\,\bar{\Gamma}_1 + 8\,\bar{\Gamma}_2 - 4\,E| \ .$$

Since $Y$ and $\widetilde{S}$ are smooth and $\psi$ is finite, the branch locus $B_Y$ is smooth.

## 2 – The number of moduli of $S$

It is possible to compute the number of moduli of the surface $\widetilde{S}$ (and therefore of $S$) by applying the projection formula to the tangent sheaf:

$$h^i(\widetilde{S}, T_{\widetilde{S}}) \,=\, h^i\Big(Y, T_Y(-\log B_Y)\Big) + h^i\Big(Y, T_Y(-D)\Big) \,, \qquad i = 0, 1, 2 \ , \tag{3}$$

where $2D \sim B_Y$ (cf. [6]). Note that

$$D \in |5\,\bar\Gamma_1 + 5\,\bar\Gamma_2 - 3\,E|\ .$$

**Proposition 2.1.**

$$h^2\Big(Y, T_Y(-\log B_Y)\Big) = 0\ .$$

**Proof:** Consider the exact sequence

$$0 \to T_Y(-\log B_Y) \to T_Y \longrightarrow \mathcal{O}_{B_Y}(B_Y) \to 0\ . \tag{4}$$

The curve $B_Y$ is the disjoint union of 5 components: there are 4 rational curves composing $E$, plus the curve $B'_Y$, of genus 43, which can be easily computed by adjunction formula. By Serre duality, $H^1(B_Y, \mathcal{O}_{B_Y}(B_Y)) = 0$. Moreover $H^2(Y, T_Y) = H^2(Q, T_Q) = 0$. Hence, the long exact sequence of cohomology coming from (4) implies that $H^2(Y, T_Y(-\log B_Y)) \cong H^2(Y, T_Y) = 0$. ∎

**Lemma 2.2.**

$$H^k\Big(Y, bl^* T_Q(-D)\Big) = 0, \qquad \textit{for}\ \ k = 0, 2\ ,$$
$$H^1\Big(Y, bl^* T_Q(-D)\Big) \cong \mathbb{C}^8\ .$$

**Proof:** One has

$$H^k\Big(Y, bl^* T_Q(-D)\Big) =$$
$$= H^k\Big(Y, \mathcal{O}_Y(-3\,\bar\Gamma_1 - 5\,\bar\Gamma_2 - 3\,E)\Big) \oplus H^k\Big(Y, \mathcal{O}_Y(-5\,\bar\Gamma_1 - 3\,\bar\Gamma_2 - 3\,E)\Big)\ .$$

In fact $T_Q = \mathcal{O}_Q(2\Gamma_1) \oplus \mathcal{O}_Q(2\Gamma_2)$. The rest follows from Riemann–Roch formula. ∎

**Proposition 2.3.** $h^1(S, T_S) \le 38$.

**Proof:** Since $h^0(S, T_S) = 0$, being $S$ of general type, then $h^1(S, T_S) = h^2(S, T_S) - \chi(T_S) = 34 + h^2(S, T_S)$, by (1). The proposition follows once we prove that $h^2(S, T_S) \le 4$. It is sufficient to verify that $h^2(\tilde S, T_{\tilde S}) \le 4$. In fact, it is $h^2(S, T_S) = h^2(\widetilde S, T_{\widetilde S})$, see for instance [5].

Consider now the exact sequence

$$0 \to T_Y(-D) \to bl^* T_Q(-D) \to N^*_{E/Y}(-D) \to 0\ . \tag{5}$$

Since $N^*_{E/Y}(-D) = \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 4}$, we get

$$H^0\Big(Y, N^*_{E/Y}(-D)\Big) = 0 \quad \text{and} \quad H^1\Big(Y, N^*_{E/Y}(-D)\Big) = \mathbb{C}^4 \ .$$

From (5) and lemma 2.2 one has the following exact sequence:

$$0 \to H^1\Big(Y, T_Y(-D)\Big) \to \mathbb{C}^8 \to \mathbb{C}^4 \to H^2\Big(Y, T_Y(-D)\Big) \to 0 \ .$$

In particular, $h^2(Y, T_Y(-D)) \le 4$. From (3) and proposition 2.1 one finally has:

$$h^2(\tilde{S}, T_{\tilde{S}}) \ = \ h^2\Big(Y, T_Y(-\log B_Y)\Big) + h^2\Big(Y, T_Y(-D)\Big) \ \le \ 4 \ . \ \blacksquare$$

### 2.1. Proof of Theorem 1.3

The irreducibility has been proved in [2].

Consider the family $\mathcal{F}$ of surfaces as in theorem 1.1. It is sufficient to show that $\dim \mathcal{F} = h^1(S, T_S) = 38$. Since the general surface $S$ of $\mathcal{F}$ is the double cover of a nonsingular quadric $Q$ of $\mathbb{P}^3$ branched on a divisor $B$, we can compute the dimension $\dim \mathcal{F}$ by computing the dimension of the linear system $\Sigma(B)$ of the divisors $B$. We recall that $B = \eta_1 + \eta_2 + \eta_1' + \eta_2' + B'$, where $\eta_1 + \eta_1'$ and $\eta_2 + \eta_2'$ are lines of the same pencil on the quadric, $B'$ belongs to the sublinear system $\Sigma(B')$ cut on $Q$ by the surfaces of degree 8, having quadruple points at the 4 intersection points of the 4 lines. Thus $\dim \Sigma(B) = 4 + (\dim \Sigma(B'))$. Since each of the quadruple points gives 10 conditions then

$$\dim \Sigma(B') \ = \ h^0(Q, \mathcal{O}_Q(8)) - 40 - 1 \ = \ 40 \ .$$

Hence

$$\dim \mathcal{F} \ = \ \dim \Sigma(B) - \dim \mathrm{Aut}\,(Q) \ = \ 40 + 4 - 6 \ = \ 38 \ .$$

Therefore $h^1(S, T_S) \ge \dim \mathcal{F} = 38$. By proposition 2.3, the equality holds. $\blacksquare$

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