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ON 3D SLIGHTLY COMPRESSIBLE EULER EQUATIONS

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Abstract: This paper is concerned with the Euler equations of a barotropic inviscid compressible fluid in the three dimensional space $\mathbb{R}^3$.

Following the method of decomposition in [4], [5], we show the existence of a smooth compressible flow on an arbitrary time interval $[0,T]$ for any Mach number sufficiently small and almost constant initial densities, when the incompressible limit flow is assumed to exist up to T as well.

The life span $O\left(1/\epsilon^{\mu-1}\right)$ for the compressible solution is obtained assuming also that the incompressible part of the solution itself has a life span of order $O(1/\epsilon^{\mu-1})$ and is $O(\epsilon^{\mu-1})$ for suitable $\mu>1$.

1 – Introduction

In this paper we study smooth solutions to the Euler equations of a barotropic inviscid compressible fluid in $\mathbb{R}^3$ for small Mach numbers.

Setting $Q_T := (0,T)\times\mathbb{R}^3$, in a suitable nondimensional form the compressible Euler equations read as the following initial value problem (ivp):

$$(1)\qquad \begin{cases} \partial_t\rho^\epsilon + \nabla\cdot(\rho^\epsilon v^\epsilon) \;=\; 0\;, \\ \rho^\epsilon\Big(\partial_t v^\epsilon + (v^\epsilon\cdot\nabla)v^\epsilon\Big) + \frac{1}{\epsilon^2}\nabla p^\epsilon \;=\; 0 \qquad \text{in } Q_T\;, \\ \rho^\epsilon(0,x) = \rho_0^\epsilon(x)\;, \\ v^\epsilon(0,x) = v_0^\epsilon(x) \qquad \text{in } \mathbb{R}^3\;. \end{cases}$$

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The density $\rho^\epsilon = \rho^\epsilon(t,x)$, the velocity $v^\epsilon = v^\epsilon(t,x) = (v^{\epsilon,1}(t,x), v^{\epsilon,2}(t,x), v^{\epsilon,3}(t,x))$ and the pressure $p^\epsilon = p^\epsilon(t,x)$ are unknown functions of the time t and the space variables $x = (x_1, x_2, x_3)$ and $\epsilon > 0$ is essentially the Mach number. We assume that the density and the pressure are related by the equation of state $p^\epsilon = p(\rho^\epsilon) = (\rho^\epsilon)^\gamma$ for $\gamma > 1$; with standard notations we set $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$, $\nabla = (\partial_1, \partial_2, \partial_3)$ and $v \cdot \nabla = v^1\partial_1 + v^2\partial_2 + v^3\partial_3$.

Solutions $(\rho^\epsilon, v^\epsilon)$ to (1) are sought as a perturbation from an equilibrium state $(\bar\rho, 0)$, where $\bar\rho > 0$.

If $\rho_0^\epsilon(x) \to \bar\rho$, as $\epsilon \to 0$, then one expects that $\rho^\epsilon \to \bar\rho$ and $v^\epsilon \to w$, where w is a solution to the Euler equations of an incompressible ideal fluid flow; namely there exists $\pi = \pi(t,x)$ such that (π, w) is a solution to the 4×4 system

$$\begin{cases} \nabla \cdot w = 0 \ , \\ \partial_t w + (w \cdot \nabla)w + \nabla \pi \ = \ 0 \qquad \text{in } Q_T \ , \\ w(0,x) = w_0(x) \qquad \text{in } \mathbb{R}^3 \ , \end{cases} \tag{2}$$

where $\nabla \cdot w_0 = 0$ in $\mathbb{R}^3$.

Solutions of (1) can be viewed as the nonlinear coupling of irrotational solutions to (1), with as initial data the initial density and the gradient part of the initial velocity, and incompressible solutions to (2), with as initial data the divergence-free part of the initial velocity. In a normal mode analysis, the study of the interaction between such irrotational and incompressible components of the solutions to (1) corresponds to studying the nonlinear interaction between, respectively, the genuinely nonlinear eigenvalues $v \cdot \nu \pm c(\rho)$ ($c(\rho)$ denotes the sound velocity, ν is a unit vector) and the double linearly degenerate eigenvalue $v \cdot \nu$. The analysis of the present paper, through the study of the life span of solutions, shows that this nonlinear coupling is rather weak.

In three dimensions the global solvability in time of (2) is an outstanding open problem, even if many numerical computations appear to exhibit blow up in finite time, due to concentration of vorticity. However, if the solution of (2) exists up to an arbitrary time T, in view of the above incompressible limit, it is natural to expect that also the solution to (1) exists up to T, for all Mach numbers sufficiently small and almost constant initial densities.

After reformulating the problem (1) in an appropriate functional setting, in the next sections 2, 3 such a result is proved under suitable assumptions on the size of the initial data $(\rho_0^\epsilon, v_0^\epsilon)$. Following [4], [5], we decompose the solution $(\rho^\epsilon, v^\epsilon)$ as the sum of irrotational flow, incompressible flow and the remainder giving the nonlinear interaction between the first two parts. By our assumption on the incompressible component and the "almost" global existence of the irrotational flow shown by Sideris [6], the solvability of (1) follows by showing the existence of the nonlinear interaction up to T. The result follows by a combination of the time decay property of the irrotational part and energy estimates.

By the same techniques, a life span of type $O(1/\epsilon^{\mu-1})$, with a suitable $\mu > 1$, is obtained for a compressible flow under stronger hypotheses; namely we assume that the solution of (2) with initial datum given by the incompressible part of the velocity v_0^ϵ exists on the time interval $[0, A/\epsilon^{\mu-1}]$, $A > 0$, and is $O(\epsilon^{\mu-1})$.

In order to simplify the forthcoming computations, it is useful to rewrite the system (1) in an appropriate symmetric hyperbolic form. Namely, following [4], we make the change of variable

$$g^\epsilon(t,x) \,=\, \log\left(\frac{\rho^\epsilon(t,x)}{\bar\rho}\right).$$

Therefore the system (1) reduces to

$$\begin{cases} \partial_t g^\epsilon + v^\epsilon\cdot\nabla g^\epsilon + \nabla\cdot v^\epsilon \,=\, 0\,, \\ \frac{1}{h(g^\epsilon)}\Big(\partial_t v^\epsilon + (v^\epsilon\cdot\nabla)v^\epsilon\Big) + \frac{1}{\epsilon^2}\nabla g^\epsilon \,=\, 0 \qquad \text{in } Q_T\,, \\ g^\epsilon(0,x) = g_0^\epsilon(x)\,, \\ v^\epsilon(0,x) = v_0^\epsilon(x) \qquad \text{in } \mathbb{R}^3\,, \end{cases} \tag{3}$$

where

$$g_0^\epsilon(x) \,=\, \log\left(\frac{\rho_0^\epsilon(x)}{\bar\rho}\right)$$

and $h(s) := p'(\bar\rho e^s) = \gamma(\bar\rho e^s)^{\gamma-1}$. Without loss of generality we assume $h(0) = p'(\bar\rho) = 1$.

It is also convenient to rescale the variables by

$$g(t,x) = g^\epsilon(\epsilon t, x)\,, \qquad v(t,x) = \epsilon v^\epsilon(\epsilon t, x)\,. \tag{4}$$

The system (3) becomes then

$$\begin{cases} \partial_t g + v\cdot\nabla g + \nabla\cdot v = 0\,, \\ \frac{1}{h(g)}\Big(\partial_t v + (v\cdot\nabla)v\Big) + \nabla g = 0 \qquad \text{in } Q_{T/\epsilon}\,, \\ g(0,x) = g_0(x)\,, \\ v(0,x) = v_0(x) \qquad \text{in } \mathbb{R}^3\,, \end{cases} \tag{5}$$

where

$$g_0 = g_0^\epsilon \quad \text{ and } \quad v_0 = \epsilon\, v_0^\epsilon\,, \tag{6}$$

or, in the vector form,

$$(7) \qquad \begin{cases} A_0(u)\,\partial_t u + \sum\limits_{j=1}^{3} A_j(u)\,\partial_j u \;=\; 0 \qquad \text{in } Q_{T/\epsilon}\ , \\ u(x,0) = u_0(x) \qquad \text{in } \mathbb{R}^3\ , \end{cases}$$

where $u =: \begin{pmatrix} g \\ v \end{pmatrix}$, $u_0 := \begin{pmatrix} g_0 \\ v_0 \end{pmatrix}$ and

$$(8) \qquad A_0(u) := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{h(g)}\mathbf{I} \end{pmatrix}\ , \qquad A_j(u) := \begin{pmatrix} v^j & e_j \\ {}^t e_j & \frac{v^j}{h(g)}\mathbf{I} \end{pmatrix} \quad j = 1,2,3\ .$$

$\mathbf{I}$ denotes the 3×3 identity matrix and $e_1 := (1,0,0),\ e_2 := (0,1,0),\ e_3 := (0,0,1)$.

Let us now fix the functional setting to be used through the following.

Given a positive integer m, we write $H^m(\mathbb{R}^3)$ (H^m for shortness) for the Sobolev space of order m and $\|.\|_m$ for the related norm; in particular $\|.\| := \|.\|_0$ is the norm in $L^2(\mathbb{R}^3)$, while $|.|_p$ denotes the norm in $L^p(\mathbb{R}^3)$ for $1 \le p \le \infty$, $p \ne 2$.

Moreover for $T > 0$ we define the space $X^m(T)$ by

$$X^m(T) \;=\; \bigcap_{k=0}^{m-1} C^k([0,T]; H^{m-k})\ ,$$

where $C^k([0,T];\mathcal{B})$ denotes the space of the k-times continuously differentiable functions on $[0,T]$ taking values in a Banach space $\mathcal{B}$. $X^m(T)$ is, in its turn, a Banach space with respect to the norm

$$\|u\|_{X^m(T)} \;=\; \sup_{t\in[0,T]} |||u(t)|||_m\ ,$$

where $|||u(t)|||_m^2 = \sum_{k=0}^{m-1} \|\partial_t^k u(t)\|_{m-k}^2$.

According to [6], let us denote by Γ the family of operators $\Gamma = \{\partial_t, \partial_j, \Omega_j, L_j, S\}_{j=1}^3 = \{\Gamma_i\}_{i=0}^{10}$, where Ω_j, $j=1,2,3$, are the generators of proper rotations, $S := \mathbf{I}(t\,\partial_t + \sum_{j=1}^3 x_j\partial_j)$, is the *scaling operator*, while $L_j := \mathbf{I}\ell_j - B_j$, $j=1,2,3$, are the generators of *Lorentz rotations* (*boosts*); the matrices B_j are defined by

$$B_j := \begin{pmatrix} 0 & e_j \\ {}^t e_j & \mathbf{0} \end{pmatrix}, \qquad j = 1,2,3$$

where $\mathbf{0}$ denotes the 3×3 zero matrix and $\ell_j = t\partial_j + x_j\partial_t$, $j=1,2,3$.

For every multi-index $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{10})$ we set $\Gamma^\alpha = \Gamma_0^{\alpha_0} ... \Gamma_{10}^{\alpha_{10}}$; moreover given a positive integer m and a real $T > 0$ we define $X_\Gamma^m(T)$ to be the space of all the functions $w(t) = w(t,x)$ such that $\|w\|_{X_\Gamma^m(T)} := \sup_{t\in[0,T]} \hat{E}_m[w(t)]$ is finite, where:

$$\hat{E}_m[w(t)] = \sum_{|\alpha|\le m} \|\Gamma^\alpha w(t,.)\| \ .$$

Let us remark that the space $X_\Gamma^m(T)$ is included into $X^m(T)$, as the set Γ contains the derivatives ∂_t, ∂_j for $j = 1, 2, 3$.

It is well-known that $L^2(\mathbb{R}^3)$ may be represented as the orthogonal sum $L^2(\mathbb{R}^3) = G \oplus H$, where the spaces G and H are defined as follows:

$$H = \left\{ u \in L^2(\mathbb{R}^3);\ \nabla \cdot u = 0 \text{ in } \mathbb{R}^3 \right\} , \qquad G = \left\{ \nabla\psi \in L^2(\mathbb{R}^3);\ \psi \in H^1_{\text{loc}}(\mathbb{R}^3) \right\} .$$

Writing P for the projection of $L^2(\mathbb{R}^3)$ onto H and $Q = I - P$, we have $P \in \mathcal{L}(H^m, H^m)$, for every integer $m \ge 0$.

The derivatives $\partial_t^k g_0^\epsilon$, $\partial_t^k v_0^\epsilon$ of the initial data $(g_0^\epsilon, v_0^\epsilon)$ are recursively defined for every $k \ge 1$ formally differentiating equations (3) $k-1$ times in t, solving the resulting equations for $\partial_t^k g^\epsilon$, $\partial_t^k v^\epsilon$ and evaluating them at $t = 0$ in order to get an expression involving only the initial data $(g_0^\epsilon, v_0^\epsilon)$ and their derivatives in t up to the order $k - 1$. Analogously, we define $\partial_t^k g_0$, $\partial_t^k v_0$ starting from (5) and the initial data (g_0, v_0).

Since $\Gamma^\alpha w(t,x)$ consists of a finite sum of derivatives in (t,x) of the components $w^j(t,x)$, $1 \le j \le 3$, multiplied by monomials in (t,x), we can consequently define $\hat{E}_k[(g_0, v_0)]$ for $k \ge 1$.

After making the change of variable

$$\xi(t,x) = \frac{2}{\gamma - 1} \left[\exp\left(\frac{\gamma - 1}{2}\ g(t,x) \right) - 1 \right] ,$$

Sideris proved in [6] (cf. Theorem 2) the existence of a solution to the ivp (5), written in terms of the variables (ξ, v), in the frame of the functional spaces $X_\Gamma^m(T)$ and for irrotational initial data (ξ_0, v_0); moreover a lower bound of type $\exp(C_0/\epsilon) - 1$ is obtained for the life span T_ϵ of the solution (ξ, v).

Relying on the result of Sideris, in the next Section 3 we will show the following theorem about the existence of the solution to the ivp (3) on arbitrary time intervals, for all sufficiently small Mach numbers and almost constant densities. Observe that there is no restriction on the size of the initial velocity.

Theorem 1.1. *Let* $(g_0^\epsilon, v_0^\epsilon)$ *satisfy:*

$$\hat{E}_5[\xi_0^\epsilon] \le C_1\,\epsilon\,, \qquad \epsilon > 0\ , \tag{9}$$

where $\xi_0^\epsilon = \frac{2}{\gamma-1}\left[\exp(\frac{\gamma-1}{2}\,g_0^\epsilon) - 1\right]$ *and*

$$\hat{E}_5[Qv_0^\epsilon] + \|Pv_0^\epsilon\|_4 \le C_1\,, \qquad \epsilon > 0\ , \tag{10}$$

with a positive constant C_1 *independent of* ϵ.

We assume moreover that the solution $(\pi^\epsilon, w^\epsilon)$ *of (2) with initial datum* $w_0(x) = Pv_0^\epsilon$ *exists up to a time* $T > 0$ *in* $X^4(T)$ *and fulfills:*

$$\|(\pi^\epsilon, w^\epsilon)\|_{X^4(T)} \le \widehat{C}\,, \qquad \epsilon > 0\ , \tag{11}$$

for some $\widehat{C} > 0$ *independent of* ϵ.

Then there exist two constants $\epsilon_0, C_2 > 0$ *such that for every* $0 < \epsilon < \epsilon_0$ *the ivp (3) with initial data* $(g_0^\epsilon, v_0^\epsilon)$ *has a unique solution* (g^ϵ, v^ϵ) *in* $X^3(T)$ *and*

$$\|g^\epsilon\|_{X^3(T)} \le C_2\,\epsilon\,, \qquad \|v^\epsilon\|_{X^3(T)} \le C_2\ . \tag{12}$$

Theorem 1.1 gives the existence of compressible fluid flow (g^ϵ, v^ϵ) on the time interval $[0,T]$, with an arbitrary $T > 0$ and $\epsilon > 0$ small, by assuming that the solution $(\pi^\epsilon, w^\epsilon)$ of the incompressible problem (2) exists up to the same time T and $\|(\pi^\epsilon, w^\epsilon)\|_{X^4(T)} = O(1)$ as $\epsilon \to 0$.

We also obtain for (g^ϵ, v^ϵ) a life span of type $O\left(1/\epsilon^{\mu-1}\right)$, with a suitable $\mu > 1$, under the hypothesis that the incompressible part $(\pi^\epsilon, w^\epsilon)$ of (g^ϵ, v^ϵ) has a life span $A/\epsilon^{\mu-1}$ for a given $A > 0$ and is small for small ϵ, in the sense that $\|(\pi^\epsilon, w^\epsilon)\|_{X^4(A/\epsilon^{\mu-1})} = O(\epsilon^{\mu-1}),\ \epsilon \to 0$.

Proposition 1.1. *Let* $(g_0^\epsilon, v_0^\epsilon)$ *satisfy*

$$\hat{E}_5\,[\xi_0^\epsilon] \le C_1\,\epsilon, \quad \hat{E}_5[Qv_0^\epsilon] \le C_1, \quad \|Pv_0^\epsilon\|_4 \le C_1\,\epsilon^{\mu-1}, \qquad \epsilon > 0\ , \tag{13}$$

where $1 < \mu < \frac{6}{5}$ *and* C_1 *is a given positive constant.*

Let us assume moreover that there exist constants $A, C_2 > 0$ *such that the solution* $(\pi^\epsilon, w^\epsilon)$ *of (2) with initial datum* Pv_0^ϵ *exists up to time* $A/\epsilon^{\mu-1}$ *in* $X^4(A/\epsilon^{\mu-1})$ *and fulfills*

$$\|(\pi^\epsilon, w^\epsilon)\|_{X^4(A/\epsilon^{\mu-1})} \le C_2\,\epsilon^{\mu-1}\ , \tag{14}$$

for any $\epsilon > 0$ *sufficiently small.*

Then we may find $0 < \epsilon_0 < 1$ *and* $0 < A' \le A$ *such that the solution* $u^\epsilon = (g^\epsilon, v^\epsilon)$ *to (3) with initial data* $(g_0^\epsilon, v_0^\epsilon)$ *exists up to time* $A'/\epsilon^{\mu-1}$ *in* $X^3(A'/\epsilon^{\mu-1})$ *for all* $0 < \epsilon < \epsilon_0$ *and satisfies*

$$\|g^\epsilon\|_{X^3(A'/\epsilon^{\mu-1})} \le C_3\,\epsilon\,, \qquad \|v^\epsilon\|_{X^3(A'/\epsilon^{\mu-1})} \le C_3\;, \tag{15}$$

with a suitable $C_3 > 0$ *independent of* ϵ.

The paper is organized as follows: in Section 2 we give an enery estimate and some inequalities which will be used in the following; Theorem 1.1 is proved in Section 3; Proposition 1.1 is proved in Section 4.

2 – An energy estimate and some useful inequalities

Let w satisfy the symmetric hyperbolic system

$$A_0(u)\,\partial_t w + \sum_{j=1}^{3} A_j(u)\,\partial_j w \;=\; \mathcal{F} \quad \text{ in } \; Q_T \tag{16}$$

where, for $u = (g, v)$, the matrices $A_h(u)$, $h = 0,1,2,3$, are defined by (8) and $\mathcal{F} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix}$ has sufficiently smooth components $\mathcal{F}_1 = \mathcal{F}_1(t,x)$, $\mathcal{F}_2 = \mathcal{F}_2(t,x)$.

We define the energy $E(t)$ by

$$E(t) = \langle A_0(u)w,\, w\rangle\;,$$

where $\langle f, g\rangle = \int_{\mathbb{R}^3} f(x)\,g(x)\,dx$ is the inner product in $L^2(\mathbb{R}^3)$.

Provided $u = (g, v)$ is sufficiently smooth, by a standard argument one can prove that $E(t)$ solves

$$\frac{d}{dt}E(t) \;=\; \langle \mathrm{div} A\, w, w\rangle + 2\,\langle \mathcal{F}, w\rangle\;, \tag{17}$$

where $\mathrm{div} A := \partial_t A_0 + \sum_{j=1}^3 \partial_j A_j$.

Let us assume now that $u = (g, v)$ satisfies

$$0 < m^{-1} \le h(g) \le M \quad \text{ in } \; Q_T\;, \tag{18}$$

for some constants m, M; notice that $h(0) = 1$ yields $m, M \ge 1$.

Applying the Cauchy–Schwarz inequality to $\langle \mathrm{div} A\, w, w\rangle$, $\langle \mathcal{F}, w\rangle$ in (17) and integrating over $[0,t]$, for any $0<t\le T$, yield

$$E(t) \le E(0) + \int_0^t |\mathrm{div} A(s)|_\infty \, \|w(s)\|^2 \, ds + 2 \max_{s\in[0,t]} \|w(s)\| \int_0^t \|\mathcal{F}(s)\| \, ds \; ;$$

hence, using $E(t) \ge \frac{1}{M}\|w(t)\|^2$ and $E(0)\le m\|w(0)\|^2$, Gronwall's inequality gives

(19)
$$\|w(t)\| \le \left(Mm\|w(0)\| + 2M\int_0^t \|\mathcal{F}(s)\| \, ds \right) \exp\left(M\int_0^t |\mathrm{div} A(s)|_\infty \, ds \right), \quad 0\le t\le T\,.$$

In the next section, we will specialize the vector $\mathcal{F}$ in the right-hand side of (16) and need to estimate $\|\mathcal{F}\|$ by means of the L^2-norm of $u=(g,v)$ and a certain number of its derivatives. At this purpose we will make use of some interpolation inequalities coming from a general interpolation formula due to Nirenberg (cf. [2]). For the sake of completeness the needed inequalities are listed below.

Let us assume $u=(g,v)$ to be a sufficiently smooth solution of (5). From

$$\begin{aligned} &|\nabla u|_4 \le C\,|u|_\infty^{\frac12}\,\|\nabla u\|_1^{\frac12}\,, \qquad |\nabla u|_8 \le C\,|u|_\infty^{\frac34}\,\|\nabla u\|_3^{\frac14}\,, \\ &|D^2 u|_4 \le C\,|u|_\infty^{\frac12}\,\|\nabla u\|_3^{\frac12} \end{aligned} \tag{20}$$

and (5) we obtain

$$\begin{aligned} |\partial_t u|_4 &\le K(1+|u|_\infty)\,|u|_\infty^{\frac12}\,\|\nabla u\|_1^{\frac12}\,, \\ |\partial_t u|_8 &\le K(1+|u|_\infty)\,|u|_\infty^{\frac34}\,\|\nabla u\|_3^{\frac14}\;; \end{aligned} \tag{21}$$

hereafter C is a suitable positive constant independent of u and K denotes different positive constants which may depend boundedly on $h(.)$ and its derivatives up to the order 4.

From the differentiation of (5), (20) and (21) give also

$$\begin{aligned} |\nabla\partial_t u|_4 &\le K(1+|u|_\infty)\,|u|_\infty^{\frac12}\,\|\nabla u\|_3^{\frac12}\,, \\ |\partial_t^2 u|_4 &\le K(1+|u|_\infty)^2\,|u|_\infty^{\frac12}\,\|\nabla u\|_3^{\frac12}\,. \end{aligned} \tag{22}$$

Similarly as before, from

$$\begin{aligned} |D^3 u|_{\frac83} &\le C\,|u|_\infty^{\frac14}\,\|\nabla u\|_3^{\frac34}\,, \\ |D^4 u|_{\frac52} &\le C\,|u|_\infty^{\frac15}\,\|\nabla u\|_4^{\frac45} \end{aligned} \tag{23}$$

and differentiation of (5) we get

$$(24)\qquad \begin{aligned} |D^{2-k}\partial_t^{k+1}u|_{\frac{8}{3}} \;&\le\; K(1+|u|_\infty)^{k+2}\,|u|_\infty^{\frac14}\,\|\nabla u\|_3^{\frac34}\,, \qquad k=0,1,2\;,\\ |D^{4-k}\partial_t^{k}u|_{\frac{5}{2}} \;&\le\; K(1+|u|_\infty)^{k+2}\,|u|_\infty^{\frac15}\,\|\nabla u\|_4^{\frac45}\,, \qquad k=1,...,4\;. \end{aligned}$$

Lastly from

$$(25)\qquad |\nabla u|_\infty \;\le\; C\,|u|_\infty^{\frac57}\,\|\nabla u\|_4^{\frac27}$$

and (5) we derive

$$(26)\qquad |\partial_t u|_\infty \le\; K(1+|u|_\infty)\,|u|_\infty^{\frac57}\,\|\nabla u\|_4^{\frac27}\;.$$

3 – Proof of Theorem 1.1

We argue on the solution $u=(g,v)$ of the rescaled problem (5) or (7) given by formula (4).

Hereafter it will be convenient to adopt the standard multi-index notation; namely for $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$ we set $\partial^\alpha := \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

For any multi-indices α,β later on we will also write

$$\beta\le\alpha \quad\text{if}\ \beta_j\le\alpha_j\ \text{for}\ j=0,1,2,3\;;\qquad \beta<\alpha\quad\text{if}\ \beta\le\alpha\ \text{and}\ \beta\ne\alpha\;;$$

$$\alpha! \;=\; \alpha_0!\,\alpha_1!\,\alpha_2!\,\alpha_3!\;.$$

For $\beta\le\alpha$, $\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}=\binom{\alpha_0}{\beta_0}\binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\binom{\alpha_3}{\beta_3}$.

Given $(g_0^\epsilon,v_0^\epsilon)$ as in (9), (10) we define (g_0,v_0) by (6) and

$$\xi_0 \;=\; \tfrac{2}{\gamma-1}\Big[\exp\Big(\tfrac{\gamma-1}{2}g_0\Big)-1\Big]\;.$$

We easily show that

$$\hat E_5\,[(\xi_0,Qv_0)]\le C\,\epsilon\,,\qquad \epsilon>0\;,$$

for a suitable constant C. Let us consider the problem

$$(27)\qquad \begin{cases} \partial_t\xi_1+\nabla\cdot v_1+v_1\cdot\nabla\xi_1+\frac{\gamma-1}{2}\,\xi_1\,\nabla\cdot v_1 \;=\;0\;,\\ \partial_t v_1+\nabla\xi_1+v_1\cdot\nabla v_1+\frac{\gamma-1}{2}\,\xi_1\,\nabla\xi_1 \;=\;0 \qquad \text{in } Q_T\;,\\ \xi_1(0,x)=\xi_0(x)\;,\\ v_1(0,x)=Qv_0(x) \qquad \text{in } \mathbb{R}^3\;. \end{cases}$$

From Theorem 2 in [6] we may find two positive constants C_0, C_1' such that (ξ_1, v_1) exists up to a time $T_\epsilon \geq \exp(C_0/\epsilon) - 1$ in the space $X^5_\Gamma(T_\epsilon)$ and satisfies

$$\|(\xi_1, v_1)\|_{X^5_\Gamma(T_\epsilon)} \leq C_1' \epsilon\,, \qquad \epsilon > 0\;. \tag{28}$$

Due to a generalized Sobolev inequality by Klainerman (see [7]), the following decay estimate is also achieved:

$$|(\xi_1(t), v_1(t))|_\infty \leq C_1' \,\epsilon (1+t)^{-1}\,, \qquad 0 \leq t \leq T_\epsilon, \quad \epsilon > 0\;. \tag{29}$$

Let us define g_1 by the formula

$$g_1(t,x) \;=\; \frac{2}{\gamma - 1}\,\log\Big(\frac{\gamma-1}{2}\,\xi_1(t,x) + 1\Big)\,. \tag{30}$$

Then $u_1 = (g_1, v_1)$ is the solution to (5) with irrotational initial data (g_0, Qv_0). u_1 is also irrotational since $v_1 = Qv_1$.

Using formula (30) and the Sobolev embedding theorem it is easy to prove that (28), (29) yield for $u_1 = (g_1, v_1)$

$$\|u_1\|_{X^5(T_\epsilon)} \leq C_1' \,\epsilon\,, \qquad \epsilon > 0 \tag{31}$$

and

$$|u_1(t)|_\infty \leq C_1' \,\epsilon\,(1+t)^{-1}\,, \qquad 0 \leq t \leq T_\epsilon, \quad \epsilon > 0\;. \tag{32}$$

On the other hand by (31), (32) and some of the interpolation inequalities (20)-(26) we obtain:

$$\begin{aligned}\sum_{|\alpha|=1}\Big(|\partial^\alpha u_1|_4 + |\partial^\alpha u_1|_8 + |\partial^\alpha u_1|_\infty\Big) + \sum_{|\alpha|=2}|\partial^\alpha u_1|_4 + \sum_{|\alpha|=3}|\partial^\alpha u_1|_{\frac{8}{3}} \;+& \\ + \sum_{|\alpha|=4}|\partial^\alpha u_1|_{\frac{5}{2}} \;\leq\; C_1'' \,\epsilon\,(1+t)^{-\delta}&\,,\end{aligned} \tag{33}$$

for $\delta = \frac{1}{5}$, $0 \leq t \leq T_\epsilon$, $\epsilon > 0$ and some $C_1'' > 0$ independent of ϵ.

For a given $T > 0$, let $0 < \epsilon_0 < 1$ be such that $\exp(C_0/\epsilon) - 1 \geq T/\epsilon$ as $0 < \epsilon < \epsilon_0$.

Let $(\pi^\epsilon, w^\epsilon)$ be the solution of (2) as in (11), with initial datum $w^\epsilon(0,x) = Pv_0^\epsilon(x)$.

Let us define

$$\pi(t,x) = \epsilon^2\,\pi^\epsilon(\epsilon t, x)\,, \qquad w(t,x) = \epsilon\, w^\epsilon(\epsilon t, x)\;.$$

Then (π, w) solves (2) in the time interval $[0, T/\epsilon]$ with initial datum $w(0,x) = \epsilon P v_0^\epsilon(x)$. From (11) it follows that

$$\sum_{\substack{|\alpha|\le 4\\ \alpha_0=k}} \|\partial^\alpha \pi(t)\| \le \widehat{C}\,\epsilon^{k+2}\,, \qquad \sum_{\substack{|\alpha|\le 4\\ \alpha_0=k}} \|\partial^\alpha w(t)\| \le \widehat{C}\,\epsilon^{k+1}, \qquad k=0,...,3 \tag{34}$$

for every $0 \le t \le T/\epsilon$.

Following [4] and [5] we seek a solution of (5) in the form

$$g = g_1 + \pi + g_2\,, \qquad v = v_1 + w + v_2\ , \tag{35}$$

for the previously defined (g_1, v_1) and (π, w).

In order that $u = (g, v)$ solves (7), the remainder $u_2 = (g_2, v_2)$ should satisfy the following symmetric hyperbolic system

$$\begin{cases} A_0(u)\,\partial_t u_2 + \sum_{j=1}^{3} A_j(u)\,\partial_j u_2 = \mathcal{F} & \text{in } Q_{T/\epsilon}\ , \\ u_2(0,x) = u_{2,0}(x) \qquad \text{in } \mathbb{R}^3\ , \end{cases} \tag{36}$$

where $A_h(u)$, $h = 0,1,2,3$, are defined in (8), $u_{2,0}(x) = \begin{pmatrix} -\pi(0,x) \\ 0 \end{pmatrix}$, $\mathcal{F} = \begin{pmatrix} \mathcal{F}^1 \\ \mathcal{F}^2 \end{pmatrix}$ and

$$\begin{aligned} \mathcal{F}^1 &= v_1 \cdot \nabla g_1 - v \cdot \nabla (g_1 + \pi) - \partial_t \pi\ , \\ \mathcal{F}^2 &= \frac{1}{h(g)} \Big\{ (v_1 \cdot \nabla) v_1 + (w \cdot \nabla) w - (v \cdot \nabla)(v_1 + w) \\ &\qquad + [h(g_1) - h(g)]\, \nabla g_1 + [1 - h(g)] \nabla \pi \Big\}\ . \end{aligned}$$

Let us assume that $u = (g, v)$ fulfills (18) on some interval $[0, T']$ with $T' \le T/\epsilon$. Then from (19) we have

(37)

$$\|u_2(t)\| \le \Big(Mm\|u_2(0)\| + 2M\int_0^t \|\mathcal{F}(s)\|\,ds \Big) \exp\Big(M\int_0^t |\mathrm{div} A(s)|_\infty\,ds \Big)\,, \quad 0 \le t \le T'\,.$$

Let α be an arbitrarily fixed multi-index. By applying ∂^α to (36) we obtain that $u_{2,\alpha} := \partial^\alpha u_2$ must solve the symmetric hyperbolic system

$$A_0(u)\,\partial_t u_{2,\alpha} + \sum_{j=1}^{3} A_j(u)\,\partial_j u_{2,\alpha} = \mathcal{F}_{(\alpha)} \quad \text{in } \ Q_{T'}\ , \tag{38}$$

where $\mathcal{F}_{(\alpha)} = \begin{pmatrix} \mathcal{F}^1_{(\alpha)} \\ \mathcal{F}^2_{(\alpha)} \end{pmatrix}$ and

$$\begin{aligned} \mathcal{F}^1_{(\alpha)} &:= \partial^\alpha \mathcal{F}^1 - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta v \cdot \partial^{\alpha-\beta} \nabla g_2 \,, \\ \mathcal{F}^2_{(\alpha)} &:= \frac{1}{h(g)} \Bigg(\partial^\alpha (h(g)\mathcal{F}^2) - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} (\partial^\beta v \cdot \nabla)\, \partial^{\alpha-\beta} v_2 \\ &\quad - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta (h(g))\, \partial^{\alpha-\beta} \nabla g_2 \Bigg) \,. \end{aligned} \tag{39}$$

Arguing on (38) as from (16) to (19) we obtain an estimate such as (37) for each derivative $u_{2,\alpha}(t)$. Summing then through all the multi-indices α with $|\alpha| \le 3$ and $\alpha_0 \le 2$, the following is derived for $|||u_2(t)|||_3$:

$$|||u_2(t)|||_3 \le \Bigg(Mm|||u_2(0)|||_3 + 2M \sum_{\substack{|\alpha|\le 3 \\ \alpha_0\le 2}} \int_0^t \|\mathcal{F}_{(\alpha)}(s)\|\, ds \Bigg) \exp\Bigg(M \int_0^t |\mathrm{div} A(s)|_\infty \, ds \Bigg), \tag{40}$$

for every $t \in [0, T']$.

From $\|g_2(0)\|_3 = \|\pi(0)\|_3 \le \widehat{C}\epsilon^2$, $v_2(0) = 0$ and equations (5), we may find a positive constant K_1, independent of ϵ, such that $|||u_2(0)|||_3 \le K_1\epsilon^2$.

Chosen a positive μ so that $1 < \mu < 1 + \delta$, for the same δ as in (33), let us define

$$\begin{aligned} T_1 &= \sup\Big\{ T' > 0 : \ |||u_2(t)|||_3 \le 2\, K_1 \epsilon^\mu, \ \text{ if } \ 0 \le t \le T' \Big\} \,, \\ T_2 &= \min\Big\{ T_1,\, T/\epsilon \Big\} \,. \end{aligned}$$

Notice that, in view of the definition of T_2, (32), (34) and the Sobolev embedding theorem, the assumption (18) is actually satisfied by u on any interval $[0, T']$ with $0 < T' \le T_2$ and $0 < \epsilon < \epsilon_0$ for suitable constants m, M; so (40) holds on $[0, T']$.

By (31), (34) we also find K_2 such that

$$|\mathrm{div} A(t)|_\infty \le K_2\, \epsilon \,, \qquad 0 \le t \le T' \,, \tag{41}$$

for any $0 < T' \le T_2$ and $0 < \epsilon < \epsilon_0$. Indeed from (18) it comes that

$$|\partial_t A_0(t)|_\infty \ \le \ K_2' \Big(|\partial_t g_1(t)|_\infty + |\partial_t \pi(t)|_\infty + |\partial_t g_2(t)|_\infty \Big) \,, \qquad 0 \le t \le T', \quad 0 < \epsilon < \epsilon_0$$

holds true for some $K_2' > 0$ independent of ϵ; thus (31), jointly with the Sobolev embedding theorem, (34) and the definition of T_2 give $|\partial_t A_0(t)|_\infty \le K_2\epsilon$ on $0 \le t \le T' \le T_2$. Analogously it may be shown that $|\partial_j A_j(t)|_\infty \le K_2\epsilon$ on $[0, T']$ for $j = 1, 2, 3$.

On the other hand, (31), (34) and (20)–(26) lead to estimate the norm $\|\mathcal{F}_{(\alpha)}(t)\|$, for $|\alpha| \le 3$ and $\alpha_0 \le 2$, as follows

$$\sum_{\substack{|\alpha|\le 3\\ \alpha_0\le 2}} \|\mathcal{F}_{(\alpha)}(t)\| \;\le\; K_3\,\epsilon^2(1+t)^{-\delta} + K_4\,\epsilon^{2\mu} + K_5\,\epsilon\,|||u_2(t)|||_3\,, \qquad 0 \le t \le T'\,, \tag{42}$$

for every $0 < T' \le T_2$ and $0 < \epsilon < \epsilon_0$.

Indeed let us come back to formula (39); by Leibnitz rule it follows that:

$$\partial^\alpha \mathcal{F}^1 \;=\; -\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta v_1 \cdot \partial^{\alpha-\beta}\nabla\pi - \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta (v_2+w)\cdot\partial^{\alpha-\beta}\nabla(g_1+\pi) - \partial^\alpha\partial_t\pi\,.$$

By (31), (34), Hölder's inequality and the definition of T_2 for all $\beta \le \alpha$ the following estimates hold

$$\|\partial^\beta v_1\cdot\partial^{\alpha-\beta}\nabla\pi\| \le \widetilde{C}\epsilon^{2\mu}, \quad \|\partial^\beta(v_2+w)\cdot\partial^{\alpha-\beta}\nabla\pi\| \le \widetilde{C}\epsilon^{2\mu}, \quad 0 \le t \le T'\,, \tag{43}$$

for every $0 < T' \le T_2$.

Using also (33), we obtain

$$\|\partial^\beta(v_2 + w)\cdot\partial^{\alpha-\beta}\nabla g_1\| \;\le\; C_1\,\epsilon^2\,(1+t)^{-\delta}\,, \qquad 0 \le t \le T' \tag{44}$$

for every $0 < T' \le T_2$ and $\beta \le \alpha$.

Lastly (34) directly gives

$$\|\partial^\alpha\partial_t\pi\| \;\le\; \hat{C}\,\epsilon^3 \;\le\; \hat{C}\,\epsilon^{2\mu} \tag{45}$$

for $0 \le t \le T'$ and $0 < T' \le T_2$.

Estimates (43)–(45) yield

$$\|\partial^\alpha\mathcal{F}^1\| \;\le\; C_1\,\epsilon^2\,(1+t)^{-\delta} + C_2\,\epsilon^{2\mu}\,, \tag{46}$$

for $0 \le t \le T' \le T_2$, $0 < \epsilon < \epsilon_0$, where $C_1, C_2 > 0$ do not depend on ϵ.

Similarly we get

$$\left\|\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta v\cdot\partial^{\alpha-\beta}\nabla g_2\right\| \le\; C_1'\,\epsilon^2\,(1+t)^{-\delta} +\; C_2'\,\epsilon^{2\mu} +\; C_3'\,\epsilon\,|||g_2(t)|||_3\,,$$

for $0 \le t \le T' \le T_2$, $0 < \epsilon < \epsilon_0$, which shows, jointly with (46), estimate (42) for the component $\mathcal{F}^1$ of $\mathcal{F}$. By similar computations (42) is proved also for $\mathcal{F}^2$.

Estimating the right-hand side of (40) by means of (41), (42), using Gronwall's lemma and $t \le T/\epsilon$ give

$$(47) \quad |||u_2(t)|||_3 \le \epsilon^\mu \exp(K_1'\epsilon\, t)\Big(K_2'\,\epsilon^{2-\mu} + K_3'\,\epsilon^{2-\mu}\,t^{1-\delta} + K_4'\,\epsilon^\mu\, t\Big), \qquad 0 \le t \le T',$$

for any $0 < T' \le T_2$, $0 < \epsilon < \epsilon_0$ and positive constants K_1', K_2', K_3', K_4' independent of ϵ.

Since $t \le T/\epsilon$ and $0 < \epsilon < \epsilon_0$ give $\exp(K_1'\epsilon\, t) \le \exp(K_1' T)$, $\epsilon^{2-\mu} < \epsilon_0^{2-\mu}$, $\epsilon^{2-\mu}\,t^{1-\delta} < \epsilon_0^{1+\delta-\mu}\,T^{1-\delta}$ and $\epsilon^\mu\, t < \epsilon_0^{\mu-1}\,T$, from (47) we derive

$$(48) \qquad |||u_2(t)|||_3 < 2\,K_1\,\epsilon^\mu\,,$$

if $0 \le t \le T' \le T_2$ and $0 < \epsilon < \epsilon_0$, provided ϵ_0 is taken suitably small. This shows that $T/\epsilon \le T_1$, i.e. $u_2 = (g_2, v_2)$ exists up to T/ϵ. Consequently, also $u = (g, v)$ exists up to T/ϵ.

Then we conclude that the solution $u^\epsilon = (g^\epsilon, v^\epsilon)$ to (3) exists on the time-interval $[0, T]$, provided $0 < \epsilon < \epsilon_0$, and here satisfies (12) as a direct consequence of (31), (34) and (48). ∎

4 – Proof of Proposition 1.1

The proof of Proposition 1.1 essentially relies on the computations in Section 3. As we did in proving Theorem 1.1 we make the rescaling (4). If (g^ϵ, v^ϵ) is a solution to (3) with initial data $(g_0^\epsilon, v_0^\epsilon)$ then $u = (g, v)$ solves the ivp (5) or (7) with the initial data $u_0 = (g_0, v_0)$ given by (6).

Once again we write the solution $u = (g, v)$ in the form:

$$g = g_1 + \pi + g_2\,, \qquad v = v_1 + w + v_2\,,$$

where $u_1 = (g_1, v_1)$ is the solution of (5) or (7) corresponding to the irrotational initial data (g_0, Qv_0), while

$$\pi(t,x) := \epsilon^2\,\pi^\epsilon(\epsilon\, t, x)\,, \qquad w(t,x) := \epsilon\, w^\epsilon(\epsilon\, t, x)$$

and $u_2 = (g_2, v_2)$ solves (36).

Due to (13), by Theorem 2 in [6] u_1 satisfies (31)–(33) with life span $\exp(C_0/\epsilon) - 1$. Let $0 < \epsilon_0 < 1$ be such that $\exp(C_0/\epsilon) - 1 > A/\epsilon^\mu$ as $0 < \epsilon < \epsilon_0$.

In view of (14), (π, w) satisfies now:

$$(49) \qquad \sum_{\substack{|\alpha|\le 4\\ \alpha_0=k}} \|\partial^\alpha \pi(t)\| \le \widehat{C}\,\epsilon^{k+1+\mu}\,, \qquad \sum_{\substack{|\alpha|\le 4\\ \alpha_0=k}} \|\partial^\alpha w(t)\| \le \widehat{C}\,\epsilon^{k+\mu}\,, \qquad k=0,...,3$$

for $0 \le t \le A/\epsilon^\mu$.

Moreover from our hypotheses we derive that there exists a constant $K_1 > 0$ such that $|||u_2(0)|||_3 \le K_1\,\epsilon^2$. Taken an arbitrary $0 < A' \le A$, analogously to the proof of Theorem 1.1 we define T_3 and T_4 by:

$$(50) \qquad \begin{aligned} T_3 &:= \sup\Big\{T'\!>\!0:\ |||u_2(t)|||_3 \le 2\,K_1\,\epsilon^\mu,\ 0\!\le\! t\!\le\! T'\Big\}\ ;\\ T_4 &:= \min\Big\{T_3,\ A'/\epsilon^\mu\Big\}\ . \end{aligned}$$

Since in any interval $[0,T']$ for $0 < T' \le T_4$ the solution $u = (g,v)$ satisfies (18), the *a priori* estimate (40) holds.

By (25), (26) (for $u = u_1$), (49) and the definition of T_4 we may find a constant $K_2 > 0$ independent of ϵ_0 and A' such that:

$$(51) \qquad |\mathrm{div} A(t)|_\infty \le K_2\Big(\epsilon(1\!+\!t)^{-\delta} + \epsilon^\mu\Big)\,, \qquad 0 \le t \le T'\,,$$

for every $0 < T' \le T_4,\ 0 < \epsilon < \epsilon_0$ and $\delta = \frac{1}{5}$.

Since $1 < \mu < \frac{6}{5} = 1+\delta$, $T_4 \le A'/\epsilon^\mu$ and $\epsilon_0 < 1$, from (51) it follows

$$(52) \qquad \int_0^t |\mathrm{div} A(s)|_\infty\, ds \ \le\ K_2(\epsilon\, t^{1-\delta} + \epsilon^\mu\, t) \ \le\ K_2({A'}^{1-\delta}\epsilon_0^{\delta^2} + A')\,, \qquad 0\!\le\! t\!\le\! T'\,,$$

for every $0 < T' \le T_4$ and $0 < \epsilon < \epsilon_0$.

It remains to estimate the norm $\|\mathcal{F}_{(\alpha)}\|$, for every $|\alpha| \le 3$ and $\alpha_0 \le 2$, in the right-hand side of (40).

Arguing as from (31)–(34) to (42), with (49) instead of (34), leads now to the following estimate:

$$\|\mathcal{F}_{(\alpha)}\| \ \le\ K_3\,\epsilon^{\mu+1}(1+t)^{-\delta} + K_4\,\epsilon^{2\mu}\ ,$$

hence

$$(53) \qquad \sum_{\substack{|\alpha|\le 3,\\ \alpha_0\le 2}} \int_0^t \|\mathcal{F}_{(\alpha)}(s)\|\, ds \ \le\ K_3\,\epsilon^{\mu+1}\, t^{1-\delta} + K_4\,\epsilon^{2\mu}\, t\,, \qquad 0 \le t \le T'\ ,$$

for $0 < T' \le T_4,\ 0 < \epsilon < \epsilon_0$ and positive constants K_3, K_4 independent of ϵ, ϵ_0 and A'.

Estimating the right-hand side of (40) by means of (52), (53) we obtain now:

$$|||u_2(t)||| \leq K_3' \, e^{K_2(A'^{1-\delta}\epsilon_0^{\delta^2}+A')} \, \epsilon^\mu (\epsilon^{2-\mu} + \epsilon \, t^{1-\delta} + \epsilon^\mu \, t) \,, \quad 0 \leq t \leq T' \leq T_4 \,,$$

hence, taking into account $0 < \epsilon < \epsilon_0 < 1$ and $T_4 \leq A'/\epsilon^\mu$,

$$(54) \quad |||u_2(t)|||_3 \leq K_3' \, e^{K_2(A'^{1-\delta}\epsilon_0^{\delta^2}+A')} \, \epsilon^\mu (\epsilon_0^{2-\mu} + A'^{1-\delta}\epsilon_0^{\delta^2} + A') \,, \quad 0 \leq t \leq T' \leq T_4$$

for all $0 < \epsilon < \epsilon_0$.

This ends the proof, since in view of (54) we may restrict ϵ_0 and A', if necessary, in order to obtain that $|||u_2(t)|||_3 < 2\,K_1 \epsilon^\mu$ for $0 \leq t \leq T' \leq T_4$ and $0 < \epsilon < \epsilon_0$. Thus it follows that $A'/\epsilon^\mu < T_3$ which yields that the solution $u_2 = (g_2, v_2)$ exists up to time A'/ϵ^μ. Coming back to the solution $u^\epsilon = (g^\epsilon, v^\epsilon)$ of (3), this means that u^ϵ exists up to time $A'/\epsilon^{\mu-1}$; moreover (15) is fulfilled on $[0, A'/\epsilon^{\mu-1}]$ as a consequence of (31), (49) and (48). ∎

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