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ENERGY DECAY ESTIMATES FOR THE DAMPED EULER–BERNOULLI EQUATION WITH AN UNBOUNDED LOCALIZING COEFFICIENT

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Abstract: We consider the Euler–Bernoulli equation in a bounded domain Ω with a local dissipation ay'. The localizing coefficient a is of the form $a(x) = \alpha(x)/(d(x,\Gamma))^s$, $(0 < s \leq 1)$, where Γ is the boundary of Ω, $d(x,\Gamma)$ is the distance from x to Γ, and α is a bounded nonnegative function such that a is unbounded. Using integral inequalities and multiplier techniques, we prove exponential and polynomial decay estimates for the energy of each solution of this equation. In particular, since the localizing coefficient a is unbounded, an important technical difficulty occurs adding to the difficulty of dealing with a local dissipation. A judicious application of Hardy inequality enables us to overcome this difficulty. The results obtained improve existing results where the boundedness of the function a is critical.

1 – Introduction and statement of the main results

The main purpose of this paper is to give precise decay estimates for the energy of Euler–Bernoulli equations with a linear damping term localized in a neighborhood of a suitable subset of the domain under consideration. For the sequel, we need some notations. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ having a sufficiently smooth boundary $\Gamma = \partial\Omega$. We denote by ν the unit normal

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pointing into the exterior of Ω. We fix $x^0 \in \mathbb{R}^N$ and we set $m(x) = x - x^0$,

$$R = \sup\Big\{|m(x)|,\ x \in \Omega\Big\}, \quad \Gamma_+ = \Big\{x \in \Gamma;\ m(x)\cdot\nu(x) > 0\Big\} \quad \text{and} \quad \Gamma_- = \Gamma\backslash\Gamma_+$$

$(u\cdot v = \sum_1^N u_i v_i$ for all $u, v \in \mathbb{R}^N)$. Let $a = \frac{\alpha}{d^s}$ with $d(x) = \operatorname{dist}(x, \Gamma)$, $0 < s \leq 1$, and $\alpha \in L^\infty(\Omega)$ is a nonnegative function such that:

$$\alpha(x) \geq a_0 \quad \text{a.e. in } \ \omega\ , \tag{1.1}$$

or

$$\exists\, p > 0 : \ \int_\omega \frac{dx}{\alpha(x)^p} < \infty\ , \tag{1.2}$$

where ω is a neighborhood of Γ_+ contained in Ω, and a_0 is a positive constant. By neighborhood of Γ_+, we actually mean the intersection of Ω and a neighborhood of Γ_+. Throughout the paper, we denote by $|u|_r$ the norm of a function $u \in L^r(\Omega)$, $1 \leq r \leq \infty$, and by $|1/\alpha|_p$, the quantity $\Big(\int_\omega \frac{dx}{\alpha(x)^p}\Big)^{\frac{1}{p}}$, $p > 0$.

Now consider the following damped Euler–Bernoulli equation

$$\begin{cases} y'' + \Delta^2 y + a y' = 0 & \text{in } \ \Omega\times(0,\infty) \\ y = \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \ \Gamma\times(0,\infty) \\ y(0) = y^0 & \text{in } \ \Omega \\ y'(0) = y^1 & \text{in } \ \Omega\ . \end{cases} \tag{1.3}$$

Condition (1.1) or (1.2) ensures that the damping term ay' is effective on the set ω. Let $\{y^0, y^1\} \in H_0^2(\Omega)\times L^2(\Omega)$. System (1.3) is then well-posed in the space $H_0^2(\Omega)\times L^2(\Omega)$; in fact, there exists a unique weak solution of (1.3) with

$$y \in \mathcal{C}\Big([0,\infty); H_0^2(\Omega)\Big) \cap \mathcal{C}^1\Big([0,\infty); L^2(\Omega)\Big)\ . \tag{1.4}$$

This result can be proved using the Hille–Yosida Theorem [2].

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \Big\{|y'(x,t)|^2 + |\Delta y(x,t)|^2\Big\}\, dx\,, \qquad \forall t \geq 0\ . \tag{1.5}$$

The energy E is a nonincreasing function of the time variable t and we have for almost every $t \geq 0$,

$$E'(t) = -\int_\Omega a(x)\, |y'(x,t)|^2\, dx\ . \tag{1.6}$$

Although the literature on the decay estimates of the energy of the wave equation with locally distributed damping is quite impressive [3, 4, 8, 9, 13, 16–20, 22–28,...], little is known on the decay estimate of the energy of plate equations with a locally distributed damping; to the knowledge of the author, only a few papers [5, 9, 24, 27] address this issue for a bounded localizing coefficient. The boundedness of this coefficient is critical in the approaches developed in those four papers. In fact when the coefficient is bounded, its L^∞-norm is trivially absorbed by constants in the estimations. In the case at hand which involves an unbounded localizing coefficient we have to handle the estimates with a great care, then apply the Hardy inequality to get estimates similar to those established in the case of a bounded coefficient.

Before stating our main results, let us recall the following regularity result

Theorem 1.0. *Let $\{y^0, y^1\} \in H^4(\Omega) \cap H_0^2(\Omega) \times H_0^2(\Omega)$, and let a be given as above. Then the solution y of (1.3) satisfies*

$$y \in \mathcal{C}\Big([0,\infty); H^4(\Omega) \cap H_0^2(\Omega)\Big) \cap \mathcal{C}^1\Big([0,\infty); H_0^2(\Omega)\Big) \cap \mathcal{C}^2\Big([0,\infty); L^2(\Omega)\Big) . \tag{1.7}$$

Moreover, if we set

$$F_0 = \Big(\|y^1\|^2_{H_0^2(\Omega)} + |\Delta^2 y^0|_2^2\Big)^{\frac{1}{2}} \tag{1.8}$$

then there exits a positive constant c depending only on Ω and a such that

$$|\Delta y'(t)|_2 \le c\, F_0, \quad |\Delta^2 y(t)|_2 \le c\, F_0\,, \qquad \forall t \ge 0 \ . \tag{1.9}$$

The proof of this result relies on Hille–Yosida Theorem and Hardy inequality [2]; for the reader convenience, it is provided in Section 3 below.

We are now in the position to state our main results:

Theorem 1.1. *Let $\{y^0, y^1\} \in H_0^2(\Omega) \times L^2(\Omega)$. Let ω be a neighborhood of Γ_+. Assume that $\alpha \in L_+^\infty(\Omega)$ satisfies (1.1) for some $a_0 > 0$. Then there exists a positive constant τ independent of the initial data such that*

$$E(t) \le \Big[\exp(1 - \tau t)\Big] E(0)\,, \qquad \forall t \ge 0 \ . \tag{1.10}$$

Theorem 1.2. *Let $\{y^0, y^1\} \in H^4(\Omega) \cap H_0^2(\Omega) \times H_0^2(\Omega)$. Let ω be a neighborhood of Γ_+. Assume that $\alpha \in L_+^\infty(\Omega)$ satisfies (1.2) for some $p > 0$. Then for every space dimension $N \ne 4$, the energy E satisfies*

$$E(t) \le K_1\Big(|(1/\alpha)|_p\, F_0^{\frac{N}{2p}} + E(0)^{\frac{N}{4p}}\Big)^{\frac{4p}{N}} t^{-\frac{4p}{N}}\,, \qquad \forall t > 0 \ , \tag{1.11}$$

where K_1 is a positive constant independent of the initial data.

When $N=4$, we have the decay estimate

(1.12)
$$E(t) \le \left(c_r |(1/\alpha)|_p \, F_0^{\frac{2(r+p)}{p(r-1)}} + c\, E(0)^{\frac{(r+p)}{p(r-1)}}\right)^{\frac{p(r-1)}{r+p}} t^{-\frac{p(r-1)}{r+p}}\,, \quad \forall t>0,\ \ \forall\, 1<r<\infty\,,$$

where c_r and c are positive constants independent of the initial data.

The remainder of the paper is organized as follows. In section 2, we discuss some technical lemmas that are later used in the proofs of Theorems 1.1 and 1.2. Section 3 is devoted to the proofs of Theorems 1.0, 1.1 and 1.2.

2 – Some Technical Lemmas

The proofs of Theorems 1.1 and 1.2 rely on the following lemmas.

Lemma 2.1 (Gagliardo–Nirenberg)**.** *Let $1 \le q \le s \le \infty$, $1 \le r \le s$, $0 \le k < m < \infty$, where k and m are nonnegative integers, and $\theta \in [0,1]$. Let $v \in W^{m,q}(\Omega) \cap L^r(\Omega)$. Suppose that*

$$k - \frac{N}{s} \le \theta\left(m - \frac{N}{q}\right) - \frac{N(1-\theta)}{r}\,. \tag{2.1}$$

Then $v \in W^{k,s}(\Omega)$, and there exists a positive constant C such that

$$\|v\|_{W^{k,s}(\Omega)} \le C\, \|v\|^{\theta}_{W^{m,q}(\Omega)}\, |v|_r^{1-\theta}\,. \ \blacksquare \tag{2.2}$$

Lemma 2.2. *Let $E\colon [0,\infty[\ \to [0,\infty[$ be a nonincreasing locally absolutely continuous function such that there are constants $\beta \ge 0$ and $A > 0$ with*

$$\int_S^\infty E(t)^{\beta+1}\,dt \le A E(S)\,, \qquad \forall S \ge 0\,. \tag{2.3}$$

Then we have

$$E(t) \le \begin{cases} \left[\exp\!\left(1-\dfrac{t}{A}\right)\right] E(0), & \forall t \ge 0 \quad \textit{if } \beta=0, \\[2ex] \left(A\left(1+\dfrac{1}{\beta}\right)\right)^{\frac{1}{\beta}} t^{-\frac{1}{\beta}}, & \forall t>0 \quad \textit{if } \beta>0\,. \ \blacksquare \end{cases} \tag{2.4}$$

This lemma is due to Haraux and its proof can be found in [6, 7] or [10, 11], [12], [17]. This lemma reduces the proofs of Theorems 1.1–1.2 to the proofs of estimates of type (2.4).

From now on, we denote by S and T two real numbers such that $0 \le S < T < \infty$, and we write E instead of $E(t)$.

Lemma 2.3. *Let $\mu \geq 0$, and let $q \in (W^{2,\infty}(\Omega))^N$, $\delta \in \mathbb{R}$ and $\xi \in W^{2,\infty}(\Omega)$. Let y be a weak solution of (1.3). We have the identities*

$$\begin{aligned}
&\int_\Omega y'\Big\{2q\cdot\nabla y+\delta y\Big\}\,dx\ E^\mu\Big]_S^T + 4\int_{\Omega\times]S,T[}\frac{\partial q_k}{\partial x_i}\,\frac{\partial^2 y}{\partial x_i\partial x_k}\,E^\mu\ dx\,dt\ + \\
&\quad+\int_{\Omega\times]S,T[}\Big(\operatorname{div}(q)-\delta\Big)\Big\{|y'|^2-|\Delta y|^2\Big\}\,E^\mu\ dx\,dt \\
(2.5)\quad&\quad-2\int_{\Omega\times]S,T[}\Delta y\,\nabla y\cdot\Delta q\,E^\mu\ dx\,dt\ -\ \mu\int_{\Omega\times]S,T[}E^{\mu-1}E'y'\Big\{2q\cdot\nabla y+\delta y\Big\}\,dx\,dt \\
&\quad+\int_{\Omega\times]S,T[}ay'\Big\{2q\cdot\nabla y+\delta y\Big\}E^\mu\ dx\,dt\ =\ \int_{\Gamma\times]S,T[}E^\mu(q\cdot\nu)\,(\Delta y)^2\ d\Gamma\,dt\ .
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega y'\xi\,y\,dx\ E^\mu\Big]_S^T-\int_{\Omega\times]S,T[}\xi\Big\{|y'|^2-|\Delta y|^2\Big\}\,E^\mu\ dx\,dt\ - \\
(2.6)\qquad&\quad-\ \mu\int_{\Omega\times]S,T[}E^{\mu-1}E'y'y\,\xi\ dx\,dt\ +\ 2\int_{\Omega\times]S,T[}\Delta y\,\nabla y\cdot\nabla\xi\,E^\mu\ dx\,dt \\
&\quad+\int_{\Omega\times]S,T[}y\,\Delta y\,\Delta\xi\,E^\mu\ dx\,dt\ +\int_{\Omega\times]S,T[}ay'\,\xi y\,E^\mu\ dx\,dt\ =\ 0\ .\ \blacksquare
\end{aligned}$$

The proof of Lemma 2.3 is based on standard multipliers technique, the interested reader should refer to Lions [14] or Komornik [10] for the details.

Throughout the remaining part of the paper, c denotes different positive constants independent of the initial data.

Lemma 2.4. *Let α satisfy (1.2) for some $p > 0$. Let y be any strong solution of (1.3). Then for all $t \geq 0$, we have the estimates*

$$(2.7)\qquad \int_\omega |y'|^2\,dx\ \le\ \begin{cases} c\,|(1/\alpha)|_p^{\frac{p}{p+1}}\,F_0^{\frac{N}{2(p+1)}}\,|E'|^{\frac{p}{p+1}}\,E^{\frac{4-N}{4(p+1)}}, & \text{if } 1\le N\le 3\ ,\\ c_r\,|(1/\alpha)|_p^{\frac{p(r-1)}{r(p+1)}}\,F_0^{\frac{2(r+p)}{r(p+1)}}\,|E'|^{\frac{p(r-1)}{r(p+1)}}, \quad \forall\,1<r<\infty, & \text{if } N=4\ ,\\ c\,|(1/\alpha)|_p^{\frac{4p}{N+4p}}\,F_0^{\frac{2N}{N+4p}}\,|E'|^{\frac{4p}{N+4p}}, & \text{if } N\ge 5\ . \end{cases}$$

Let y be any weak solution of (1.3). Then for all $t \geq 0$, we have the estimate

$$(2.8)\qquad \int_\Omega a\Big(|\nabla y|^2+|y|^2\Big)\,dx\ \le\ c\int_\Omega|\Delta y|^2\,dx\ \le\ c\,E(t)\,,\qquad \forall t\ge 0\ .$$

Remark. The proof of (2.7) will follow from a judicious application of Hölder inequality, Gagliardo–Nirenberg interpolation inequalities, and Sobolev imbedding theorem. As for (2.8), its proof is a direct consequence of Hardy inequality once we observe that y and ∇y lie in $H_0^1(\Omega)$ and $(H_0^1(\Omega))^N$ respectively. ▫

Proof of Lemma 2.4: We begin with the proof of (2.7). For this proof we need different approaches for the three cases involved, so we will proceed by cases.

Case 1: $1 \le N \le 3$. Since y is a strong solution of (1.3), it is known that $y_t(.,t) \in H_0^2(\Omega)$, for all $t \ge 0$. On the other hand, Sobolev imbedding theorems show that $H_0^2(\Omega)$ is continuously embedded in $L^\infty(\Omega)$ for $1 \le N \le 3$. Applying Hölder inequality and this imbedding result, we find

$$\begin{aligned}
\int_\omega |y'|^2\,dx &= \int_\omega (1/a)^{\frac{p}{p+1}}\, a^{\frac{p}{p+1}}\, |y'|^2\,dx \\
&\le |(1/a)|_p^{\frac{p}{p+1}} \left(\int_\omega a|y'|^{\frac{2(p+1)}{p}} dx\right)^{\frac{p}{p+1}} \\
&\le c\,|(1/\alpha)|_p^{\frac{p}{p+1}}\, |y'(.,t)|_\infty^{\frac{2}{p+1}} \left(\int_\Omega a|y'|^2\,dx\right)^{\frac{p}{p+1}} \\
&\le c\,|(1/\alpha)|_p^{\frac{p}{p+1}}\, |y'(.,t)|_\infty^{\frac{2}{p+1}}\, |E'(t)|^{\frac{p}{p+1}}\ .
\end{aligned} \tag{2.9}$$

Thanks to Lemma 2.1, we have the interpolation inequality

$$\begin{aligned}
|y'(.,t)|_\infty &\le c\,\|y'(.,t)\|_{H^2(\Omega)}^{\frac{N}{4}}\, |y'(.,t)|_2^{\frac{4-N}{4}} \\
&\le c\,|\Delta y'(.,t)|_2^{\frac{N}{4}}\, E^{\frac{4-N}{8}} \\
&\le c\,F_0^{\frac{N}{4}}\, E^{\frac{4-N}{8}}\ .
\end{aligned} \tag{2.10}$$

Reporting (2.10) in (2.9), we get the claimed estimate, and we are done with this case.

Case 2: $N = 4$. Let $r > 1$, and $\tau \in (0,2)$. We have by a twofold application of Hölder inequality

$$\begin{aligned}
\int_\omega |y'|^2\,dx &= \int_\omega |y'|^{2-\tau}\,|y'|^\tau\,dx \\
&\le \left(\int_\Omega |y'|^{(2-\tau)r}\,dx\right)^{\frac{1}{r}} \left(\int_\omega (1/a)^{\frac{p}{p+1}}\,(a)^{\frac{p}{p+1}}\,|y'|^{\frac{\tau r}{r-1}}\,dx\right)^{\frac{r-1}{r}} \le
\end{aligned} \tag{2.11}$$

$$\leq\ |(1/a)|_p^{\frac{p(r-1)}{r(p+1)}} \left(\int_\Omega |y'|^{\frac{2(r+p)}{p+1}}\,dx\right)^{\frac{1}{r}} \left(\int_\Omega a|y'|^2\,dx\right)^{\frac{p(r-1)}{r(p+1)}}, \quad \text{with } \tau = \frac{2p(r-1)}{r(p+1)}$$

$$\leq\ c_r\,|(1/\alpha)|_p^{\frac{p(r-1)}{r(p+1)}}\,\|y'(.,t)\|_{H^2(\Omega)}^{\frac{2(r+p)}{r(p+1)}}\,|E'|^{\frac{p(r-1)}{r(p+1)}}$$

$$\leq\ c_r\,|(1/\alpha)|_p^{\frac{p(r-1)}{r(p+1)}}\,F_0^{\frac{2(r+p)}{r(p+1)}}\,|E'|^{\frac{p(r-1)}{r(p+1)}}\ .$$

It should be noted that in (2.11), we use in an essential manner the Sobolev imbedding theorem: $H^2(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $1\leq q<\infty$, when $N=4$. To complete the proof of (2.7), it remains to deal with the last case.

Case 3: $N\geq 5$. First of all, we note the Sobolev imbedding theorem: $H^2(\Omega)$ is continuously embedded in $L^s(\Omega)$ for $1\leq s\leq \frac{2N}{N-4}$ when $N\geq 5$. Choosing $r=\frac{N+4p}{N-4}$ in (2.11), we get the claimed estimate, and we are done with the proof of (2.7). Let us turn now to the proof of (2.8).

We have

$$\begin{aligned}
\int_\Omega a\Big(|\nabla y|^2+|y|^2\Big)\,dx\ &=\int_\Omega \alpha\,d(x)^s\Big(|\nabla y/d(x)^s|^2+|y/d(x)^s|^2\Big)\,dx\\
&\leq\ c\int_\Omega\Big(|\nabla y/d(x)|^2+|y/d(x)|^2\Big)\,dx, \quad \text{since } 0<s\leq 1\\
&\leq\ c\int_\Omega |\Delta y|^2\,dx, \quad \text{by Hardy inequality}\\
&\leq\ c\,E(t)\,, \quad \forall t\geq 0\ ,
\end{aligned}\tag{2.12}$$

which completes the proof of Lemma 2.4. ∎

Remark. It should be noted that one may prove the last cases of (2.7), ($N=4$ and $N\geq 5$), by combining the Gagliardo–Nirenberg interpolation inequalities with Hölder inequality as in the first case or as in [23, 25]. However, in doing so, one gets in the end much weaker estimates under rather severe restrictions on the degeneracy of the localizing function a. Therefore the new approach developed here, which is based on an astute application of Hölder inequality, can be used to strongly improve earlier results established in [18, 23, 25] in the case of the wave equation. □

3 – Proofs of Theorems 1.0, 1.1 and 1.2.

3.1. Proof of Theorem 1.0

We may rewrite the first equation of (1.3) in the form

$$\begin{cases} y' - z = 0 & \text{in } \Omega\times(0,\infty) , \\ z' + \Delta^2 y + a z = 0 & \text{in } \Omega\times(0,\infty) . \end{cases} \tag{3.1.1}$$

Setting $Z = \begin{pmatrix} y \\ z \end{pmatrix}$, (3.1.1) becomes $Z' + \mathcal{A} Z = 0$, so that (1.3) is equivalent to

$$\begin{cases} Z' + \mathcal{A} Z = 0 & \text{in } (0,\infty) , \\ Z(0) = \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} , \end{cases} \tag{3.1.2}$$

where the unbounded operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ \Delta^2 & aI \end{pmatrix} \tag{3.1.3}$$

with $D(\mathcal{A}) = H^4(\Omega) \cap H^2_0(\Omega)\times H^2_0(\Omega)$.

Now we are going to apply the Hille–Yosida theory in the Hilbert space $\mathcal{H} = H^2_0(\Omega)\times L^2(\Omega)$ endowed with the norm

$$\|Z\|^2_{\mathcal{H}} = \int_\Omega |\Delta y|^2\, dx + \int_\Omega |z|^2\, dx . \tag{3.1.4}$$

Let us show that the operator $\mathcal{A}$ is maximal monotone. This amounts to proving that:

(i) $(\mathcal{A}Z, Z) \geq 0$, $\forall Z = \begin{pmatrix} y \\ z \end{pmatrix} \in D(\mathcal{A})$,

(ii) $\mathcal{A} + \mathcal{I}$ is surjective, ($\mathcal{I}$ is the identity operator)

where in (i), $(.,.)$ denotes the scalar product induced by the norm defined in (3.1.4).

Proof of (i): Since for all $Z \in D(\mathcal{A})$, we have

$$\mathcal{A} Z = \begin{pmatrix} -z \\ \Delta^2 y + a z \end{pmatrix} ,$$

it follows that

$$
(3.1.5) \qquad \begin{aligned} (\mathcal{A}Z, Z) &= -\int_\Omega \Delta z\, \Delta y\, dx + \int_\Omega z\, \Delta^2 y\, dx + \int_\Omega a z^2\, dx \\ &= \int_\Omega a z^2\, dx \;\geq\; 0 \,, \end{aligned}
$$

which establishes (i).

Proof of (ii): We shall prove that for all $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathcal{H}$, there exists Z in $D(\mathcal{A})$ such that

$$
(3.1.6) \qquad \mathcal{A}Z + Z = \begin{pmatrix} u \\ v \end{pmatrix} \;.
$$

If we set $Z = \begin{pmatrix} y \\ z \end{pmatrix}$, then (3.1.6) may be rewritten

$$
(3.1.7) \qquad \begin{cases} -z + y = u & \text{in } \Omega \,, \\ \Delta^2 y + a z = v & \text{in } \Omega \,. \end{cases}
$$

The first equation of (3.1.7) gives z as a function of y and u. Reporting this in the second equation, we get

$$
(3.1.8) \qquad \Delta^2 y + a y \;=\; v + a u \;.
$$

Since v lies in $L^2(\Omega)$, and u lies in $H_0^2(\Omega)$, Hardy inequality shows that the right hand side of (3.1.8) belongs to $L^2(\Omega)$. Since we are looking for y in $H_0^2(\Omega)$, the application of the theory of elliptic problems [2, 15] gives the existence and uniqueness of y. The existence of z follows immediately, and (ii) is proved.

The theorem of Hille–Yosida shows that (3.1.2) has a unique solution

$$
(3.1.9) \qquad Z \in \mathcal{C}\Big([0,\infty); D(\mathcal{A})\Big) \cap \mathcal{C}^1\Big([0,\infty); \mathcal{H}\Big) \;,
$$

hence (1.7). The proof of (1.9) follows by standard energy method. ∎

We now turn to the proofs of Theorems 1.1 and 1.2. This proof essentially relies on the multiplier techniques as developed in [10, 14, 27]. This method introduces lower order terms that need to be absorbed to get the energy decay estimates announced. In general, authors rely on a unique continuation argument to get rid of these lower order terms [18, 19, 27, 28]. However the compactness-uniqueness approach introduces in the estimates constants on which one has no control. Recently a direct approach, which consists in introducing a suitable auxiliary stationary system and using its solution to build a multiplier in order to absorb the lower order terms, was introduced in [22], and was subsequently used with success in [17, 23, 24, 25,...]. The direct approach was developed as an alternative to the compactness-uniqueness method, and it turns out that it is much simpler to use, and it provides explicit decay estimates, which makes it very suitable for numerical experiments.

3.2. Proof of Theorems 1.1, and 1.2.

The first steps of the proofs of these two theorems are similar. This explains why we are proving these two theorems simultaneously.

We proceed in several steps.

Step 1. Applying (2.5) with $\delta = N-2$, $q(x) = m(x)$, observing that $\operatorname{div}(m) = N$ and using (1.5), we find

$$\begin{aligned} 4\int_S^T E^{\mu+1}dt \;=\; & -\int_\Omega y'\Big\{2m\cdot\nabla y + (N-2)y\Big\}\,dx\ E^\mu\Big]_S^T \\ & + \mu\int_{\Omega\times]S,T[} E^{\mu-1}E'\,y'\Big\{2m\cdot\nabla y + (N-2)y\Big\}\,dx\,dt \\ & - \int_{\Omega\times]S,T[} ay'\Big\{2m\cdot\nabla y + (N-2)y\Big\}\,E^\mu\ dx\,dt \\ & + \int_{\Gamma\times]S,T[} E^\mu (m\cdot\nu)\,(\Delta y)^2\ d\Gamma\,dt\ . \end{aligned} \tag{3.2.1}$$

Since the energy is nonincreasing, it follows that

$$\left| -\int_\Omega y'\Big\{2m\cdot\nabla y + (N-2)y\Big\}\,dx\ E^\mu\Big]_S^T \right| \;\le\; \frac{4R}{\lambda_0}\,E(0)^\mu\,E(S)\ , \tag{3.2.2}$$

and

$$\left|\mu\int_{\Omega\times]S,T[} E^{\mu-1}E'y'\Big\{2m\cdot\nabla y + (N-2)y\Big\}\,dx\,dt\right| \;\le\; \frac{2\,\mu R}{\lambda_0}\,E(0)^\mu\,E(S)\ , \tag{3.2.3}$$

where λ_0^2 is the first eigenvalue of the eigenvalue problem

$$\begin{cases} \Delta^2 u = -\lambda^2\Delta u & \text{in } \ \Omega\ , \\ u = \dfrac{\partial u}{\partial\nu} = 0 & \text{on } \ \partial\Omega\ . \end{cases} \tag{3.2.4}$$

By Hölder inequality we have

$$
\begin{aligned}
&\left| \int_{\Omega\times]S,T[} a y' \Big\{ 2m\cdot\nabla y + (N-2)y \Big\} E^{\mu}\, dx\, dt \right| \le \\
&\qquad \le\; c \int_S^T E^{\mu} \Big(\int_\Omega a|y'|^2\, dx \Big)^{\frac12} \Big(\int_\Omega a\Big(|\nabla y|^2 + |y|^2\Big)\, dx \Big)^{\frac12} dt \\
(3.2.5)\qquad &\qquad \le\; c \int_S^T E^{\mu}\, |E'|^{\frac12} \Big(\int_\Omega |\Delta y|^2\, dx \Big)^{\frac12} dt, \qquad \text{by (2.8)} \\
&\qquad \le\; c \int_S^T E^{\frac{2\mu+1}{2}}\, |E'|^{\frac12}\, dt, \qquad \text{by (1.6)}\ .
\end{aligned}
$$

Now, using Young inequality and the fact that E is noincreasing, we get for every $t \ge 0$

$$
(3.2.6)\qquad c\, E^{\frac{2\mu+1}{2}}\, |E'|^{\frac12} \;\le\; E^{\mu+1} + c\, E(0)^{\mu}\, |E'|\ .
$$

Reporting (3.2.6) in (3.2.5) and combining (3.2.1)–(3.2.5), we get

$$
(3.2.7)\qquad \int_S^T E^{\mu+1}\, dt \;\le\; c\Big(1 + E(0)^{\mu}\Big)\, E(S) \;+\; c \int_{\Gamma_+\times]S,T[} E^{\mu} (\Delta y)^2\, d\Gamma\, dt\ .
$$

At this stage, we observe, thanks to Lemma 2.2, that it suffices to obtain judicious estimates of the last term of the right hand side of (3.2.7) in terms of $E(S)$ and $\int_S^T E^{\mu+1} dt$ to complete the proof of Theorems 1.1, 1.2.

Step 2. Let $h \in (W^{2,\infty}(\Omega))^N$ such that

$$
(3.2.8)\qquad h = \nu \ \text{ on } \Gamma_+\,, \qquad h\cdot\nu \ge 0 \ \text{ on } \Gamma\,, \qquad h = 0 \ \text{ in } \Omega\setminus\omega_1
$$

where ω_1 is another neighborhood of Γ_+ strictly contained in ω.

Choose $\delta = 0$ and $q = h$ in (2.5). Following Zuazua [27], we can show that there exists a positive constant c_0 depending only on Ω and ω such that

$$
\begin{aligned}
&\bar c \int_{\Gamma_+\times]S,T[} E^{\mu} (\Delta y)^2\, d\Gamma\, dt \;\le \\
&\qquad \le\; \bar c \int_{\Gamma\times]S,T[} E^{\mu} (h\cdot\nu)\, (\Delta y)^2\, d\Gamma\, dt \\
(3.2.9)\qquad &\qquad \le\; c_0 \int_{\omega_1\times]S,T[} \Big\{ |y'|^2 + |\Delta y|^2 \Big\} E^{\mu}\, dx\, dt \;+\; 2\,\bar c \Big[\int_\Omega y' h\cdot\nabla y\, dx\ E^{\mu} \Big]_S^T \\
&\qquad\quad -\, 2\,\mu\,\bar c \int_{\Omega\times]S,T[} E^{\mu-1} E' y' h\cdot\nabla y\, dx\, dt \;+\; 2\,\bar c \int_{\Omega\times]S,T[} a y' h\cdot\nabla y\, E^{\mu}\, dx\, dt
\end{aligned}
$$

where $\bar c$ is the constant in (3.2.7).

Simple calculations using Young inequality show that

$$\left| 2\,\bar{c} \int_\Omega y' h\cdot\nabla y \, dx \; E^\mu \Big]_S^T \right| \le c\,E(0)^\mu E(S) \tag{3.2.10}$$

and

$$\left| 2\,\mu\,\bar{c} \int_{\Omega\times]S,T[} E^{\mu-1} E' y' h\cdot\nabla y \; dx\,dt \right| \le c\,E(0)^\mu E(S)\ . \tag{3.2.11}$$

In the last term of the right hand side of (3.2.9), we proceed as in (3.2.5) to get

$$\left| 2\,\bar{c} \int_{\Omega\times]S,T[} a y' h\cdot\nabla y\, E^\mu \; dx\,dt \right| \le c \int_S^T E^{\frac{2\mu+1}{2}} \, |E'|^{\frac12}\, dt\ . \tag{3.2.12}$$

Thanks to (3.2.6) we easily derive from (3.2.12) that

$$\left| 2\,\bar{c} \int_{\Omega\times]S,T[} a y' h\cdot\nabla y\, E^\mu \; dx\,dt \right| \le \frac{1}{2} \int_S^T E^{\mu+1} dt \,+\, c\,E(0)^\mu E(S)\ . \tag{3.2.13}$$

Combining (3.2.9) to (3.2.13) and reporting the obtained result in (3.2.7) yield

$$\int_S^T E^{\mu+1}\, dt \;\le\; c\,E(0)^\mu E(S) \,+\, c \int_{\omega_1\times]S,T[} \Big\{ |y'|^2 + |\Delta y|^2 \Big\} E^\mu \; dx\,dt\ . \tag{3.2.14}$$

Thanks to Lemma 2.2, it remains to get rid of the second term in the right hand side of (3.2.14) to complete the proof of Theorems 1.1, 1.2.

Step 3. Let η be a function satisfying

$$\eta \in W^{2,\infty}(\Omega)\,, \qquad 0\le\eta\le 1\,, \qquad \eta = 1 \;\text{ in }\; \omega_1\,, \qquad \eta = 0 \;\text{ in }\; \Omega\backslash\omega\ . \tag{3.2.15}$$

Applying (2.6) with $\xi = \eta^4$, (we choose $\xi = \eta^4$ instead of $\xi = \eta$ as in [16, 17] or [25, 26] to make our computations easy to understand) we find

$$\int_{\Omega\times]S,T[} \eta^4\, |\Delta y|^2 E^\mu \, dx\,dt \;= \tag{3.2.16}$$

$$\begin{aligned}
= -\int_\Omega y' \eta^4 y \, dx \; E^\mu \Big]_S^T &+ \int_{\Omega\times]S,T[} \eta^4 |y'|^2 E^\mu \, dx\,dt \,+\, \mu \int_{\Omega\times]S,T[} E^{\mu-1} E' y' y\, \eta^2 \, dx\,dt \\
&- 8 \int_{\Omega\times]S,T[} \eta^3 \Delta y (\nabla y\cdot\nabla\eta) E^\mu \, dx\,dt \,-\, 2 \int_{\Omega\times]S,T[} y\,\eta^2 \Delta y\, \Delta(\eta^2) E^\mu \, dx\,dt \\
&- 8 \int_{\Omega\times]S,T[} y\,\eta^2 \Delta y\, |\nabla\eta|^2 E^\mu \, dx\,dt \,-\, \int_{\Omega\times]S,T[} a y' \eta^4 y E^\mu \, dx\,dt\ .
\end{aligned}$$

It follows from (3.2.16) that for all $\varepsilon > 0$,

$$\begin{aligned}\tilde{c}\int_{\Omega\times]S,T[}\eta^4|\Delta y|^2E^\mu\,dx\,dt \ \le\\ \le\ C_\varepsilon\,E(0)^\mu E(S)\,+\,\varepsilon\,\tilde{c}\int_{\Omega\times]S,T[}\eta^4|\Delta y|^2E^\mu\,dx\,dt\\ +\ \varepsilon\int_S^T E^{\mu+1}dt\,+\,C_\varepsilon\int_{\omega\times]S,T[}\Big\{|y'|^2+\eta^2|\nabla y|^2+|y|^2\Big\}E^\mu\,dx\,dt\ ,\end{aligned}\tag{3.2.17}$$

where $\tilde{c}$ is the constant appearing in the right hand side of (3.2.14).

On the other hand Green's formula yields

$$\begin{aligned}C_\varepsilon\int_{\omega\times]S,T[}\eta^2|\nabla y|^2E^\mu\,dx\,dt\ \le\\ \le\ \varepsilon\,\tilde{c}\int_{\Omega\times]S,T[}\eta^4|\Delta y|^2E^\mu\,dx\,dt\,+\,C_\varepsilon\int_{\omega\times]S,T[}|y|^2E^\mu\,dx\,dt\ .\end{aligned}\tag{3.2.18}$$

Therefore

$$\begin{aligned}\tilde{c}\int_{\Omega\times]S,T[}\eta^4|\Delta y|^2E^\mu\,dx\,dt\ \le\ c\,E(0)^\mu E(S)+\frac{1}{2}\int_S^T E^{\mu+1}dt\\ +\ c\int_{\omega\times]S,T[}|y'|^2E^\mu\,dx\,dt\,+\,c\int_{\omega\times]S,T[}|y|^2E^\mu\,dx\,dt\,.\end{aligned}\tag{3.2.19}$$

Reporting (3.2.19) in (3.2.14), we find

$$\int_S^T E^{\mu+1}dt\ \le\ c\,E(0)^\mu E(S)+c\int_{\omega\times]S,T[}|y'|^2E^\mu\,dx\,dt+c\int_{\omega\times]S,T[}|y|^2E^\mu\,dx\,dt\ .\tag{3.2.20}$$

Now, we are going to use a special multiplier to absorb the third term in the right hand side of (3.2.20).

Step 4. Introduce $z(t)\in H_0^2(\Omega)$ solution of

$$\begin{cases}\Delta^2 z=\chi_\omega y & \text{in}\ \ \Omega\ ,\\ z=\dfrac{\partial z}{\partial\nu}=0 & \text{on}\ \ \Gamma\ ,\end{cases}\tag{3.2.21}$$

where χ_ω is the characteristic function of ω.

It is easy to check that $z'=\frac{dz}{dt}$ satisfies

$$\begin{cases}\Delta^2 z'=\chi_\omega y' & \text{in}\ \ \Omega\ ,\\ z'=\dfrac{\partial z'}{\partial\nu}=0 & \text{on}\ \ \Gamma\ .\end{cases}\tag{3.2.22}$$

Some elementary calculations show that

$$\begin{aligned} &\int_\Omega |\Delta z|^2\,dx \;\le\; c\int_\omega |y|^2\,dx\,, \qquad \int_\Omega |\Delta z'|^2\,dx \;\le\; c\int_\omega |y'|^2\,dx \\ &\int_\Omega \Delta z\,\Delta y\,dx \;=\; \int_\omega |y|^2\,dx\;. \end{aligned} \tag{3.2.23}$$

Now multiplying the first equation of (1.3) by zE^μ, integrating by parts over $\Omega\times]S,T[$ and using the second line of (3.2.23); we find

$$\begin{aligned} \int_{\omega\times]S,T[} |y|^2 E^\mu\,dx\,dt \;=\; &-\int_\Omega y'z\,dx\;E^\mu\Big]_S^T + \int_{\Omega\times]S,T[} E^\mu\,y'z'\,dx\,dt \\ &+\mu\int_{\Omega\times]S,T[} E^{\mu-1}E'y'z\,dx\,dt - \int_{\Omega\times]S,T[} ay'zE^\mu\,dx\,dt\;, \end{aligned} \tag{3.2.24}$$

from which we derive that

$$\check{c}\int_{\omega\times]S,T[} |y|^2E^\mu\,dx\,dt \;\le\; c\,E(0)^\mu E(S) + \frac{1}{2}\int_S^T E^{\mu+1}dt + c\int_{\omega\times]S,T[} |y'|^2E^\mu\,dx\,dt\;, \tag{3.2.25}$$

where $\check{c}$ stands for the constant in (3.2.20).

Reporting (3.2.25) in (3.2.20) we get

$$\int_S^T E^{\mu+1}dt \;\le\; c\,E(0)^\mu E(S) + c\int_{\omega\times]S,T[} |y'|^2E^\mu\,dx\,dt\;. \tag{3.2.26}$$

From now on we separate the proofs of Theorems 1.1, 1.2.

Proof of Theorem 1.1 (continued): Since the localizing coefficient a satisfies (1.1), it is easy to check that

$$\int_{\omega\times]S,T[} |y'|^2E^\mu\,dx\,dt \;\le\; c\,E(0)^\mu E(S)\;. \tag{3.2.27}$$

Reporting (3.2.27) in (3.2.26), taking the limit as $T\to\infty$, and applying Lemma 2.2, we obtain (1.10) and the proof of Theorem 1.1 is complete. ∎

It remains to complete the proof of Theorem 1.2.

Proof of Theorem 1.2 (continued): To this end we shall absorb the second term in (3.2.26). Applying (2.7), we find

$$\begin{aligned} &\int_S^T E^{\mu+1}dt \;\le \\ &\quad\le\; c\,E(0)^\mu E(S) \;+\; c\,|(1/\alpha)|_p^{\frac{p}{p+1}}\,F_0^{\frac{N}{2(p+1)}}\int_S^T |E'|^{\frac{p}{p+1}}E^{\mu+\frac{4-N}{4(p+1)}}\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if}\;\;1\le N\le 3\;, \end{aligned} \tag{3.2.28}$$

$$
\begin{aligned}
&\int_S^T E^{\mu+1} dt \;\le \\
&\le\; c\,E(0)^{\mu} E(S) \;+\; c_r |(1/\alpha)|_p^{\frac{p(r-1)}{r(p+1)}} F_0^{\frac{2(r+p)}{r(p+1)}} \int_S^T E^{\mu}\, |E'|^{\frac{p(r-1)}{r(p+1)}}\,, \\
&\hfill \forall\, 1<r<\infty, \quad \text{if } N=4\,,
\end{aligned} \tag{3.2.29}
$$

and

$$
\begin{aligned}
&\int_S^T E^{\mu+1} dt \;\le \\
&\le\; c\,E(0)^{\mu} E(S) \;+\; c|(1/\alpha)|_p^{\frac{4p}{N+4p}} F_0^{\frac{2N}{N+4p}} \int_S^T E^{\mu}\, |E'|^{\frac{4p}{N+4p}}, \quad \text{if } N \ge 5\,.
\end{aligned} \tag{3.2.30}
$$

Choosing $\mu = N/4p$ in (3.2.28) and (3.2.30), and using Young inequality, we find

$$
\begin{aligned}
&\int_S^T E^{\frac{N}{4p}+1} dt \;\le \\
&\le\; c\,E(0)^{\frac{N}{4p}} E(S) + \frac{1}{p+1} \int_S^T E^{\frac{N}{4p}+1} dt \;+\; c\,|(1/\alpha)|_p\, F_0^{\frac{N}{2p}} E(S)\,, \\
&\hfill \text{if } 1 \le N \le 3\,,
\end{aligned} \tag{3.2.31}
$$

and

$$
\begin{aligned}
&\int_S^T E^{\frac{N}{4p}+1} dt \;\le \\
&\le\; c\,E(0)^{\frac{N}{4p}} E(S) + \frac{N}{N+4p} \int_S^T E^{\frac{N}{4p}+1} dt \;+\; c\,|(1/\alpha)|_p\, F_0^{\frac{N}{2p}} E(S)\,, \\
&\hfill \text{if } N \ge 5\,.
\end{aligned} \tag{3.2.32}
$$

Therefore letting $T \to \infty$, and applying Lemma 2.2, one gets (1.11). It remains to prove (1.12). For this purpose, choosing $\mu = (r+p)/p(r-1)$ in (3.2.29), and using Young inequality, we get

$$
\begin{aligned}
&\int_S^T E^{\frac{(r+p)}{p(r-1)}+1} dt \;\le \\
&\le\; c\,E(0)^{\frac{(r+p)}{p(r-1)}} E(S) + c_r |(1/\alpha)|_p\, F_0^{\frac{2(r+p)}{p(r-1)}} E(S) + \frac{r+p}{r(p+1)} \int_S^T E^{\frac{(r+p)}{p(r-1)}+1} dt\,, \\
&\hfill \forall\, 1<r<\infty, \quad \text{if } N=4\,,
\end{aligned} \tag{3.2.33}
$$

from which one derives (1.12) by the application of Lemma 2.2, and the proof of Theorem 1.2 is complete. ∎

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