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ON THE CONCENTRATION OF SOLUTIONS OF SINGULARLY PERTURBED HAMILTONIAN SYSTEMS IN $\mathbb{R}^N$

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Abstract: We consider a system of the form $-\varepsilon^2\Delta u + a(x)\,u = g(v)$, $-\varepsilon^2\Delta v + a(x)\,v = f(u)$ in $\mathbb{R}^N$, $N \geqslant 3$ and f and g are power-type nonlinearities having superlinear and subcritical growth at infinity. We establish that the least energy solutions to such a system concentrate at global minimum points of a as $\varepsilon \to 0$.

1 – Introduction

We consider

$$(1.1) \qquad -\varepsilon^2\Delta u + a(x)\,u \;=\; g(v)\,, \quad -\varepsilon^2\Delta v + a(x)\,v \;=\; f(u)\;, \qquad u,v \in H^1(\mathbb{R}^N)\;,$$

where $a(x) \in C(\mathbb{R}^N)$ is such that

$$(1.2) \qquad 0 \;<\; a(0) \;=\; \min_{x\in\mathbb{R}^N} a(x) \;<\; \liminf_{|x|\to\infty} a(x) \;\in\;]0,+\infty]\;.$$

Concerning f and g, we will assume the following.

(H) $f(0)=0=f'(0)$, $g(0)=0=g'(0)$ and there exist real numbers $\ell_1, \ell_2 > 0$ and $p,q > 2$ such that $\frac{1}{p}+\frac{1}{q} > \frac{N-2}{N}$ and

$$(1.3) \qquad \lim_{|s|\to\infty} \frac{f'(s)}{|s|^{p-2}} \;=\; \ell_1\;, \qquad \lim_{|s|\to\infty} \frac{g'(s)}{|s|^{q-2}} \;=\; \ell_2\;.$$

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Moreover, for some $\delta > 0$ and every $s \in \mathbb{R}$, $s \neq 0$,

$$f(s)s \geqslant (2+\delta)\,F(s) > 0 \quad \text{and} \quad f^2(s) \leqslant 2\,f'(s)\,F(s)\ , \tag{1.4}$$

and

$$g(s)s \geqslant (2+\delta)\,G(s) > 0 \quad \text{and} \quad g^2(s) \leqslant 2\,g'(s)\,G(s)\ , \tag{1.5}$$

where $F(s) := \int_0^s f(\sigma)\,d\sigma$ and $G(s) := \int_0^s g(\sigma)\,d\sigma$. We look for positive solutions of (1.1), and therefore we let $f(s) = g(s) = 0$ for $s \leqslant 0$.

Our motivation for the study of such a problem goes back to the works of Rabinowitz [10] and Wang [13] concerning the single equation

$$-\varepsilon^2 \Delta u + a(x)\,u \;=\; f(u) \qquad \text{in} \quad \mathbb{R}^N\,. \tag{1.6}$$

In [10], Rabinowitz uses a mountain-pass type argument to find a ground state for $\varepsilon > 0$ sufficiently small, when a satisfies the global assumption

$$\liminf_{|x|\to\infty} a(x) \;>\; \inf_{x\in\mathbb{R}^N} a(x) \;>\; 0\ .$$

In [13], Wang proves that the mountain-pass solutions found in [10] concentrate around a global minimum of a as ε tends to 0. A local version of these results was proved by Del Pino and Felmer in [6, 7].

It is known that the extension of these results to systems such as (1.1) presents some difficulties. Roughly, they are due to the strongly indefinite character of the energy functional associated to the system, that is

$$\int_{\mathbb{R}^N} \Big(\varepsilon^2 \langle \nabla u, \nabla v\rangle + a(x)\,uv - F(u) - G(v)\Big)\,dx\ .$$

Other difficulties have to do with the assumption that $\frac{1}{P} + \frac{1}{q} > \frac{N-2}{N}$ (which is more general than to assume that $p, q < 2N/(N-2)$) and to the unclear picture of the "limit problem" associated to (1.1) (nonexistence and uniqueness results). We refer the reader to the Introduction in [3, 8] for more details on this.

To the best of our knowledge, a first approach to the singularly perturbed system in a bounded domain, with Neumann boundary, and $a(x) \equiv 1$ appeared in [3] by means of a dual variational formulation of the problem (restricted to the case where $f(s) = s^{p-1}$, $g(s) = s^{q-1}$, $2 < p,\, q < 2N/(N-2)$). In [1], the authors employ a similar variational setting to the system (1.1) in $\mathbb{R}^N$ which allows to consider two different functions $a(x)$ and $b(x)$ in the equations of (1.1).

Related results for Hamiltonian systems can be found in [2]. A more direct approach was proposed in [11] and was subsequently developed in [8, 9]; in these papers the authors extend to system (1.1) (in a bounded domain, with Dirichlet or Neumann boundary conditions, and $a(x) \equiv 1$) the elementary point of view of the paper [6] for the single equation (1.6).

In the present paper our goal is to prove the following.

Theorem 1.1. *Under assumptions* (1.2) *and* (H) *and for any sufficiently small* $\varepsilon > 0$, *the system admits a ground-state solution* $(u_\varepsilon, v_\varepsilon) \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, $u_\varepsilon, v_\varepsilon > 0$ *with the following properties: both functions* u_ε *and* v_ε *attain their maximum value at some unique and common point* $x_\varepsilon \in \mathbb{R}^N$*; the sequence* $(x_\varepsilon)_\varepsilon$ *is bounded and, whenever it converges to* $x_0 \in \mathbb{R}^N$ *along a subsequence, we have that* $a(x_0) = a(0) = \min_{x\in\mathbb{R}^N} a(x)$.

The proof of Theorem 1.1 is postponed to Section 4. Before we prove it we study some auxiliary problems involving appropriate truncated functions in the place of $a(x)$; in particular, Section 3 contains the core of the proof of our main result, where we deal with the indefinite sign of the quadratic term $\int_{\mathbb{R}^N} \big(\langle \nabla u, \nabla v\rangle + a(x)\, uv\big)$. We point out that this is in contrast with the single equation case, where no such problem arises.

2 – Preliminaries

In this section we establish some preliminary results needed for the proof of Theorem 1.1. Given $f, g \in C^2(\mathbb{R};\mathbb{R})$ satisfying condition (H), we consider the system

$$-\Delta u + a(x)\, u \;=\; g(v)\,, \quad -\Delta v + a(x)\, v = f(u)\;, \qquad u, v \in H^1(\mathbb{R}^N)\;. \tag{2.1}$$

Let X be the Hilbert space defined by $X = H^1(\mathbb{R}^N) \cap \big\{u \colon \int a(x)\, u^2 < \infty\big\}$ with the inner product

$$\langle u, v\rangle \;=\; \int \big(\langle \nabla u, \nabla v\rangle + a(x)\, uv\big)\;,$$

whose associated norm we denote by $\|\cdot\|$. Under the hypothesis (1.2), X is continuously embedded in $H^1(\mathbb{R}^N)$.

For $(u, v) \in E \doteq X{\times}X$ we define the energy functional

$$I(u, v) \;=\; \int \langle \nabla u, \nabla v\rangle \;+ \int a(x)\, uv \;- \int F(u) \;- \int G(v)\;, \tag{2.2}$$

where $F(s) \doteq \int_0^s f(t)\,dt$ and $G(s) \doteq \int_0^s g(t)\,dt$. Under our assumptions, it can happen that, for instance, $q > 2N/(N-2) > p$ and so I may not be well defined for $(u,v) \in E$. In order to overcome this problem, as explained in [8] and [11], we may assume without loss of generality that the numbers p, q in (H) are such that $2 < p = q < 2N/(N-2)$. This is due to the fact that in our case we will always work with ground-states having a bounded Morse index; these solutions are a priori bounded for the L^∞ norm and this bound is unchanged whenever we truncate f and g in a neighborhood of infinity in such a way that (H) is still satisfied for the modified functions with $2 < p = q < 2N/(N-2)$ in condition (1.3) (see Theorem 1.1 in [11] for details). Taking this remark into account, the energy functional I is well defined and belongs to $C^2(E,\mathbb{R})$. Furthermore,

$$I'(u,v)(\phi,\psi) \;=\; \langle u,\psi\rangle + \langle v,\phi\rangle - \int \big(f(u)\phi + g(v)\psi\big)\;, \qquad \forall\,(\phi,\psi) \in E\;.$$

Thus, every critical point of I corresponds to a solution of problem (2.1).

We start by proving a lemma which will play a significant role in the sequel.

Lemma 2.1. *Let (u_n,v_n) be a Palais–Smale sequence for the functional I, namely*

$$I(u_n,v_n) \;\to\; c \in \mathbb{R}^+$$

and

$$\mu_n := \sup\Big\{\big|I'(u_n,v_n)(\phi,\psi)\big|,\;\; \phi,\psi \in X,\;\; \|\phi\| + \|\psi\| \leqslant 1\Big\} \;\to\; 0\;.$$

Then (u_n,v_n) is bounded and

$$\sup\Big\{I\big(s(u_n,v_n) + (\phi,-\phi)\big)\colon\;\; s \in \mathbb{R}_0^+,\;\; \phi \in X\Big\} \;=\; I(u_n,v_n) + \mathrm{O}(\mu_n^2)\;.$$

Proof: 1. Since

$$2\,I(u_n,v_n) \;=\; I'(u_n,v_n)(u_n,v_n) + \int \big(f(u_n)u_n - 2\,F(u_n)\big) + \int \big(g(v_n)v_n - 2\,G(v_n)\big)$$

we have from (1.4) and (1.5) that

$$\int \big(f(u_n)u_n + g(v_n)v_n\big) \;\leqslant\; C + \mu_n\big(\|u_n\| + \|v_n\|\big)\;. \tag{2.3}$$

From (1.4) and (1.5) again, for any $\delta > 0$ we have

$$\int \big(|u_n|^p + |v_n|^p\big) \;\leqslant\; \delta \int \big(|u_n|^2 + |v_n|^2\big) + C_\delta + \mu_n\big(\|u_n\| + \|v_n\|\big)\;. \tag{2.4}$$

Now, since

$$\|u_n\|^2 + \|v_n\|^2 = I'(u_n, v_n)(v_n, u_n) + \int \big(f(u_n)v_n + g(v_n)u_n\big) \tag{2.5}$$

we have by (H) that, for any given $\delta' > 0$,

$$\begin{aligned} \|u_n\|^2 + \|v_n\|^2 \leqslant\ & \delta' \int \big(|u_n|^2 + |v_n|^2\big) + C_{\delta'} \int \big(|u_n|^{p-1}\,|v_n| + |v_n|^{p-1}\,|u_n|\big) \\ & + \mu_n\big(\|u_n\| + \|v_n\|\big)\ . \end{aligned} \tag{2.6}$$

Combining (2.4) and (2.6) yields that

$$\|u_n\|^2 + \|v_n\|^2 \leqslant \delta \int \big(|u_n|^2 + |v_n|^2\big) + C_\delta + \mu_n\big(\|u_n\| + \|v_n\|\big)\ .$$

For a sufficiently small δ, this implies that $\|u_n\| + \|v_n\| \leqslant C$.

2. Let $s_n \geqslant 0$, $t_n \in \mathbb{R}$ and $\phi_n \in X$ be such that $\|\phi_n\| = 1$ and

$$\sup\Big\{I\big(s(u_n, v_n) + (\phi, -\phi)\big)\colon\ s \in \mathbb{R}_0^+,\ \phi \in X\Big\} = I\big(s_n(u_n, v_n) + t_n(\phi_n, -\phi_n)\big)\ .$$

We show next that the sequences (s_n) and (t_n) are bounded. In view of a contradiction, assume first that *both* sequences are unbounded along a subsequence. We observe that

$$\begin{aligned} I_n &:= I\big(s_n(u_n, v_n) + t_n(\phi_n, -\phi_n)\big) \\ &= s_n^2\mathrm{O}(1) - t_n^2 + t_n s_n \mathrm{O}(1) - \int F(s_n u_n + t_n\phi_n) - \int G(s_n v_n - t_n\phi_n)\ . \end{aligned} \tag{2.7}$$

Case a). Suppose $|t_n|/s_n \to \infty$. Then we see from (2.7) that $I_n \leqslant t_n^2(\mathrm{o}(1) - 1) \to -\infty$ and this contradicts the fact that $\liminf I_n > 0$.

Case b). Suppose $|t_n|/s_n \to 0$. From (2.7) we have that

$$0 < I_n \leqslant s_n^p \left(\mathrm{o}(1) - \int \frac{F(s_n u_n + t_n\phi_n)}{s_n^p}\right).$$

This implies that $\int \frac{F(s_n u_n + t_n\phi_n)}{s_n^p} \to 0$. We decompose $\int = \int_{A_n} + \int_{\mathbb{R}^N \setminus A_n}$ where $A_n = \big\{x\colon\ |s_n u_n(x) + t_n\phi_n(x)| \geqslant 1\big\}$. From (1.3),

$$\int_{A_n} \frac{F(s_n u_n + t_n\phi_n)}{s_n^p} \geqslant C \int_{A_n} \left|u_n + \frac{t_n}{s_n}\,\phi_n\right|^p,$$

for some positive constant. Thus, $\int_{A_n} |u_n + \frac{t_n}{s_n}\phi_n|^p \to 0$. Consequently,

$$\int_{A_n} |u_n|^p \;\to\; 0 \;. \tag{2.8}$$

On the other hand, over $\mathbb{R}^N \backslash A_n$ we have that $|u_n| \leqslant \frac{1}{s_n} + \frac{|t_n|}{s_n}\,|\phi_n|$ and so

$$\int_{\mathbb{R}^N\backslash A_n} |u_n|^p \;\to\; 0 \;. \tag{2.9}$$

Hence, from (2.8) and (2.9) we have $\int |u_n|^p \to 0$. In a similar way we deduce that $\int |v_n|^p \to 0$. It then follows from (2.5) that $\|u_n\| + \|v_n\| \to 0$ and this contradicts the fact that $\liminf I(u_n, v_n) > 0$.

Case c). Suppose $|t_n|/s_n \to \ell \in \mathbb{R}^+$. Similarly to case b) we deduce that

$$\int \left|u_n + \frac{t_n}{s_n}\,\phi_n\right|^p + \int \left|v_n - \frac{t_n}{s_n}\,\phi_n\right|^p \;\to\; 0$$

and so $\int |u_n + v_n|^p \to 0$. As a consequence, since $\int (u_n^2 + v_n^2)$ is bounded,

$$\int \big|f(u_n)\,(u_n+v_n)\big| + \int \big|g(v_n)\,(u_n+v_n)\big| \;\to\; 0 \;.$$

Since $I'(u_n, v_n)(u_n + v_n, u_n + v_n) \to 0$ this implies that $\|u_n + v_n\| \to 0$. But then

$$2\,I(u_n,v_n) \;=\; \|u_n+v_n\|^2 - \left(\|u_n\|^2 + \|v_n\|^2 + 2\int \big(F(u_n) + G(v_n)\big)\right) \;\leqslant\; \mathrm{o}(1)$$

and this contradicts the fact that $\liminf I(u_n, v_n) > 0$.

In any of the cases a), b) and c) we arrive at a contradiction and so either (t_n) or (s_n) is bounded. Now, assume that one (and only one) of these sequences is unbounded. In case $|t_n|/s_n \to \infty$ we must have that $|t_n| \to \infty$ and we arrive at a contradiction as in case a) above; while if $|t_n|/s_n$ is bounded then $s_n \to \infty$, $t_n/s_n \to 0$ and again we get a contradiction arguing as in case b). In conclusion, both sequences (t_n) and (s_n) are bounded.

3. Once we know that (t_n) and (s_n) are bounded sequences, the rest of the proof follows as in [11] and [8]. Indeed, since $\mu_n \to 0$ and since $\liminf \int \big(f(u_n)u_n + g(v_n)v_n\big) > 0$, it follows from Theorem 3.5 in [11] or Lemma 3.3 in [8] that $s_n \to 1$. Then, as explained in [8] (cf. Lemma 3.3), we have in fact that $|s_n - 1| + |t_n| = \mathrm{O}(\mu_n)$. It remains to observe that, by Taylor's formula,

$$\begin{aligned} I_n \;=\; & I(u_n,v_n) + (s_n-1)\,I'(u_n,v_n)\,(u_n,v_n) + t_n\,I'(u_n,v_n)\,(\phi_n,-\phi_n) \\ & + \mathrm{O}\big((s_n-1)^2 + t_n^2\big)\big) \end{aligned}$$

yielding that

$$\big|I_n - I(u_n, v_n)\big| \leqslant C\,\mu_n^2 \;.$$

This completes the proof of Lemma 2.1. ∎

3 – Continuous dependence of the ground-state critical levels

For any constant $\lambda > 0$ we denote by $c(\lambda) > 0$ the ground-state critical level associated to the problem

$$-\Delta u + \lambda u = g(v), \quad -\Delta v + \lambda v = f(u)\;, \qquad u, v \in H^1(\mathbb{R}^N)\;.$$

Lemma 3.1. *The map* $\lambda \mapsto c(\lambda)$ *is continuous and increasing, and we have that* $\lim_{\lambda\to\infty} c(\lambda) = \infty$.

Proof: 1. Let I_λ denote the energy functional associated to the above system. By Theorem 3 in [12], I_λ admits indeed a least positive critical level $c(\lambda)$. By the maximum principle, any nonzero solutions u, v of the system are positive in $\mathbb{R}^N$. Moreover, according to Theorem 2 in [5], u and v are radially symmetric with respect to some (common) point of $\mathbb{R}^N$. We also recall that, as a consequence of Benci–Rabinowitz's linking theorem [4], $c(\lambda)$ is positive, uniformly in λ, as long as λ remains bounded away from the origin, and that

$$c(\lambda) \leqslant \sup\Big\{ I_\lambda\big(s(u_0, v_0) + (\phi, -\phi)\big) \colon\; s \in \mathbb{R}_0^+,\; \phi \in H^1(\mathbb{R}^N)\Big\}\;, \tag{3.1}$$

for any given functions $u_0, v_0 \in H^1(\mathbb{R}^N)$ such that $v_0 \neq -u_0$; the required compactness property of the problem is obtained by using the invariance by translation.

2. For a given $\lambda > 0$, suppose $\lambda_n \to \lambda$. Let (u_n, v_n) be ground-state solutions for I_{λ_n} with $u_n > 0$, $v_n > 0$. Fix any functions $u_0, v_0 \in H^1(\mathbb{R}^N)$ such that $v_0 \neq -u_0$. According to (3.1) we have that

$$c(\lambda_n) \leqslant I_{\lambda_n}\big(s_n(u_0, v_0) + t_n(\phi_n, -\phi_n)\big)$$

for some $s_n \in \mathbb{R}_0^+$, $t_n \in \mathbb{R}$, $\phi_n \in H^1(\mathbb{R}^N)$ with $\|\phi_n\| = 1$. By using an argument similar (in fact, easier, since u_0 and v_0 are fixed) to the one in step 2 of the proof of Lemma 2.1, this implies that (s_n) and (t_n) are bounded sequences, and thus so is $(c(\lambda_n))$. Then, as in step 1 of the proof of Lemma 2.1, this implies that (u_n) and (v_n) are bounded. In particular, we have that

$$I_\lambda(u_n, v_n) \;=\; c(\lambda_n) + (\lambda - \lambda_n) \int u_n v_n \;=\; c(\lambda_n) + \mathrm{o}(1)$$

and, for any $\phi, \psi \in H^1(\mathbb{R}^N)$,

$$\left| I'_\lambda(u_n, v_n)(\phi, \psi) \right| \;=\; |\lambda - \lambda_n| \left| \int (u_n \psi + v_n \phi) \right| \;\leqslant\; \mathrm{o}(1) \left(\|\phi\| + \|\psi\| \right) .$$

Thus, it follows from (3.1) (with $u_0 = u_n$ and $v_0 = v_n$) and Lemma 2.1 that

$$c(\lambda) \;\leqslant\; I_\lambda(u_n, v_n) + C\,(\lambda - \lambda_n)^2 \;=\; c(\lambda_n) + \mathrm{o}(1) \ ,$$

yielding that

$$c(\lambda) \;\leqslant\; \liminf c(\lambda_n) \ . \tag{3.2}$$

Concerning now the reversed inequality, let (u, v) be the ground-state solution associated to I_λ, with $u > 0$, $v > 0$. Then

$$I_{\lambda_n}(u, v) \;=\; I_\lambda(u, v) + (\lambda_n - \lambda) \int uv$$

and, for any $\phi, \psi \in H^1(\mathbb{R}^N)$,

$$I'_{\lambda_n}(u, v)(\psi, \psi) \;=\; (\lambda_n - \lambda) \int (u\psi + v\phi) \ .$$

Then (3.1) (with $u_0 = u$ and $v_0 = v$) and a minor modification of the proof of Lemma 2.1 implies that

$$c(\lambda_n) \;\leqslant\; I_{\lambda_n}(u, v) + C(\lambda_n - \lambda)^2 \ .$$

As a consequence,

$$c(\lambda_n) \;\leqslant\; c(\lambda) + (\lambda_n - \lambda) \left(C|\lambda_n - \lambda| + \int uv \right) . \tag{3.3}$$

In particular,

$$\limsup c(\lambda_n) \leqslant c(\lambda) \ . \tag{3.4}$$

By combining (3.2) and (3.4) we conclude that $c(\lambda_n) \to c(\lambda)$.

3. The inequality in (3.3) also shows that $c(\lambda_n) < c(\lambda)$ whenever $\lambda_n < \lambda$ and λ is close enough to λ, that is, the map $\lambda \mapsto c(\lambda)$ is locally increasing. Since this map is continuous, we conclude that this map is increasing in $\mathbb{R}^+$.

4. Finally, assume by contradiction that $c(\lambda_n) = I_n(u_n, v_n)$ remains bounded for some sequence $\lambda_n \to \infty$. The argument in the first step of the proof of Lemma 2.1 shows that $\int |\nabla u_n|^2 + \lambda_n \int u_n^2$ is bounded and similarly for v_n. Thus $\int (u_n^2 + v_n^2) \to 0$ and $\int (|u_n|^{2^\star} + |v_n|^{2^\star})$ is bounded. By interpolation, $\int (|u_n|^p + |v_n|^p) \to 0$. Thus

$$2\, I_n(u_n, v_n) = \int \big(f(u_n) u_n - 2\, F(u_n)\big) + \int \big(g(v_n) v_n - 2\, G(v_n)\big) \;\to\; 0 \;,$$

and this contradicts the fact that $\liminf I(u_n, v_n) > 0$. ∎

We would like to state a result similar to Lemma 3.1 in the case where λ is no longer a constant function. To that purpose, we consider first the case where we deal with functions having a finite limit at infinity. Namely, let $b \in C(\mathbb{R}^N)$ be such that

$$b(x) \geqslant \bar{b} > 0 \quad \forall x \qquad \text{and} \qquad b_\infty := \lim_{|x| \to \infty} b(x) \in \mathbb{R} \;. \tag{3.5}$$

We let

$$I_b(u, v) = \int \langle \nabla u, \nabla v \rangle + \int b(x)\, uv - \int F(u) - \int G(v)$$

and we denote by I_∞ the corresponding functional with b_∞ in place of $b(x)$. Of course, here we work in the space $X = H^1(\mathbb{R}^N) \cap \big\{u \colon \int b(x)\, u^2 < \infty\big\}$.

Lemma 3.2. *Under assumptions* (3.5), *the Palais–Smale condition holds for* I_b *at any level* $0 < c < c(b_\infty)$.

Proof: This follows from standard arguments. Indeed, let (u_n, v_n) be such that $I_b(u_n, v_n) \to c \in \,]0, c(b_\infty)[$ and $I'_b(u_n, v_n) \to 0$. Arguing as in the first step of the proof of Lemma 2.1 we see that, up to a subsequence, $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in X. Clearly, $I'_b(u, v) = 0$. In particular,

$$2\, I_b(u, v) = \int \big(f(u) u - 2\, F(u)\big) + \int \big(g(v) v - 2\, G(v)\big) \geqslant 0 \;.$$

Next we observe that

$$\int F(u_n - u) = \int F(u_n) - \int F(u) + \mathrm{o}(1) \tag{3.6}$$

and similarly for $\int G(v_n - v)$. Indeed, we have strong convergence $u_n \to u$ in $H^1_{\rm loc}(\mathbb{R}^N)$, while, for any given $R>0$,

$$\int_{|x|\geqslant R} \big|F(u_n - u) - F(u_n)\big| \;\leqslant\; C\int_{|x|\geqslant R} \big(|u_n u| + u^2 + |u_n|^{p-1}|u| + |u|^p\big) \;=\; {\rm o}(1)$$

as $R\to\infty$. Similarly, since $\big|f(u_n - u) - f(u_n) + f(u)\big|^{p/(p-1)} \to 0$ in $L^1_{\rm loc}(\mathbb{R}^N)$, we have that, for any $\phi \in X$,

$$\left|\int f(u_n - u)\,\phi - \int f(u_n)\,\phi + \int f(u)\,\phi\right| \;\leqslant\; {\rm o}(1)\,\|\phi\| \tag{3.7}$$

and similarly for

$$\left|\int g(v_n - v)\,\phi - \int g(v_n)\,\phi + \int g(v)\,\psi\right| .$$

(A similar argument can be found for instance in [14], Lemma 8.1.)

Now, let $\overline{u}_n = u_n - u$ and $\overline{v}_n = v_n - v$. It follows from (3.6) that

$$I_b(\overline{u}_n, \overline{v}_n) \;=\; I_b(u_n, v_n) - I_b(u,v) + {\rm o}(1) \;\leqslant\; c + {\rm o}(1)\ ,$$

while (3.7) implies that

$$I_b'(\overline{u}_n, \overline{v}_n)(\phi,\psi) \;=\; I_b'(u_n, v_n)(\phi,\psi) - I_b'(u,v)(\phi,\psi) + {\rm o}(1) \;=\; {\rm o}(1)\ ,$$

uniformly for bounded ϕ and ψ. Since moreover $\overline{u}_n \rightharpoonup 0$, $\overline{v}_n \rightharpoonup 0$ weakly in X and since $b_\infty = \lim_{|x|\to\infty} b(x)$, a similar conclusion holds for $I_\infty(\overline{u}_n, \overline{v}_n)$ and $I'_\infty(\overline{u}_n, \overline{v}_n)$. Thus, in case $\liminf I_\infty(\overline{u}_n, \overline{v}_n) > 0$, Lemma 2.1 and (3.1) imply that

$$c(b_\infty) \;\leqslant\; I_\infty(\overline{u}_n, \overline{v}_n) + {\rm o}(1) \;\leqslant\; c + {\rm o}(1)\ .$$

This contradicts the assumption that $c < c(b_\infty)$. Thus $\liminf I_\infty(\overline{u}_n, \overline{v}_n) \leqslant 0$ and then the argument in the first step of the proof of Lemma 2.1 shows that $\liminf(\|\overline{u}_n\| + \|\overline{v}_n\|) = 0$. ∎

Lemma 3.3. *Under assumptions* (3.5), *suppose there exist* $u \neq -v$ *such that*

$$\sup\Big\{I_b\big(s(u,v) + (\phi,-\phi)\big)\colon\ s\in\mathbb{R}_0^+,\ \phi\in X\Big\} \;<\; c(b_\infty)\ . \tag{3.8}$$

Then I_b *admits a ground-state critical level* c_b *and* $c(\overline{b}) \leqslant c_b$.

Proof: It follows from Lemma 3.2 and [4] that I_b admits a critical point (u_1, v_1) such that

$$0 < I_b(u_1, v_1) \leqslant \sup\Big\{ I_b\big(s(u,v) + (\phi - \phi)\big) \colon\ s \in \mathbb{R}^+,\ \phi \in X \Big\} .$$

In particular,

$$c_b := \inf\Big\{ I_b(u,v) \colon\ (u,v) \neq (0,0),\ I_b'(u,v) = 0 \Big\} < c(b_\infty) . \tag{3.9}$$

But then, again by Lemma 3.2 and Theorem 3 in [12] (namely, its argument in page 1460), the infimum in (3.9) is actually a minimum, and it follows that I_b admits a ground-state critical level c_b.

Assume by contradiction that $c(\bar{b}) > c_b$. For $t \in [0,1]$, let $b_t(x) := (1-t)\, b(x) + t\bar{b}$ and denote by c_t the corresponding ground-state level. As in Lemma 3.1, for t close to zero, c_t is well-defined and $c_t < c((1-t)b_\infty + t\bar{b})$; moreover, c_t decreases with t. As a consequence, using Lemma 3.1, if t is close to zero,

$$c_t \leqslant c_b < c(\bar{b}) \leqslant c\big((1-t)\, b_\infty + t\bar{b}\big) .$$

Using a continuation argument we conclude that $c_t < c((1-t)\, b_\infty + t\bar{b})$ for every $t \in [0,1]$. This is a contradiction for $t = 1$, since $c_1 = c(\bar{b})$. ∎

Lemma 3.4. *Let $b(x) \geqslant \bar{b} > 0$ and suppose (u_n, v_n) is a Palais–Smale sequence for I_b with $\liminf I_b(u_n, v_n) > 0$. Then $c(\bar{b}) \leqslant I_b(u_n, v_n) + \mathrm{o}(1)$.*

Proof: Denote

$$I_n := \sup\Big\{ I_b\big(s(u_n, v_n) + (\phi, -\phi)\big) \colon\ s \in \mathbb{R}_0^+,\ \phi \in X \Big\} .$$

We know from Lemma 2.1 that $I_n = I_b(u_n, v_n) + \mathrm{o}(1)$ and that (u_n, v_n) is bounded. Moreover, according to Lemma 3.1, we can fix M so large that $I_n < c(M)$ for every n. For a given sequence $R_n \to \infty$, let $b_n(x)$ be a continuous function such that $b_n(x) = b(x)$ if $|x| \leqslant R_n$ and $b_n(x) = M$ if $|x| \geqslant R_n + 1$; we can take R_n so large that $I_{b_n}(u_n, v_n) = I_b(u_n, v_n) + \mathrm{o}(1)$ and $I'_{b_n}(u_n, v_n)(\phi, \psi) = \mathrm{o}(1)$ uniformly for all bounded functions ϕ and ψ. It follows then as in Lemma 2.1 that

$$\bar{I}_n := \sup\Big\{ I_{b_n}\big(s(u_n, v_n) + (\phi, -\phi)\big) \colon\ s \in \mathbb{R}_0^+,\ \phi \in X \Big\} = I_{b_n}(u_n, v_n) + \mathrm{o}(1) .$$

On the other hand, we see from Lemma 3.3 (applied to $b_n(x)$) that

$$c(\bar{b}) \leqslant c_{b_n} \leqslant \bar{I}_n$$

and the conclusion follows. ∎

4 – Existence and concentration of ground state solutions

Now we come back to our original problem. According to (1.2), we fix $\overline{a} \in \mathbb{R}$ such that

$$0 < a(0) = \min_{x\in\mathbb{R}^N} a(x) < \overline{a} < \liminf_{|x|\to\infty} a(x) .$$

Lemma 4.1. *Suppose (u_n, v_n) is a Palais–Smale sequence for the functional I such that $\liminf I(u_n, v_n) > 0$. If $u_n \rightharpoonup 0$ and $v_n \rightharpoonup 0$ weakly in X then $c(\overline{a}) \leqslant I(u_n, v_n) + o(1)$.*

Proof: Fix R_0 such that $a(x) > \overline{a}$ for every $|x| > R_0$ and let $b(x)$ be any continuous function such that $b(x) = a(x)$ for every $|x| > R_0$ and $b(x) \geqslant \overline{a}$ for every $x \in \mathbb{R}^N$. Since $u_n \rightharpoonup 0$ and $v_n \rightharpoonup 0$, we see that $I_b(u_n, v_n) = I(u_n, v_n) + o(1)$ and (u_n, v_n) is a Palais–Smale sequence for I_b. The conclusion follows then from Lemma 3.4. ∎

Lemma 4.2. *The Palais–Smale condition holds for I at any level $0 < c < c(\overline{a})$.*

Proof: The proof follows as in Lemma 3.2, by using also Lemma 4.1. ∎

For any small $\varepsilon > 0$, we denote by I_ε (resp. I_0) the energy functional defined in (2.2) with $a(\varepsilon x)$ (resp. $a(0)$) in place of $a(x)$.

Lemma 4.3. *The functional I_ε admits a ground-state critical level c_ε and $c_\varepsilon \to c_0$ as $\varepsilon \to 0$.*

Proof: Let (u, v) be a ground-state solution for I_0. Since $I_\varepsilon(u, v) = I_0(u, v) + o(1)$ as $\varepsilon \to 0$ and also $I'_\varepsilon(u, v)(\phi, \psi) = I'_0(u, v)(\phi, \psi) + o(1)$ uniformly for bounded ϕ and ψ, a minor modification of the proof of Lemma 2.1 shows that

$$d_\varepsilon := \sup\Big\{I_\varepsilon\big(s(u, v) + (\phi, -\phi)\big) \colon\ s \in \mathbb{R}_0^+,\ \phi \in X\Big\} = I_\varepsilon(u, v) + o(1) ;$$

in particular,

$$d_\varepsilon = c_0 + o(1) \quad \text{as } \varepsilon \to 0 . \tag{4.1}$$

Moreover, according to Lemma 3.1, we know that $c_0 < c(\overline{a})$; in particular, it follows from Lemma 4.2 that I_ε satisfies the Palais–Smale condition at any positive level not larger than d_ε. It follows then as in Theorem 3 in [12] that I_ε admits a ground-state critical level $c_\varepsilon \leqslant d_\varepsilon$. Clearly, (4.1) shows that $\limsup c_\varepsilon \leqslant c_0$. Similarly we can show that $c_0 \leqslant \liminf c_\varepsilon$. ∎

According to the above lemma, for any sufficiently small $\varepsilon > 0$ we can find least energy positive solutions u_ε and v_ε for the system

$$(4.2) \qquad -\varepsilon^2 \Delta u + a(x)\, u = g(v)\,, \quad -\varepsilon^2 \Delta v + a(x)\, v = f(u)\;, \qquad u, v \in H^1(\mathbb{R}^N)\;,$$

By elliptic estimates, $u_\varepsilon, v_\varepsilon \in C^2(\mathbb{R}^N)$ and they decay at infinity. Moreover, using standard elliptic estimates we have that $\|u_\varepsilon\|_{L^\infty} + \|v_\varepsilon\|_{L^\infty} \leqslant C$ uniformly in ε.

Lemma 4.4. *Both functions u_ε and v_ε can be chosen in such a way that they attain their maximum value at some unique and common point $x_\varepsilon \in \mathbb{R}^N$; the sequence $(x_\varepsilon)_\varepsilon$ is bounded and, whenever it converges to $x_0 \in \mathbb{R}^N$ along a subsequence, we have that $a(x_0) = \min_{x \in \mathbb{R}^N} a(x)$.*

Proof: As explained in Theorem 4.1 in [11], u_ε and v_ε can be chosen with the further property that their relative Morse index is less or equal than 1. Now, suppose that for some sequence $\varepsilon_j \to 0$ we have that $\|u_{\varepsilon_j}\|_{L^\infty} = u(x_j)$ with $|x_j| \to \infty$. By letting $u_j(x) := u_{\varepsilon_j}(\varepsilon_j x + x_j)$ and $v_j(x) := v_{\varepsilon_j}(\varepsilon_j x + x_j)$, by usual arguments we have that, up to a subsequence, $u_j \rightharpoonup u$, $v_j \rightharpoonup v$ weakly in X, for some nonzero functions $u, v \in X$. We also recall that

$$(4.3) \qquad 0 \;<\; \liminf I_j(u_j, v_j) \;\leqslant\; \limsup I_j(u_j, v_j) \;<\; c(\overline{a})$$

where

$$(4.4) \qquad I_j(u_j, v_j) \;:=\; \int \langle \nabla u_j, \nabla v_j \rangle \;+\; \int a_j(x)\, u_j v_j \;-\; \int F(u_j) \;-\; \int G(v_j)$$

and $a_j(x) := a(\varepsilon_j x + x_j)$. Moreover, as proved in Proposition 1.6 in [11], the information on the relative Morse index of (u_j, v_j) yields that

$$(4.5) \qquad \forall\, \delta > 0 \quad \exists\, j_0 \in \mathbb{N},\ \exists\, R > 0\colon \quad \int_{|x| \geqslant R} a_j(x)\, (u_j^2 + v_j^2) \;\leqslant\; \delta\,, \quad \forall\, j \geqslant j_0\;.$$

So, let $b_j := \max\{\overline{a}, a_j\}$ and denote by $\overline{I}_j$ the corresponding energy functional, defined as in (4.4) with $a_j(x)$ replaced by $b_j(x)$. Then (4.5) and our assumption that $|x_j| \to \infty$ imply that (u_j, v_j) is a Palais–Smale sequence for $\overline{I}_j$ and that $\overline{I}_j(u_j, v_j) = I_j(u_j, v_j) + \mathrm{o}(1)$. Since $b_j(x) \geqslant \overline{a}$ for every x, it follows then from (a minor modification of) Lemma 3.4 that $c(\overline{a}) \leqslant \overline{I}_j(u_j, v_j) + \mathrm{o}(1)$. This is in contradiction with (4.3).

In conclusion, any maximum points x_ε of u_ε (and of v_ε as well) must remain bounded. Moreover, whenever $x_\varepsilon \to x_0 \in \mathbb{R}^N$, by the preceding argument and since $a(\varepsilon x + x_\varepsilon) \to \lambda := a(x_0)$ pointwise, we have that $c_\varepsilon \to c(\lambda)$. Thus, according to Lemma 4.3 and Lemma 3.1 we have that $\lambda = a(0)$.

The uniqueness of the maximum points of u_ε and v_ε can be deduced exactly as in Theorem 2.1 in [11]. ∎

Proof of Theorem 1.1**:** For a given sequence $R_n \to \infty$, let f_n and g_n be C^1 functions satisfying our general assumption (H), with the further property that (H) holds with $2 < p = q < 2N/(N-2)$ and moreover $f_n(s) = f(s)$ and $g_n(s) = g(s)$ for every $|s| \leqslant R_n$. For large n, the ground-state critical levels of the problems

$$-\Delta u + \overline{a}\, u = g_n(v)\,, \quad -\Delta v + \overline{a}\, v = f_n(u)\ , \qquad u, v \in H^1(\mathbb{R}^N)\ ,$$

are bounded above independently of n. On the other hand, we have proved that for every n there exists ε_n such that the conclusions of Theorem 1.1 hold for the modified problems

$$-\varepsilon^2 \Delta u + a(x)\, u = g_n(v)\,, \quad -\varepsilon^2 \Delta v + a(x)\, v = f_n(u)\ , \qquad u, v \in H^1(\mathbb{R}^N)\ ,$$

provided $0 < \varepsilon \leqslant \varepsilon_n$. Since the corresponding ground-state critical levels remain bounded independently of ε and n, it follows as in [12], page 1457, that the H^1-norms of the corresponding (rescaled) ground-state solutions are also bounded independently of ε and n. Thus, by regularity arguments (see Theorem 1 (a) in [12]), the same holds true for their L^∞-norms and the conclusion follows. ∎

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