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Bivariate Generalization of the Kummer-Beta Distribution

Generalización Bivariada de la Distribución Kummer-Beta

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Abstract

In this article, we study several properties such as marginal and conditional distributions, joint moments, and mixture representation of the bivariate generalization of the Kummer-Beta distribution. To show the behavior of the density function, we give some graphs of the density for different values of the parameters. Finally, we derive the exact and approximate distribution of the product of two random variables which are distributed jointly as bivariate Kummer-Beta. The exact distribution of the product is derived as an infinite series involving Gauss hypergeometric function, whereas the beta distribution has been used as an approximate distribution. Further, to show the closeness of the approximation, we have compared the exact distribution and the approximate distribution by using several graphs. An application of the results derived in this article is provided to visibility data from Colombia.

***Key words*:** Beta distribution, Bivariate distribution, Dirichlet distribution, Hypergeometric function, Moments, Transformation.

Resumen

En este artículo, definimos la función de densidad de la generalización bivariada de la distribución Kummer-Beta. Estudiamos algunas de sus propiedades y casos particulares, así como las distribuciones marginales y condicionales. Para ilustrar el comportamiento de la función de densidad, mostramos algunos gráficos para diferentes valores de los parámetros. Finalmente, encontramos la distribución del producto de dos variables cuya distribución conjunta es Kummer-Beta bivariada y utilizamos la distribución beta como

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una aproximación. Además, con el fin de comparar la distribución exacta y la aproximada de este producto, mostramos algunos gráficos. Se presenta una aplicación a datos climáticos sobre niebla y neblina de Colombia.

***Palabras clave*:** distribución Beta, distribución bivariada, distribución Dirichlet, función hipergeométrica, momentos, transformación.

1. Introduction

The beta random variable is often used for representing processes with natural lower and upper limits. For example, refer to Hahn & Shapiro (1967). Indeed, due to a rich variety of its density shapes, the beta distribution plays a vital role in statistical modeling. The beta distribution arises from a transformation of the F distribution and is typically used to model the distribution of order statistics. The beta distribution is useful for modeling random probabilities and proportions, particularly in the context of Bayesian analysis. Varying within $(0, 1)$ the standard beta is usually taken as the prior distribution for the proportion p and forms a conjugate family within the beta prior-Bernoulli sampling scheme. A natural univariate extension of the beta distribution is the Kummer-Beta distribution defined by the density function (Gupta, Cardeño & Nagar 2001, Nagar & Gupta 2002, Ng & Kotz 1995),

$$\frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)} \frac{x^{a-1}(1-x)^{c-1}\exp(-\lambda x)}{{}_1F_1(a; a+c; -\lambda)} \tag{1}$$

where $a > 0, c > 0$, $0 < x < 1$, $-\infty < \lambda < \infty$ and ${}_1F_1$ is the confluent hypergeometric function defined by the integral (Luke 1969),

$$\begin{gathered} {}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)\,\mathrm{d}t, \\ \operatorname{Re}(c) > \operatorname{Re}(a) > 0 \end{gathered} \tag{2}$$

The Kummer-Beta distribution can be seen as bimodal extension of the Beta distribution (on a finite interval) and thus can help to describe real world phenomena possessing bimodal characteristics and varying within two finite bounds. The Kummer-Beta distribution is used in common value auctions where posterior distribution of "value of a single good" is Kummer-Beta (Gordy 1998). Recently, Nagar & Zarrazola (2005) derived distributions of product and ratio of two independent random variables when at least one of them is Kummer-Beta.

The random variables X and Y are said to have a bivariate Kummer-Beta distribution, denoted by $(X, Y) \sim KB(a, b; c; \lambda)$, if their joint density is given by

$$f(x, y; a, b; c; \lambda) = C(a, b; c; \lambda)x^{a-1}y^{b-1}(1-x-y)^{c-1}\exp[-\lambda(x+y)] \tag{3}$$

where $x > 0$, $y > 0$, $x + y < 1$, $a > 0, b > 0$, $c > 0$, $-\infty < \lambda < \infty$ and

$$C(a, b; c; \lambda) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\{{}_1F_1(a+b; a+b+c; -\lambda)\}^{-1} \tag{4}$$

For $\lambda = 0$, the density (3) slides to a Dirichlet density with parameters a, b and c. In Bayesian analysis, the Dirichlet distribution is used as a conjugate prior distribution for the parameters of a multinomial distribution. However, the Dirichlet family is not sufficiently rich in scope to represent many important distributional assumptions, because the Dirichlet distribution has few number of parameters. We provide a generalization of the Dirichlet distribution with added number of parameters. Several other bivariate generalizations of Beta distribution are available in Mardia (1970), Barry, Castillo & Sarabia (1999), Kotz, Balakrishnan & Johnson (2000), Balakrishnan & Lai (2009), Hutchinson & Lai (1991), Nadarajah & Kotz (2005), and Gupta & Wong (1985).

The matrix variate generalization of Beta and Dirichlet distributions have been defined and studied extensively. For example, see Gupta & Nagar (2000).

It can also be observed that bivariate generalization of the Kummer-Beta distribution defined by the density (3), belongs to the Liouville family of distributions proposed by Marshall & Olkin (1979) and Sivazlian (1981), (also see Gupta & Song (1996), Gupta & Richards (2001) and Song & Gupta (1997)).

In this article we study several properties such as marginal and conditional distributions, joint moments, correlation, and mixture representation of the bivariate Kummer-Beta distribution defined by the density (3). We also derive the exact and approximate distribution of the product XY where $(X, Y) \sim KB(a, b; c; \lambda)$. Finally, an application of the results derived in this article is provided to visibility data about fog and mist from Colombia.

2. Properties

In this section we study several properties of the bivariate Kummer-Beta distribution defined in Section 1.

Using the Kummers relation,

$$
{}_1F_1(a; c; -z) = \exp(-z)\, {}_1F_1(c-a; c; z) \tag{5}
$$

the density given in (3) can be rewritten as

$$
C(a, b; c; \lambda) \exp(-\lambda) x^{a-1} y^{b-1} (1-x-y)^{c-1} \exp[\lambda(1-x-y)] \tag{6}
$$

Expanding $\exp[\lambda(1-x-y)]$ in power series and rearranging certain factors, the joint density of X and Y can also be expressed as

$$
\{{}_1F_1(c; a+b+c; \lambda)\}^{-1} \sum_{j=0}^{\infty} \frac{\Gamma(a+b+c)\Gamma(c+j)}{\Gamma(a+b+c+j)\Gamma(c)} \frac{\lambda^j}{j!} \frac{x^{a-1} y^{b-1} (1-x-y)^{c+j-1}}{B(a, b, c+j)}
$$

where

$$
B(\alpha, \beta, \gamma) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)}
$$

Thus the bivariate Kummer-Beta distribution is an infinite mixture of Dirichlet distributions.

In Bayesian probability theory, if the posterior distributions are in the same family as the prior probability distribution, the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior. In case of multinomial distribution, the usual conjugate prior is the Dirichlet distribution. If

$$P(r,s,f|x,y) = \binom{r+s+f}{r,s,f} x^r y^s (1-x-y)^f$$

and

$$p(x,y) = C(a,b;c;\lambda) x^{a-1} y^{b-1} (1-x-y)^{c-1} \exp[-\lambda(x+y)]$$

where $x > 0$, $y > 0$, and $x + y < 1$, then

$$p(x,y \mid r,s,f) = C(a+r, b+s; c+f; \lambda) \times x^{a+r-1} y^{b+s-1} (1-x-y)^{c+f-1} \exp[-\lambda(x+y)]$$

Thus, the bivariate family of distributions considered in this article is the conjugate prior for the multinomial distribution.

A distribution is said to be negatively likelihood ratio dependent if the density $f(x,y)$ satisfies

$$f(x_1,y_1)f(x_2,y_2) \le f(x_1,y_2)f(x_2,y_1)$$

for all $x_1 > x_2$ and $y_1 > y_2$ (see Lehmann (1966)). In the case of bivariate generalization of the Kummer-Beta distribution the above inequality reduces to

$$(1-x_1-y_1)(1-x_2-y_2) < (1-x_1-y_2)(1-x_2-y_1)$$

which clearly holds. Hence, the bivariate distribution defined by the density (3) is negatively likelihood ratio dependent.

If $(X,Y) \sim KB(a,b;c;\lambda)$, then Ng & Kotz (1995) have shown that $Y/(X+Y)$ and $X+Y$ are mutually independent, $Y/(X+Y) \sim B(b,a)$ and $X+Y \sim KB(a+b;c;\lambda)$. Here we give a different proof of this result based on angular transformation.

Theorem 1. *Let* $(X,Y) \sim KB(a,b;c;\lambda)$ *and define* $X = R^2\cos^2\Theta$ *and* $Y = R^2\sin^2\Theta$. *Then,* R^2 *and* Θ *are independent,* $R^2 \sim KB(a+b;c;\lambda)$ *and* $\sin^2\Theta \sim B(b,a)$.

Proof. Using the transformation $X = R^2\cos^2\Theta$ and $Y = R^2\sin^2\Theta$ with the Jacobian $J(x,y \to r^2,\theta) = 2r^2\cos\theta\sin\theta$, in the joint density of X and Y, we obtain the joint density of R and Θ as

$$C(a,b;c;\lambda)(r^2)^{a+b}(1-r^2)^{c-1}\exp(-\lambda r^2)(\cos\theta)^{2a-1}(\sin\theta)^{2b-1}, \tag{7}$$

where $0 < r^2 < 1$ and $0 < \theta < \pi/2$. From (7), it is clear that R^2 and Θ are independent. Now, transforming $S = R^2$ and $U = \sin^2\Theta$ with the Jacobian $J(r^2,\theta \to s,u) = J(r^2 \to s)J(\theta \to u) = (4s)^{-1}[u(1-u)]^{-1/2}$, above we get the desired result. □

We derive marginal and conditional distributions as follows.

Theorem 2. *If* $(X, Y) \sim KB(a, b; c; \lambda)$, *then the marginal density of* X *is given by*

$$C_1(a, b; c; \lambda) \exp(-\lambda x) x^{a-1} (1-x)^{b+c-1} {}_1F_1(b; b+c; -\lambda(1-x)) \tag{8}$$

where $0 < x < 1$ *and*

$$C_1(a, b; c; \lambda) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} \{{}_1F_1(a+b; a+b+c; -\lambda)\}^{-1}$$

Proof. To find the marginal pdf of X, we integrate (3) with respect to y to get

$$C(a, b; c; \lambda) \exp(-\lambda x) x^{a-1} \int_0^{1-x} \exp(-\lambda y) y^{b-1} (1-x-y)^{c-1} \, \mathrm{d}y$$

Substituting $z = y/(1-x)$ with $\mathrm{d}y = (1-x)\,\mathrm{d}z$ above, one obtains

$$C(a, b; c; \lambda) x^{a-1} \exp(-\lambda x)(1-x)^{b+c-1} \int_0^1 \exp[-\lambda(1-x)z] z^{b-1} (1-z)^{c-1} \, \mathrm{d}z \tag{9}$$

Now, the desired result is obtained by using (2). □

Using the above theorem, the conditional density function of X given $Y = y$, $0 < y < 1$, is obtained as

$$\frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)} \frac{\exp(-\lambda x) x^{a-1} (1-x-y)^{c-1}}{(1-y)^{a+c-1} {}_1F_1(a; a+c; -\lambda(1-y))}, \quad 0 < x < 1-y$$

Graphs 1–6 of the density function for several values of a, b, c and λ corresponding to six rows of Table 1, depicted in Figure 1, show a wide range of densities. For example, large values of a, b, c give a density similar to a bivariate normal density, whereas for small values of a, b, c the density is close to a uniform density.

Table 1: Density functions for different values of a, b, c and λ.

Graph	a	b	c	λ
1	2	1	1.5	−5.0
2	2	2	5.0	−5.0
3	5	3	2.0	−5.0
4	2	1	2.0	−0.5
5	5	3	9.0	0.5
6	3	2	1.5	3.0

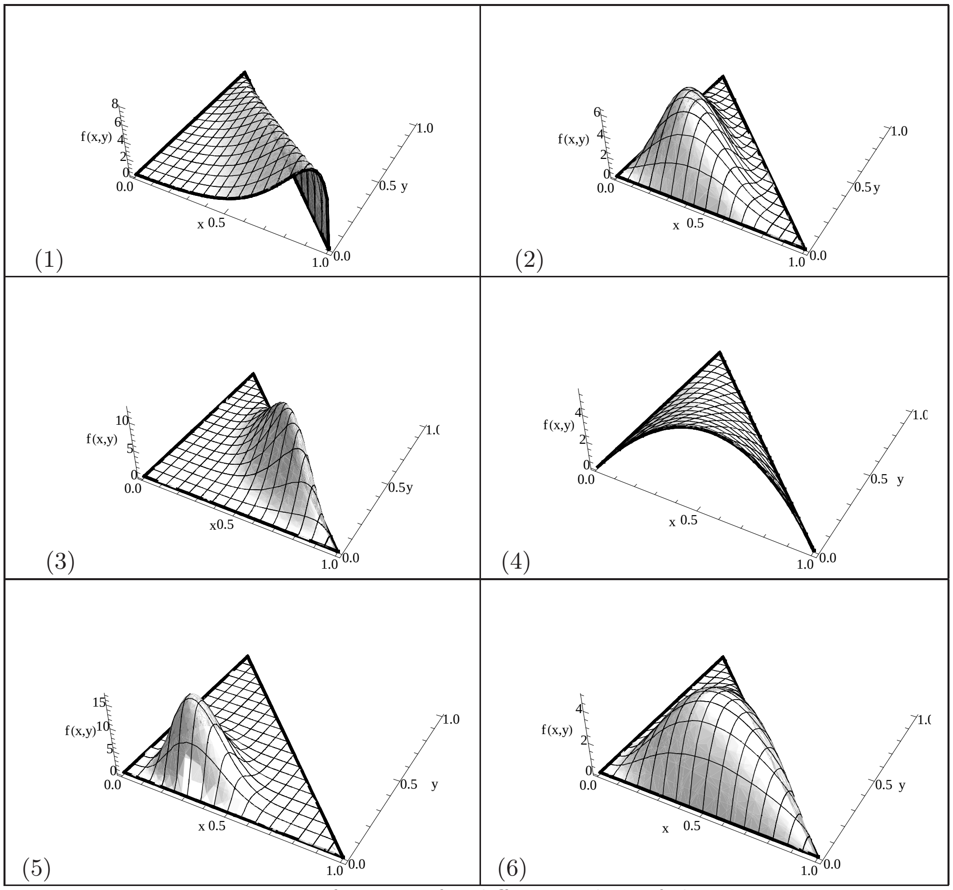

FIGURE 1: Density functions for different values of the parameters.

Further, using (3), the joint (r, s)-th moment is obtained as

$$\begin{aligned}
\mathrm{E}(X^r Y^s) &= C(a,b;c;\lambda)\int_0^1\int_0^{1-x}\exp[-\lambda(x+y)]x^{a+r-1}y^{b+s-1}\,(1-x-y)^{c-1}\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{C(a,b;c;\lambda)}{C(a+r,b+r;c;\lambda)}\\
&= \frac{\Gamma(a+r)\Gamma(b+s)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(d+r+s)}\frac{{}_1F_1(a+b+r+s;d+r+s;-\lambda)}{{}_1F_1(a+b;d;-\lambda)}
\end{aligned}$$

where $d = a + b + c$, $a + r > 0$ and $b + s > 0$. Now, substituting appropriately, we obtain

$$\mathrm{E}(X) = \frac{a}{d}\frac{{}_1F_1(a+b+1;d+1;-\lambda)}{{}_1F_1(a+b;d;-\lambda)}$$

$$\mathrm{E}(Y) = \frac{b}{d}\frac{{}_1F_1(a+b+1;d+1;-\lambda)}{{}_1F_1(a+b;d;-\lambda)}$$

$$\mathrm{E}(X^2) = \frac{a(a+1)}{d(d+1)} \frac{{}_1F_1(a+b+2; d+2; -\lambda)}{{}_1F_1(a+b; d; -\lambda)}$$

$$\mathrm{E}(Y^2) = \frac{b(b+1)}{d(d+1)} \frac{{}_1F_1(a+b+2; d+2; -\lambda)}{{}_1F_1(a+b; d; -\lambda)}$$

$$\mathrm{E}(XY) = \frac{ab}{d(d+1)} \frac{{}_1F_1(a+b+2; d+2; -\lambda)}{{}_1F_1(a+b; d; -\lambda)}$$

$$\mathrm{E}(X^2Y^2) = \frac{ab(a+1)(b+1)}{d(d+1)(d+2)(d+3)} \frac{{}_1F_1(a+b+4; d+4; -\lambda)}{{}_1F_1(a+b; d; -\lambda)}$$

$$\mathrm{Var}(X) = \frac{a}{d}\left[\frac{a+1}{d+1}\frac{{}_1F_1(a+b+2; d+2; -\lambda)}{{}_1F_1(a+b; d; -\lambda)} - \frac{a}{d}\left\{\frac{{}_1F_1(a+b+1; d+1; -\lambda)}{{}_1F_1(a+b; d; -\lambda)}\right\}^2\right]$$

$$\mathrm{Var}(Y) = \frac{b}{d}\left[\frac{b+1}{d+1}\frac{{}_1F_1(a+b+2; d+2; -\lambda)}{{}_1F_1(a+b; d; -\lambda)} - \frac{b}{d}\left\{\frac{{}_1F_1(a+b+1; d+1; -\lambda)}{{}_1F_1(a+b; d; -\lambda)}\right\}^2\right]$$

and

$$\mathrm{Cov}(X,Y) = \frac{ab}{d}\left[\frac{{}_1F_1(a+b+2; d+2; -\lambda)}{(d+1)\,{}_1F_1(a+b; d; -\lambda)} - \frac{1}{d}\left\{\frac{{}_1F_1(a+b+1; d+1; -\lambda)}{{}_1F_1(a+b; d; -\lambda)}\right\}^2\right]$$

Notice that $\mathrm{E}(XY)$, $\mathrm{E}(X^2)$, $\mathrm{E}(Y^2)$, $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ involve ${}_1F_1(\alpha; \mu; -\lambda)$ which can be computed using Mathematica by providing values of α, μ and λ. Table 2 provides correlations between X and Y for different values of a, b, c and λ. All the tabulated values of correlation are negative because X and Y satisfy $x+y<1$. As can be seen, the choices of a, b small and c, λ large yield correlations close to zero, whereas large values of a or b and small values of c or λ give small correlations. Further, for fixed values of a, b and c, the correlation decreases as the value of λ increases. Likewise, for fixed values of a, b and λ, the correlation decreases as c increases.

3. Entropies

In this section, exact forms of Renyi and Shannon entropies are determined for the bivariate Kummer-Beta distribution defined in this article.

Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a pdf f associated with $\mathcal{P}$, dominated by $\sigma-$finite measure μ on $\mathcal{X}$. Denote by $H_{SH}(f)$ the well-known Shannon entropy introduced in Shannon (1948). It is define by

$$H_{SH}(f) = -\int_{\mathcal{X}} f(x) \log f(x)\, d\mu \tag{10}$$

Table 2: Correlation for values of a, b, c and λ.

a	b	c	$\lambda=-5.000$	-2.000	-1.000	-0.500	0.000	0.500	1.000	2.000	5.000
3.0	2.0	0.5	−0.936	−0.888	−0.862	−0.846	−0.828	−0.808	−0.785	−0.731	−0.494
1.0	2.0	1.0	−0.848	−0.717	−0.653	−0.616	−0.577	−0.536	−0.493	−0.406	−0.172
3.0	2.0	1.5	−0.819	−0.716	−0.670	−0.644	−0.617	−0.589	−0.559	−0.497	−0.304
5.0	3.0	2.0	−0.799	−0.723	−0.690	−0.673	−0.655	−0.635	−0.616	−0.573	−0.433
0.5	1.0	1.5	−0.736	−0.499	−0.406	−0.360	−0.316	−0.275	−0.237	−0.171	−0.055
1.0	2.0	2.0	−0.712	−0.543	−0.477	−0.442	−0.408	−0.374	−0.341	−0.279	−0.135
0.5	1.0	2.0	−0.654	−0.414	−0.332	−0.294	−0.258	−0.225	−0.195	−0.144	−0.054
1.0	2.0	3.0	−0.598	−0.429	−0.371	−0.343	−0.316	−0.290	−0.265	−0.219	−0.118
2.0	4.0	5.0	−0.535	−0.428	−0.391	−0.374	−0.356	−0.339	−0.322	−0.290	−0.204
2.0	2.0	5.0	−0.494	−0.365	−0.324	−0.305	−0.286	−0.267	−0.250	−0.218	−0.141
1.0	0.5	5.0	−0.322	−0.185	−0.151	−0.136	−0.123	−0.111	−0.100	−0.082	−0.046

One of the main extensions of the Shannon entropy was defined by Rényi (1961). This generalized entropy measure is given by

$$H_R(\eta, f) = \frac{\log G(\eta)}{1-\eta} \qquad (\text{for } \eta > 0 \text{ and } \eta \neq 1) \tag{11}$$

where

$$G(\eta) = \int_{\mathcal{X}} f^\eta d\mu$$

The additional parameter η is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in η, while Shannon entropy (10) is obtained from (11) for $\eta \uparrow 1$. For details see Nadarajah & Zografos (2005), Zografos and Nadarajah (2005) and Zografos (1999).

First, we give the following lemma useful in deriving these entropies.

Lemma 1. *Let* $g(a,b,c,\lambda) = \lim_{\eta\to 1} h(\eta)$, *where*

$$h(\eta) = \frac{\mathrm{d}}{\mathrm{d}\eta} {}_1F_1(\eta(a+b-2)+2; \eta(a+b+c-3)+3; -\lambda\eta) \tag{12}$$

Then,

$$\begin{aligned} g(a,b,c,\lambda) = \sum_{j=1}^{\infty} \frac{\Gamma(a+b+j)\Gamma(a+b+c)}{\Gamma(a+b)\Gamma(a+b+c+j)} \frac{(-\lambda)^j}{j!} \Big[& j + (a+b-2)\psi(a+b+j) \\ & + (a+b+c-3)\psi(a+b+c) - (a+b-2)\psi(a+b) \\ & - (a+b+c-3)\psi(a+b+c+j)\Big] \end{aligned} \tag{13}$$

where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ *is the digamma function.*

***Proof*.** Expanding ${}_1F_1$ in series form, we write

$$h(\eta) = \frac{\mathrm{d}}{\mathrm{d}\eta}\sum_{j=0}^{\infty}\Delta_j(\eta)\frac{(-\lambda)^j}{j!} = \sum_{j=0}^{\infty}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\Delta_j(\eta)\right], \frac{(-\lambda)^j}{j!} \tag{14}$$

where

$$\Delta_j(\eta) = \frac{\Gamma[\eta(a+b-2)+2+j]\Gamma[\eta(a+b+c-3)+3]}{\Gamma[\eta(a+b-2)+2]\Gamma[\eta(a+b+c-3)+3+j]}\eta^j$$

Now, differentiating the logarithm of $\Delta_j(\eta)$ w.r.t. to η, one obtains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}\Delta_j(\eta) \quad = \quad & \Delta_j(\eta)\Big[\frac{j}{\eta} + (a+b-2)\psi(\eta(a+b-2)+2+j) \\ & +(a+b+c-3)\psi(\eta(a+b+c-3)+3) \\ & -(a+b-2)\psi(\eta(a+b-2)+2) \\ & -(a+b+c-3)\psi(\eta(a+b+c-3)+3+j)\Big]\end{aligned} \tag{15}$$

Finally, substituting (15) in (14) and taking $\eta \to 1$, one obtains the desired result. □

Theorem 3. *For the bivariate Kummer-Beta distribution defined by the pdf (3), the Rényi and the Shannon entropies are given by*

$$\begin{aligned}H_R(\eta, f) \quad = \quad & \frac{1}{1-\eta}\Big[\eta\log C(a,b;c;\lambda) + \log\Gamma[\eta(a-1)+1] \\ & + \log\Gamma[\eta(b-1)+1] + \log\Gamma[\eta(c-1)+1] \\ & - \log\Gamma[\eta(a+b+c-3)+3] \\ & + \log {}_1F_1(\eta(a+b-2)+2;\eta(a+b+c-3)+3;-\lambda\eta)\Big]\end{aligned} \tag{16}$$

and

$$\begin{aligned}H_{SH}(f) \quad = \quad & -\log C(a,b;c;\lambda) - [(a-1)\psi(a) + (b-1)\psi(b) + (c-1)\psi(c) \\ & -(a+b+c-3)\psi(a+b+c)] - \frac{g(a,b,c,\lambda)}{{}_1F_1(a+b;a+b+c;-\lambda)},\end{aligned} \tag{17}$$

respectively, where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ is the digamma function and $g(a,b,c,\lambda)$ is given by (13).

***Proof*.** For $\eta > 0$ and $\eta \neq 1$, using the joint density of X and Y given by (3), we have

$$\begin{aligned}
G(\eta) &= \int_0^1 \int_0^{1-x} f^{\eta}(x, y; a, b; c; \lambda)\,\mathrm{d}x\,\mathrm{d}y \\
&= [C(a, b; c; \lambda)]^{\eta} \int_0^1 \int_0^{1-x} x^{\eta(a-1)} y^{\eta(b-1)} \\
&\quad (1 - x - y)^{\eta(c-1)} \exp[-\eta\lambda(x + y)]\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{[C(a, b; c; \lambda)]^{\eta}}{C(\eta(a-1)+1, \eta(b-1)+1; \eta(c-1)+1; \lambda)} \\
&= \frac{\Gamma^{\eta}(a+b+c)\Gamma[\eta(a-1)+1]\Gamma[\eta(b-1)+1]\Gamma[\eta(c-1)+1]}{\Gamma^{\eta}(a)\Gamma^{\eta}(b)\Gamma^{\eta}(c)\Gamma[\eta(a+b+c-3)+3]} \\
&\quad \times \frac{{}_1F_1(\eta(a+b-2)+2; \eta(a+b+c-3)+3; -\lambda\eta)}{\{{}_1F_1(a+b; a+b+c; -\lambda)\}^{\eta}},
\end{aligned}$$

where the last line has been obtained by using (4). Now, taking logarithm of $G(\eta)$ and using (11) we get (16). The Shannon entropy is obtained from (16) by taking $\eta \uparrow 1$ and using L'Hopital's rule. □

4. Exact and Approximate Distribution of the Product

If $(X, Y) \sim KB(a, b; c; \lambda)$, then Ng & Kotz (1995) have shown that $X/(X+Y)$ and $X+Y$ are mutually independent, $X/(X+Y) \sim B(a, b)$ and $X+Y \sim KB(a+b; c; \lambda)$. In this section we derive the density of XY when $(X, Y) \sim KB(a, b; c; \lambda)$. The distribution of XY, where X and Y are independent random variables, $X \sim KB(a_1, b_1, \lambda_1)$ and $Y \sim KB(a_2, b_2, \lambda_2)$ has been derived in Nagar & Zarrazola (2005). In order to derive the density of the product we essentially need the integral representation of the Gauss hypergeometric function given by Luke (1969),

$$\begin{aligned}
{}_2F_1(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,\mathrm{d}t, \\
&\qquad \mathrm{Re}(c) > \mathrm{Re}(a) > 0,\ |\arg(1-z)| < \pi. \qquad (18)
\end{aligned}$$

Theorem 4. *If $(X, Y) \sim KB(a, b; c; \lambda)$, then the pdf of $W = XY$ is given by*

$$\begin{aligned}
&\frac{\sqrt{\pi}C(a, b; c; \lambda)\exp(-\lambda)}{2^{a+c-b-1}} \frac{w^{b-1}(1-4w)^{c-1/2}}{\left(1+\sqrt{1-4w}\right)^{b+c-a}} \\
&\times \sum_{i=0}^{\infty} \frac{\Gamma(c+i)}{\Gamma(c+1/2+i)\,2^i\,i!} \left(\frac{1-4w}{1+\sqrt{1-4w}}\right)^i \\
&\times {}_2F_1\left(c+i, c+b-a+i; 2c+2i; \frac{2\sqrt{1-4w}}{1+\sqrt{1-4w}}\right),\ 0 < w < \frac{1}{4}. \qquad (19)
\end{aligned}$$

***Proof*.** Making the transformation $W = XY$ with the Jacobian $J(x, y \to x, w) = x^{-1}$ in (3), we obtain the joint density of X and W as

$$C(a,b;c;\lambda)\exp(-\lambda)\frac{w^{b-1}(-x^2+x-w)^{c-1}}{x^{b+c-a}}\exp\left[\frac{\lambda(-x^2+x-w)}{x}\right]$$

where $p < x < q$ with

$$p = \frac{1-\sqrt{1-4w}}{2}, \qquad q = \frac{1+\sqrt{1-4w}}{2},$$

and $0 < w < 1/4$. Now, expanding $\exp\left[\lambda(-x^2+x-w)/x\right]$ in power series and integrating x in the above expression, we obtain the marginal density of W as

$$\begin{aligned} &C(a,b;c;\lambda)\exp(-\lambda)w^{b-1}\int_p^q \frac{[(x-p)(q-x)]^{c-1}}{x^{b+c-a}}\exp\left(\frac{\lambda(x-p)(q-x)}{x}\right)\mathrm{d}x \\ &= C(a,b;c;\lambda)\exp(-\lambda)w^{b-1}\sum_{i=0}^{\infty}\frac{(q-p)^{2i+2c-1}\lambda^i}{q^{i+b+c-a}\, i!}\int_0^1 \frac{t^{c+i-1}(1-t)^{c+i-1}\,\mathrm{d}t}{[1-t\,(1-p/q)]^{b+c-a+i}} \end{aligned}$$

where we have used the substitution $t = (q-x)/(q-p)$. Now, evaluating the above integral using (18) and simplifying the resulting expression, we get the desired result. □

In the rest of this section, we derive the approximate distribution of the product XY. It is clear from Theorem 4, that the random variable $4W = 4XY$ has support on $(0, 1)$. We, therefore, are motivated to use the Beta distribution of two parameters as an approximation to the exact distribution. Equating the first and the second moments of $4W$, with those of the Beta distribution with parameters α and β, it is easy to see that

$$\alpha = \frac{\mathrm{E}(W)[\mathrm{E}(W) - 4\mathrm{E}(W^2)]}{\mathrm{E}(W^2) - (\mathrm{E}(W))^2} \tag{20}$$

and

$$\beta = \frac{[\mathrm{E}(W) - 4\mathrm{E}(W^2)][1 - 4\mathrm{E}(W)]}{4[\mathrm{E}(W^2) - (\mathrm{E}(W))^2]}$$

The moments $\mathrm{E}(W)$ and $\mathrm{E}(W^2)$ are available in Section 2, and can be computed numerically for given values of a, b, c and λ. To demonstrate the closeness of the approximation we, in Figure 2, graphically compare the exact and approximated pdf of $4W$. First, for different values of the parameters (a, b, c, λ) we compute the corresponding estimates for (α, β), using (20) and (21). These estimates are given in Table 3, and corresponding graphics are given in Figure 2, showing comparison between exact and approximate densities. The exact pdf corresponds to the solid curve and approximate pdf corresponds to the broken curve. It is evident that the approximate density is quite close to the exact density.

TABLE 3: Estimated values of α and β.

Figure	a	b	c	λ	α	β
1	3.0	1.0	0.5	0.5	0.9567	1.0527
2	3.0	1.0	3.0	0.5	0.9514	3.7098
3	3.0	3.0	1.0	0.5	2.6239	1.5259
4	0.5	0.5	1.0	1.0	0.2646	1.8184
5	3.0	3.0	1.0	1.0	2.5250	1.5410
6	3.0	3.0	0.5	3.0	2.2502	1.0365

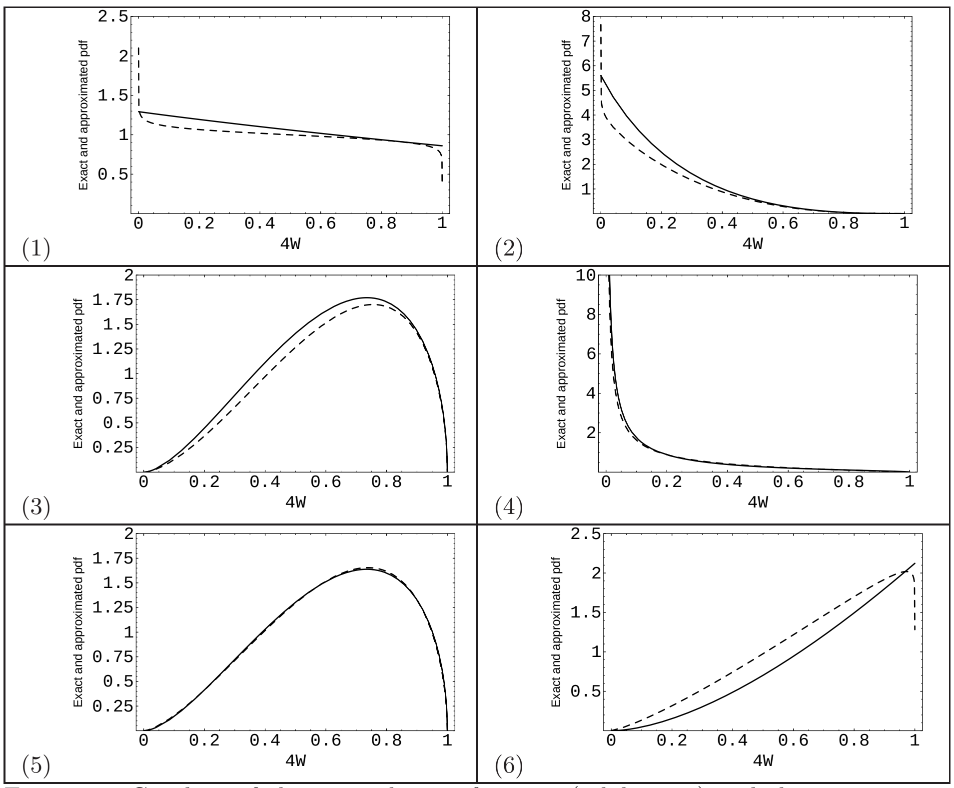

FIGURE 2: Graphics of the exact density function (solid curve) and the approximate (broken curve).

5. Application

In this section, we consider the data of fog and mist collect from five Colombian airports and present an application of the model given by (3).

Fog or mist is a collection of water droplets or ice crystals suspended in the air at or near the Earth's surface. The only difference between mist and fog is visibility. The phenomenon is called fog if the visibility is one kilometer or less; otherwise it is known as mist.

We consider data available at the website of IDEAM (Institute Hydrology, Meteorology and Environmental Studies, Colombia) collected from the following 5 major Colombian airports regarding the fog and mist:

- Ernesto Cortissoz Airport (Barranquilla)
- El Dorado Airport (Bogota)
- Alfonso Bonilla Aragón Airport (Cali)
- Rafael Núñez Airport (Cartagena)
- José María Córdova Airport (Medellin)

The data comprises average number of days each month in which mist or fog appeared during the period from 1975 to 1991. We consider the following variables:

X: the proportion of days with mist (the phenomenon weather provides a visibility of more than 1 km)

Y: proportion of days with fog (the phenomenon weather provides a visibility of 1 km or less)

In addition the following variables are of interest:

$X+Y$: proportion of days with the weather phenomenon (mist or fog)

$X/(X+Y)$: proportion of days with visibility greater than 1 km with respect to the total proportion of days exhibiting the phenomenon (mist or fog)

$Y/(X+Y)$: proportion of days with visibility less than 1 km with respect to the total proportion of days exhibiting the phenomenon (mist or fog)

Table 4, gives the estimates of a, b, c and λ, which were obtained using the maximum likelihood method, and by implementing Fisher scoring method (Kotz et al. (2000), p. 504). Table 5, gives estimated values of the moments $\mathrm{E}[X/(X+Y)]$, $\mathrm{E}[Y/(X+Y)]$ and $\mathrm{E}(X+Y)$ for five airports.

Table 4: Estimated values of a, b, c and λ.

Airport	a	b	c	λ
Barranquilla	0.620	0.266	153.00	−176.0
Bogota	8.290	3.370	3.82	12.3
Cali	0.303	0.088	70.80	−94.4
Cartagena	0.206	0.091	396.00	−407.0
Medellin	12.300	6.580	3.41	18.5

6. Conclusions of the Application

As conclusions, we can say that the proportion of days with visibility less than 1 km with respect to the total number of days presenting the phenomenon is similar for Barranquilla, Bogota and Cartagena airports. This ratio is a little lower for

Table 5: Estimated values of the moments.

Airport	$E[X/(X+Y)]$	$E[Y/(X+Y)]$	$E(X+Y)$
Barranquilla	0.700	0.300	0.129
Bogotá	0.711	0.289	0.572
Cali	0.775	0.225	0.221
Cartagena	0.695	0.305	0.023
Medellín	0.651	0.349	0.675

the Cali and Medellin airports, the value of this ratio is higher. For example, we can say that the airport at Barranquilla has 30% of total days (with phenomenon) with fog. For Medellin, this percentage corresponds to 34.9% and for Cali to 22.5%. The proportion of days with phenomenon (mist or fog) is higher for the Medellin airport followed by the Bogota airport. Cartagena airport presents the lower proportion.

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