Revista Colombiana de Estadística
Junio 2014, volumen 37, no. 1, pp. 223 a 243

The Poisson-Lomax Distribution

Distribución Poisson-Lomax

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Abstract

In this paper we propose a new three-parameter lifetime distribution with upside-down bathtub shaped failure rate. The distribution is a compound distribution of the zero-truncated Poisson and the Lomax distributions (PLD). The density function, shape of the hazard rate function, a general expansion for moments, the density of the rth order statistic, and the mean and median deviations of the PLD are derived and studied in detail. The maximum likelihood estimators of the unknown parameters are obtained. The asymptotic confidence intervals for the parameters are also obtained based on asymptotic variance-covariance matrix. Finally, a real data set is analyzed to show the potential of the new proposed distribution.

***Key words*:** Asymptotic variance-covariance matrix, Compounding, Lifetime distributions, Lomax distribution, Poisson distribution, Maximum likelihood estimation.

Resumen

En este artículo se propone una nueva distribución de sobrevida de tres parámetros con tasa fallo en forma de bañera. La distribución es una mezcla de la Poisson truncada y la distribución Lomax. La función de densidad, la función de riesgo, una expansión general de los momentos, la densidad del r-ésimo estadístico de orden, y la media así como su desviación estándar son derivadas y estudiadas en detalle. Los estimadores de máximo verosímiles de los parámetros desconocidos son obtenidos. Los intervalos de confianza asintóticas se obtienen según la matriz de varianzas y covarianzas asintótica. Finalmente, un conjunto de datos reales es analizado para construir el potencial de la nueva distribución propuesta.

***Palabras clave*:** mezclas, distribuciones de sobrevida, distribució n Lomax, distribución Poisson, estomación máximo-verosímil.

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1. Introduction

Marshall & Olkin (1997) introduced an effective technique to add a new parameter to a family of distributions. A great deal of papers have appeared in the literature used this technique to propose new distributions. In their paper, Marshall & Olkin (1997) generalized the exponential and Weibull distributions. Alice & Jose (2003) followed the same approach and introduced Marshall-Olkin extended semi-Pareto model and studied its geometric extreme stability. Ghitany, Al-Hussaini & Al-Jarallah (2005) studied the Marshall-Olkin Weibull distribution and established its properties in the presence of censored data. Marshall-Olkin extended Lomax distribution was introduced by Ghitany, Al-Awadhi & Alkhalfan (2007). Compounding Poisson and exponential distributions have been considered by many authors; e.g. Kus (2007) proposed the Poisson-exponential lifetime distribution with a decreasing failure rate function. Al-Awadhi & Ghitany (2001) used the Lomax distribution as a mixing distribution for the Poisson parameter and obtained the discrete Poisson-Lomax distribution. Cancho, Louzada-Neto & Barriga (2011) introduced another modification of the Poisson-exponential distribution.

Let $Y_1, Y_2, \dots, Y_Z$ be independent and identically distributed random variables each has a density function f, and let Z be a discrete random variable having a zero-truncated Poisson distribution with probability mass function

$$P_Z(z) \equiv P_Z(z, \lambda) = \frac{e^{-\lambda}\lambda^z}{z!(1-e^{-\lambda})}, \quad z \in \{1, 2, \dots\},\ \lambda > 0. \tag{1}$$

Suppose that X is a random variable representing the lifetime of a parallel-system of Z components, i.e. $X = \max\{Y_1, Y_2, \dots, Y_z\}$, and Y's and Z are independent. The conditional distribution function of $X|Z$ has the probability density function (pdf)

$$f_{X|Z}(x|z) = zf(x)[F(x)]^{z-1}. \tag{2}$$

where $F(x)$ is the cumulative distribution function (cdf) corresponding to $f(x)$.

A compound probability function (pdf) of $f_{X|Z}(x|z)$ and $P_Z(z)$, where X is a continuous random variable (r.v.) and Z a discrete r.v. is defined by

$$g_X(x) = \sum_{z=1}^{\infty} f_{X|Z}(x|z)P_Z(z). \tag{3}$$

Substitution of (1) and (2) in (3) then yields

$$\begin{aligned} g_X(x) &= \sum_{z=1}^{\infty} zf(x)[F(x)]^{z-1} \left(\frac{\lambda^z e^{-\lambda}}{z!(1-e^{-\lambda})}\right) \\ &= \frac{\lambda f(x)e^{-\lambda(1-F(x))}}{(1-e^{-\lambda})}, \quad x > 0,\ \lambda > 0. \end{aligned}$$

The reliability and the hazard rate functions of X are, respectively, given by

$$\bar{G}(x,\lambda) = \frac{1-e^{-\lambda \bar{F}(x)}}{(1-e^{-\lambda})}, \quad x>0, \tag{4}$$

$$h_G(x,\lambda) = \frac{\lambda f(x) e^{-\lambda \bar{F}(x)}}{1-e^{-\lambda \bar{F}(x)}} = \frac{\lambda f(x)}{e^{\lambda \bar{F}(x)}-1}. \tag{5}$$

In this paper we propose a new lifetime distribution by compounding Poisson and Lomax distributions. As we have mentioned in the previous chapters, the Lomax distribution with two parameters is a special case of the generalized Pareto distribution, and ti is also known as the Pareto of the second type. A random variable X is said to have the Lomax distribution, abbreviated as $X \sim \mathrm{LD}(\alpha,\beta)$, if it has the pdf

$$f_{LD}(x;\alpha,\beta) = \alpha\beta\left(1+\beta x\right)^{-(\alpha+1)}, \quad x>0,\ \alpha,\beta>0. \tag{6}$$

Here α and β are the shape and scale parameters, respectively. Analogous tu above, the survival and hazard functions associated with (6) are given by

$$\bar{F}_{LD}(x;\alpha,\beta) = \left(1+\beta x\right)^{-\alpha}, \quad x>0, \tag{7}$$

$$h_{LD}(x;\alpha,\beta) = \frac{\alpha\beta}{1+\beta x}, \quad x>0. \tag{8}$$

The rest of the paper is organized as follows. In Section 2, we give explicit forms and interpretation for the distribution function and the probability density function. In Section 3, we discuss the distributional properties of the proposed distribution. Section 4 discusses the estimation problem using the maximum likelihood estimation method. In Section 5, an illustrative example, model selections, goodness-of-fit tests for the distribution with estimated parameters are all presented. Finally, we conclude in Section 6.

2. Model Formulation

Substitution of (7) in (4) yields the following reliability function:

$$\bar{G}(x;\alpha,\beta,\lambda) = \frac{1-e^{-\lambda(1+\beta x)^{-\alpha}}}{(1-e^{-\lambda})}, \quad x>0,\alpha,\beta,\lambda>0. \tag{9}$$

The pdf associated with (9) is expressed in a closed form and is given by

$$g(x;\alpha,\beta,\lambda) = \frac{\alpha\beta\lambda\left(1+\beta x\right)^{-(\alpha+1)} e^{-\lambda(1+\beta x)^{-\alpha}}}{(1-e^{-\lambda})}, \quad x>0,\alpha,\beta,\lambda>0. \tag{10}$$

The density function given by (10) can be interpreted as a compound of the zero-truncated Poisson distribution and the Lomax distribution. Suppose that $X = \max\{Y_1, Y_2, \cdots, Y_z\}$, and each Y is distributed according to the Lomax distribtion.

The variable Z has zero-truncated Poisson distribution and the variables Y's and Z are independent. Then the conditional distribution function of $X|Z$ has the pdf

$$f_{X|Z}(x|z;\alpha,\beta) = z\alpha\beta(1+\beta x)^{-(\alpha+1)}[1-(1+\beta x)^{-\alpha}]^{z-1}. \tag{11}$$

The joint distribution of the random variables X and Z, denoted by $f_{X,Z}(x,z)$, is given by

$$f_{X,Z}(x,z) = \frac{z}{z!(1-e^{-\lambda})}\ \alpha\beta(1+\beta x)^{-(\alpha+1)}[1-(1+\beta x)^{-\alpha}]^{z-1}e^{-\lambda}\lambda^z, \tag{12}$$

the marginal pdf of X is as follows.

$$\begin{aligned} f_X(x;\alpha,\beta,\lambda) &= \frac{\alpha\beta\lambda e^{-\lambda}(1+\beta x)^{-(\alpha+1)}}{(1-e^{-\lambda})}\sum_{z=1}^{\infty}\frac{[(1-(1+\beta x)^{-\alpha})\lambda]^{z-1}}{(z-1)!} \\ &= \frac{\alpha\beta\lambda e^{-\lambda}(1+\beta x)^{-(\alpha+1)}e^{\lambda(1-(1+\beta x)^{-\alpha})}}{(1-e^{-\lambda})} \\ &= \frac{\alpha\beta\lambda(1+\beta x)^{-(\alpha+1)}e^{-\lambda(1+\beta x)^{-\alpha}}}{(1-e^{-\lambda})}, \end{aligned}$$

which is the distribution with the pdf given by (10). The distribution of X may be referred to as the Poisson-Lomax distribution. Symbolically it is abbreviated by $X \sim PLD(\alpha,\beta,\lambda)$ to indicate that the random variable X has the Poisson-Lomax distribution with parameters α, β and λ.

3. Distributional Properties

In this section, we study the distributional properties of the PLD. In particular, if $X \sim PLD(\alpha,\beta,\lambda)$ then the shapes of the density function, the shapes of the hazard function, moments, the density of the rth order statistics, and the mean and median deviations of the PLD are derived and studied in detail.

3.1. Shapes of pdf

The limit of the Poisson-Lomax density as $x \to \infty$ is 0 and the limit as $x \to 0$ is $\alpha\beta\lambda/(e^{\lambda}-1)$. The following theorem gives simple conditions under which the pdf is decreasing or unimodal.

Theorem 1. *The pdf, $g(x)$, of $X \sim PLD(\alpha,\beta,\lambda)$ is decreasing (unimodal) if the function $\xi(x) \geq 0$ (<0) where $\xi(x) = \alpha(1-\lambda(1+\beta x)^{-\alpha})+1$, independent of β.*

***Proof*.** The first derivative of $g(x)$ is given by

$$g'(x) = -\frac{\alpha\beta^2\lambda}{1-e^{-\lambda}}\ (1+\beta x)^{-(\alpha+2)}\ e^{-\lambda(1+\beta x)^{-\alpha}}\xi((1+\beta x)^{-\alpha}),$$

where $\xi(y) = \alpha(1-\lambda y)+1$, and $y = (1+\beta x)^{-\alpha} < 1$. Then we have the following:

(i) If $\xi(1) = \alpha(1-\lambda)+1 > 0$, then $\xi(y) > 0$ for all $y < 1$, and hence, $g'(x) \leq 0$ for all $x > 0$, i.e. the function $g(x)$ is decreasing.

(ii) If $\xi(1) < 0$, then $\xi(y)$ has a unique zero at $y_\xi = \frac{\alpha+1}{\alpha\lambda} < 1$. Since $y = (1+\beta x)^{-\alpha}$ is one to one transformation, it follows that $g(x)$ has also a unique critical point at $x_g = \frac{1}{\beta}(y_\xi^{-1/\alpha} - 1)$.

Finally, since $g(0) = \alpha\beta\lambda/(e^\lambda - 1)$ and $g(\infty) = 0$ then x_g must be a point of absolute maximum for $g(x)$. ■

***Note* 1.** It should be noted that:

(i) When $\lambda \in (0,1]$, $g(x)$ is decreasing in $x > 0$ for all values of $\alpha, \beta > 0$.

(ii) When $\lambda > 1$, $g(x)$ may still exhibit a decreasing behavior, depending on the values of α, λ such that $\alpha(1-\lambda)+1 > 0$.

(iii) The mode of the Poisson-Lomax distribution is given by

$$Mode(x) = \begin{cases} 0, & \text{if} \quad \alpha(1-\lambda)+1 \geq 0, \\ \frac{1}{\beta}\left[\left(\frac{\alpha\lambda}{\alpha+1}\right)^{1/\alpha} - 1\right] & \text{otherwsie.} \end{cases} \tag{13}$$

Figure 1 shows the pdf curves for the $PLD(\alpha, \beta, \lambda)$ for selected values of the parameters α, β and λ. From the curves, it is quite evident that the PLD is positively skewed distribution. It becomes highly positively skewed for large values of the involved parameters.

3.2. Hazard Rate Function

The hazard rate function (hrf) of a random variable X is defined by $h(x) = f(x)/\bar{F}(x)$, where $\bar{F} = 1 - F$. The hazard function of $X \sim PLD(\alpha, \beta, \lambda)$ is given by

$$h(x) = \frac{\alpha\beta\lambda(1+\beta x)^{-(\alpha+1)}}{e^{\lambda(1+\beta x)^{-\alpha}} - 1}, \quad x > 0. \tag{14}$$

The following theorem gives simple conditions under which the hrf, given in (14), is decreasing or unimodal.

Theorem 2. *The hrf,* $h(x)$*, of* $X \sim PLD(\alpha, \beta, \lambda)$ *is decreasing (unimodal) if* $\eta(x) \geq 0 (< 0)$ *where* $\eta(x) = -(\alpha+1) + (\alpha + 1 - \alpha\lambda(1+\beta x)^{-\alpha})\ e^{\lambda(1+\beta x)^{-\alpha}}$*, independent of* β*.*

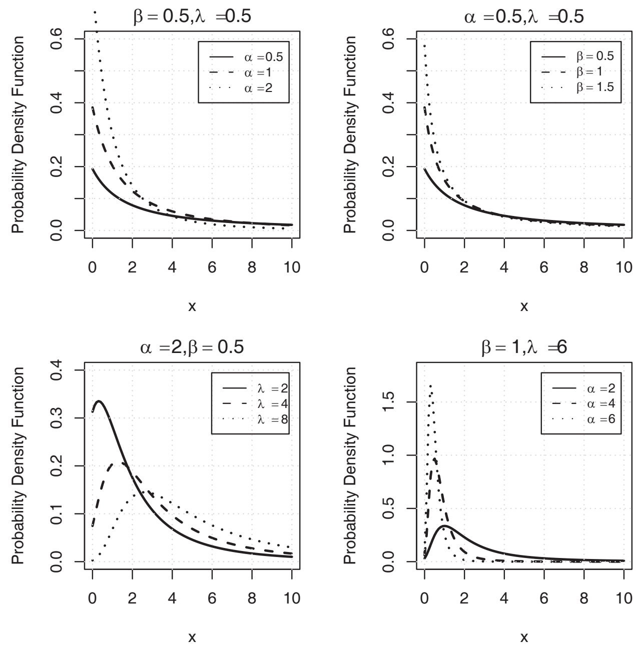

FIGURE 1: Plot of the probability density function for different values of the parameters α, β and λ.

***Proof*.** The first derivative of $h(x)$ with respect to x is given by

$$\begin{aligned} h'(x) &= \frac{-\alpha\beta^2\lambda(1+\beta x)^{-(\alpha+2)}}{(e^{\lambda(1+\beta x)^{-\alpha}}-1)^2}\left[(\alpha+1)\left(e^{\lambda(1+\beta x)^{-\alpha}}-1\right)-\alpha\lambda(1+\beta x)^{-\alpha}e^{\lambda(1+\beta x)^{-\alpha}}\right] \\ &= \frac{-\alpha\beta^2\lambda(1+\beta x)^{-(\alpha+2)}}{(e^{\lambda(1+\beta x)^{-\alpha}}-1)^2}\left[\left(\alpha+1-\alpha\lambda(1+\beta x)^{-\alpha}\right)e^{\lambda(1+\beta x)^{-\alpha}}-(\alpha+1)\right] \\ &= \frac{-\alpha\beta^2\lambda(1+\beta x)^{-(\alpha+2)}}{(e^{\lambda(1+\beta x)^{-\alpha}}-1)^2}\,\eta((1+\beta x)^{-\alpha}), \end{aligned}$$

where $\eta(y) = -(\alpha+1)+(\alpha+1-\alpha\lambda y)e^{\lambda y}$, and $y = (1+\beta x)^{-\alpha} < 1$. The remaining of the proof is similar to that of Theorem 1. ■

***Note* 2.** The following should be noted.

(i) For $\lambda \in (0,1]$, $h(x)$ is decreasing in $x > 0$ for all values of $\alpha, \beta > 0$.

(ii) For $\lambda > 1$, $h(x)$ may still exhibit a decreasing behavior, depending on the values of α and λ such that $(1+(1-\lambda)\alpha)e^{\lambda} - (\alpha+1) \geq 0$.

(iii) Since $(1+(1-\lambda)\alpha)e^{\lambda} - (\alpha+1) \geq 0$ implies that $\alpha(1-\lambda)+1 \geq 0$, then a decreasing hrf implies decreasing pdf. The converse is not necessarily true, e.g. $\alpha = 2$, $\lambda = 2$ implies decreasing pdf but unimodal hrf.

(iv) Since $(1 + (1 - \lambda)\alpha)e^{\lambda} - (\alpha + 1) < 0$ implies that $\alpha(1 - \lambda) + 1 < 0$, then a unimodal pdf implies unimodal hrf. The converse is not necessarily true, e.g., $\alpha = 2$, $\lambda = 2$ implies unimodal hrf but decreasing pdf.

Figure 2 shows the hrf curves for the $PLD(\alpha, \beta, \lambda)$ for selected values of the parameters α, β and λ.

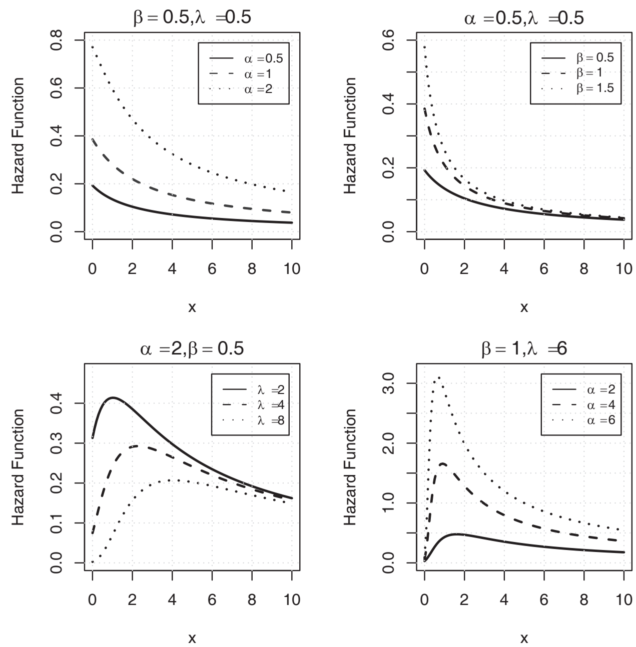

FIGURE 2: Plot of the hazard function for different values of the parameters α, β and λ.

3.3. Moments

We present an infinite sum representation for the rth moment, $\mu'_r = \mathrm{E}[X^r]$, and consequently the first four moments and variance for the PLD.

Theorem 3. *The rth moment about the origin of a random variable X, where $X \sim PLD(\alpha, \beta, \lambda)$, and $\alpha, \beta, \lambda > 0$, is given by the following:*

$$\mu'_r = \mathrm{E}[X^r] = \frac{\alpha}{\beta^r(1 - e^{-\lambda})} \sum_{n=0}^{\infty} \sum_{j=0}^{r} \binom{r}{j} \frac{\lambda^{n+1}(-1)^{n+r-j+1}}{(j - \alpha(n+1))n!}, \quad r = 1, 2, \ldots \quad (15)$$

Proof. The rth moment of X can be determined by direct integration using the pdf, i.e. $\mu'_r = \int x^r f(x)dx$. We use the Maclaurin expansion of $e^x = \sum_{n=0}^{\infty} x^n/n!$, for all x. We also use the series representation

$$(1 - w)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \; w^j, \quad \text{where } k \text{ is a positive integer.}$$

Therefore, after some transformations and integrations we have

$$\mathrm{E}\left[X^r\right]=\int_0^{\infty} x^r\ \frac{\alpha\beta\lambda\left(1+\beta x\right)^{-(\alpha+1)}e^{-\lambda(1+\beta x)^{-\alpha}}}{(1-e^{-\lambda})}\ dx.$$

Setting $y=1+\beta x$, $dx=dy/\beta$ yields

$$\begin{aligned}
\mathrm{E}\left[X^r\right] &= \frac{\alpha\lambda}{\beta^r(1-e^{-\lambda})}\int_1^{\infty}(y-1)^r y^{-(\alpha+1)}\ e^{-\lambda y^{-\alpha}}dy\\
&= \frac{\alpha\lambda}{\beta^r(1-e^{-\lambda})}\int_1^{\infty}\left\{\sum_{j=0}^{r}\binom{r}{j}y^{j-\alpha-1}(-1)^{r-j}\sum_{n=0}^{\infty}\frac{(-\lambda y^{-\alpha})^n}{n!}\right\}dy\\
&= \frac{\alpha\lambda}{\beta^r(1-e^{-\lambda})}\int_1^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{r}\binom{r}{j}\frac{\lambda^{n+1}(-1)^{n+r-j}y^{j-\alpha(n+1)-1}}{n!}\ dy\\
&= \frac{\alpha}{\beta^r(1-e^{-\lambda})}\sum_{n=0}^{\infty}\sum_{j=0}^{r}\binom{r}{j}\frac{\lambda^{n+1}(-1)^{n+r-j+1}}{(j-\alpha(n+1))n!}.
\end{aligned}$$

This completes the proof of the theorem. ■

An alternative representation formula for (15) can readily be found by expanding and substituting in the binomial expansion.

$$\mu_r'=\frac{r!}{\beta^r(1-e^{-\lambda})}\sum_{k=1}^{\infty}\frac{(-1)^{k+r-1}\lambda^k}{k!(1-k\alpha)\cdots(r-k\alpha)},\quad \alpha\neq\frac{i}{k},\ i=1,2,\cdots \tag{16}$$

One may use this representation to obtain the mean and the variance of X.

Corollary 1. *Let $X\sim PLD(\alpha,\beta,\lambda)$, where $\alpha,\beta,\lambda>0$. Then the first four moments of X are given, respectively, as follows:*

$$\left.\begin{aligned}
\mu=\mathrm{E}\left[X\right]&=\tfrac{1}{\beta(1-e^{-\lambda})}\textstyle\sum_{k=1}^{\infty}\tfrac{(-1)^k\lambda^k}{k!(1-k\alpha)},\\
\mu_2'=\mathrm{E}\left[X^2\right]&=\tfrac{2}{\beta^2(1-e^{-\lambda})}\textstyle\sum_{k=1}^{\infty}\tfrac{(-1)^{k+1}\lambda^k}{k!(1-k\alpha)(2-k\alpha)},\\
\mu_3'=\mathrm{E}\left[X^3\right]&=\tfrac{6}{\beta^3(1-e^{-\lambda})}\textstyle\sum_{k=1}^{\infty}\tfrac{(-1)^{k+2}\lambda^k}{k!(1-k\alpha)(2-k\alpha)(3-k\alpha)},\\
\mu_4'=\mathrm{E}\left[X^4\right]&=\tfrac{24}{\beta^4(1-e^{-\lambda})}\textstyle\sum_{k=1}^{\infty}\tfrac{(-1)^{k+3}\lambda^k}{k!(1-k\alpha)(2-k\alpha)(3-k\alpha)(4-k\alpha)}.
\end{aligned}\right\} \tag{17}$$

***Proof*.** Applying relations (15) or (16) for $r=1,2,3$ and $r=4$ yields the desired results. ■

Based on the results given in relations (17), the variance of X, denoted by $\sigma^2=\mu_2'-\mu^2$ is given by

$$\sigma^2=\frac{2}{\beta^2(1-e^{-\lambda})}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\lambda^k}{k!(1-k\alpha)(2-k\alpha)}-\left[\frac{1}{\beta(1-e^{-\lambda})}\sum_{k=1}^{\infty}\frac{(-1)^k\lambda^k}{k!(1-k\alpha)}\right]^2$$

It can be noticed from Table 1 that both the mean and the variance of the PL distribution are decreasing functions of α and β but they are increasing in λ. Table 2 shows the skewness and kurtosis of the PLD for various selected values of the parameters α, β and λ. The skewness is free of parameter β. Both the skewness and kurtosis are decreasing functions of α and both are increasing of λ.

Table 1: Mean and variance of PLD for various values of α, β and λ.

		$\beta = 0.5$		$\beta = 1.0$		$\beta = 2.0$	
λ	α	μ	σ^2	μ	σ^2	μ	σ^2
0.5	4.0	0.1184	1.6233	0.0592	0.4058	0.0296	0.1014
	4.5	0.1013	1.1089	0.0506	0.2772	0.0253	0.0693
	5.0	0.0885	0.8062	0.0442	0.2015	0.0221	0.0503
	5.5	0.0785	0.6128	0.0392	0.1532	0.0196	0.0383
	6.0	0.0706	0.4816	0.0353	0.1204	0.0176	0.0301
1.5	4.0	0.5890	1.9955	0.2945	0.4988	0.1472	0.1247
	4.5	0.5018	1.3402	0.2509	0.3350	0.1254	0.0837
	5.0	0.4369	0.9618	0.2184	0.2404	0.1092	0.0601
	5.5	0.3869	0.7237	0.1934	0.1809	0.0967	0.0452
	6.0	0.3471	0.5641	0.1735	0.1410	0.0867	0.0352
2.0	4.0	0.8104	2.0752	0.4052	0.5188	0.2026	0.1297
	4.5	0.6892	1.377	0.3446	0.3442	0.1723	0.0860
	5.0	0.5993	0.9791	0.2996	0.2447	0.1498	0.0611
	5.5	0.5301	0.7313	0.2650	0.1828	0.1325	0.0457
	6.0	0.4752	0.5668	0.2376	0.1417	0.1188	0.0354
4.0	4	1.4409	2.3195	0.7204	0.5798	0.3602	0.1449
	4.5	1.2179	1.4705	0.6089	0.3676	0.3044	0.0919
	5	1.0542	1.0089	0.5271	0.2522	0.2635	0.0630
	5.5	0.9289	0.7322	0.4644	0.1830	0.2322	0.0457
	6	0.8301	0.5542	0.4150	0.1385	0.2075	0.0346

3.4. L-moments

Suppose that a random sample $X_1, X_2, \ldots, X_n$ is collected from $X \sim PLD(\theta)$, where $\theta = (\alpha, \beta, \lambda)$. In what follows, we derive a general representation for the L-moments of X.

The rth population L-moments is given by

$$\begin{aligned} E[X_{r:n}] &= \int_0^\infty x f(X_{r:n})\, dx \\ &= \int_0^\infty x \sum_{i=0}^{r-1} \sum_{j=0}^{n-r+i} \binom{r-1}{i} \binom{n-r+i}{j} (-1)^{i+j} \left\{ \frac{n!\alpha\beta\lambda}{(r-1)!(n-r)!} \right. \\ &\quad \left. \times \frac{(1+\beta x)^{-(\alpha+1)} e^{-\lambda(1+\beta x)^{-\alpha}(j+1)}}{(1-e^{-\lambda})^{n-r+i+1}} \right\} dx. \end{aligned}$$

Table 2: Skewness and kurtosis of PLD for various values of α, β and λ.

		$\beta = 0.5$		$\beta = 1.0$		$\beta = 2.0$	
λ	α	γ_3	γ_4	γ_3	γ_4	γ_3	γ_4
0.5	4.5	3.6525	65.367	3.6525	16.3418	3.6525	4.0854
	5.0	3.1739	24.114	3.1739	6.0285	3.1739	1.5071
	5.5	2.8845	12.696	2.8845	3.1741	2.8845	0.7935
	6.0	2.6904	7.8535	2.6904	1.9633	2.6904	0.4908
	6.5	2.5510	5.3396	2.5510	1.3349	2.5510	0.3337
1.5	4.5	3.0490	75.423	3.049	18.855	3.0490	4.7139
	5.0	2.5371	26.405	2.5371	6.6014	2.5371	1.6503
	5.5	2.2239	13.345	2.2239	3.3362	2.2239	0.8340
	6.0	2.0116	7.9879	2.0116	1.9969	2.0116	0.4992
	6.5	1.8579	5.2867	1.8579	1.3216	1.8579	0.3304
2.0	4.5	3.0915	84.916	3.0915	21.229	3.0915	5.3072
	5.0	2.5372	29.211	2.5372	7.3029	2.5372	1.8257
	5.5	2.1963	14.561	2.1963	3.6404	2.1963	0.9101
	6.0	1.9641	8.6212	1.9641	2.1553	1.9641	0.5388
	6.5	1.7952	5.6554	1.7952	1.4138	1.7952	0.3534
4.0	4.5	3.8191	128.068	3.8191	32.017	3.8191	8.0042
	5.0	3.1425	42.525	3.1425	10.631	3.1425	2.6578
	5.5	2.7233	20.595	2.7233	5.1489	2.7233	1.2872
	6.0	2.4357	11.905	2.4357	2.9764	2.4357	0.7441
	6.5	2.2251	7.6554	2.2251	1.9138	2.2251	0.4784

Let $y = (1 + \beta x)$ so $x = (y - 1)/\beta$ and $dx = (1/\beta)dy$. After some transformation, we arrive to the formula:

$$\mathrm{E}\left[X_{r:n}\right] = \frac{1}{\beta} \sum_{m=0}^{\infty} \sum_{i=0}^{r-1} \sum_{j=0}^{n-r+i} \frac{(j+1)^m (-\lambda)^{m+1} \; A_{ij}}{(m+1)!(1-\alpha(m+1))}, \tag{18}$$

where A_{ij} is

$$A_{ij} = \frac{n!(-1)^{i+j}}{(r-1)!(n-r)!(1-e^{-\lambda})^{n-r+i+1}} \binom{r-1}{i} \binom{n-r+i}{j}.$$

One readily can use the relation (18) to obtain the first L-moments of $X_{r:n}$. For example, we take $r = n = 1$ to obtain $\lambda_1 = \mathrm{E}\left[X_{1:1}\right]$ which is the mean of the random variable X.

$$\lambda_1 = E[X_{1:1}] = \frac{1}{\beta(1-e^{-\lambda})} \sum_{m=0}^{\infty} \frac{(-\lambda)^{m+1}}{(m+1)!(1-\alpha(m+1))},$$

This result is consistent with that obtained in relation (17). The other two L-moments, λ_2 and λ_3, are respectively given by

$$\lambda_2 = \frac{1}{\beta}\Bigg[\sum_{m=0}^{\infty} \sum_{i=0}^{1} \sum_{j=0}^{i} \binom{1}{i}\binom{i}{j} \frac{(j+1)^m (-1)^{i+j+m+1} \lambda^{m+1}}{(m+1)!(1-\alpha(m+1))(1-e^{-\lambda})^{i+1}} - \sum_{m=0}^{\infty} \sum_{j=0}^{1} \binom{1}{j} \frac{(j+1)^m (-1)^{j+m+1} \lambda^{m+1}}{(m+1)!(1-\alpha(m+1))(1-e^{-\lambda})^2} \Bigg]$$

and

$$\begin{aligned}\lambda_3 &= \frac{1}{\beta}\Bigg[\sum_{m=0}^{\infty}\sum_{i=0}^{2}\sum_{j=0}^{i}\binom{2}{i}\binom{i}{j}\frac{(j+1)^m(-1)^{i+j+m+1}\lambda^{m+1}}{(m+1)!(1-\alpha(m+1))(1-e^{-\lambda})^{i+1}}\\ &-2\sum_{m=0}^{\infty}\sum_{i=0}^{1}\sum_{j=0}^{i+1}\binom{1}{i}\binom{i+1}{j}\frac{2(j+1)^m(-1)^{i+j+m+1}\lambda^{m+1}}{(m+1)!(1-\alpha(m+1))(1-e^{-\lambda})^{i+2}}\\ &+\sum_{m=0}^{\infty}\sum_{j=0}^{2}\binom{2}{j}\frac{2(j+1)^m(-1)^{j+m+1}\lambda^{m+1}}{(m+1)!(1-\alpha(m+1))(1-e^{-\lambda})^{3}}\Bigg]\end{aligned}$$

The method of L-moments consists of equating the first L-moments of a population, λ_1, λ_2 and λ_3, to the corresponding L-moments of a sample, l_1, l_2 and l_3, thus getting a number of equations that are needed to be solved, numerically, in terms of the unknown parameters, θ.

3.5. Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the PL distribution in (10) and let $X_{1:n}, \dots, X_{n:n}$ denote the corresponding order statistics. Then, the pdf of $X_{r:n}$, $1 \leq r \leq n$, is given by (see, David & Nagaraja 2003, Arnold, Balakrishnan & Nagaraja 1992)

$$g_{(r)}(x) = C_{r,n}\, g(x)[G(x)]^{r-1}[1-G(x)]^{n-r}, \quad 0 < x < \infty, \tag{19}$$

where $C_{r,n} = [B(r, n-r+1)]^{-1}$, with $B(a,b)$ being the complete beta function.

Theorem 4. *Let $G(x)$ and $g(x)$ be the cdf and pdf of a Poisson-Lomax distribution for a random variable X. The density of the rth order statistic, say $g_{(r)}(x)$ is given by*

$$g_{(r)}(x) = \alpha\beta\lambda C_{r,n}\sum_{i=0}^{r-1}\sum_{j=0}^{n-r+i}\binom{r-1}{i}\binom{n-r+i}{j} \frac{(-1)^{i+j}(1+\beta x)^{-(\alpha+1)}\ e^{-\lambda(1+\beta x)^{-\alpha}(j+1)}}{(1-e^{-\lambda})^{n-r+i+1}} \tag{20}$$

***Proof*.** First it should be noted that (19) can be written as follows:

$$g_{(r)}(x) = C_{r,n}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i g(x)[\bar{G}(x)]^{n-r+i} \tag{21}$$

then the proof follows by replacing the reliability, $\bar{G}(x)$, and the pdf, $g(x)$, of $X \sim PLD(\alpha, \beta, \lambda)$ which are obtained from (9) and (10), respectively, and substituting them into relation (21), and expanding the term $(1-e^{-\lambda(1+\beta x)^{-\alpha}})^{n-r+i}$ using the binomial expansion. ∎

3.6. Quantile Function

Let X denote a random variable with the probability density function given by (10). The quantile function, denoted by $Q(u)$, is

$$Q(u) = \inf\{x \in R : F(x) \geq u\}, \quad \text{where } 0 < u < 1$$

By inverting the distribution function, $F = 1 - \bar{F}$, we can write the following:

$$Q(u) = \frac{1}{\beta}\left[\left(\frac{-\ln(u(1-e^{-\lambda}) + e^{-\lambda})}{\lambda}\right)^{-1/\alpha} - 1\right] \tag{22}$$

The first quartile, the median and the third quartile can be obtained simply by applying (22). The quartiles; Q_1 first quartile, Q_2 second quartile, or the median, and Q_3 third quartile are obtained in Table 3.

TABLE 3: The quartile values of the PLD for different values of α, β and λ.

		$\beta = 0.5$			$\beta = 1.0$			$\beta = 2.0$		
λ	α	Q_1	Q_2	Q_3	Q_1	Q_2	Q_3	Q_1	Q_2	Q_3
0.5	4.0	0.1870	0.4583	0.9647	0.0935	0.2291	0.4824	0.0467	0.1146	0.2412
	4.5	0.1654	0.4025	0.8379	0.0827	0.2013	0.4189	0.0413	0.1006	0.2095
	5.0	0.1482	0.3589	0.7403	0.0741	0.1794	0.3701	0.0371	0.0897	0.1851
	5.5	0.1343	0.3238	0.6629	0.0672	0.1619	0.3315	0.0336	0.0809	0.1657
	6.0	0.1228	0.2949	0.6002	0.0614	0.1474	0.3001	0.0307	0.0737	0.1500
1.5	4.0	0.2893	0.6431	1.2469	0.1446	0.3216	0.6234	0.0723	0.1608	0.3117
	4.5	0.2552	0.5625	1.0767	0.1276	0.2813	0.5384	0.0638	0.1406	0.2692
	5.0	0.2282	0.4998	0.9470	0.1141	0.2499	0.4735	0.0571	0.1249	0.2368
	5.5	0.2065	0.4496	0.8450	0.1032	0.2248	0.4225	0.0516	0.1124	0.2112
	6.0	0.1885	0.4086	0.7626	0.0942	0.2043	0.3813	0.0471	0.1021	0.1907
2.0	4.0	0.3521	0.7418	1.3856	0.1760	0.3709	0.6928	0.0880	0.1855	0.3464
	4.5	0.3101	0.6474	1.1933	0.1550	0.3237	0.5966	0.0775	0.1618	0.2983
	5.0	0.2770	0.5742	1.0473	0.1385	0.2871	0.5237	0.0693	0.1435	0.2618
	5.5	0.2503	0.5158	0.9328	0.1252	0.2579	0.4664	0.0626	0.1289	0.2332
	6.0	0.2283	0.4681	0.8408	0.1142	0.2341	0.4204	0.0571	0.1170	0.2102
4.0	4.0	0.6324	1.1205	1.8827	0.3162	0.5602	0.9414	0.1581	0.2801	0.4707
	4.5	0.5533	0.9700	1.6068	0.2766	0.4850	0.8034	0.1383	0.2425	0.4017
	5.0	0.4917	0.8548	1.4003	0.2458	0.4274	0.7001	0.1229	0.2137	0.3501
	5.5	0.4424	0.7640	1.2401	0.2212	0.3820	0.6201	0.1106	0.1910	0.3100
	6.0	0.4020	0.6900	1.1125	0.2010	0.3452	0.5562	0.1005	0.1726	0.2781

3.7. Mean Deviations

The mean deviation about the mean and the mean deviation about the median are, respectively, defined by

$$\delta_1(\mu) = 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} z f(z) dz \tag{23}$$

$$\delta_2(M) = 2MF(M) - M - \mu + 2\int_{M}^{\infty} z f(z) dz \tag{24}$$

Theorem 5. *Let X be a random variable distributed according to the PL distribution. Then the mean deviation about the mean, δ_1, and the mean deviation about the median, δ_2, are given as follows:*

$$\begin{aligned}\delta_1(\mu) = \frac{2}{1-e^{-\lambda}} \Bigg\{ &\mu(e^{-\lambda(1+\beta\mu)^{-\alpha}} - 1) - \frac{\alpha}{\beta}\sum_{n=0}^{\infty}\frac{\lambda^{n+1}(-1)^n}{n!} \\ &\times \left(\frac{(1+\beta\mu)^{1-\alpha(n+1)}}{1-\alpha(n+1)} + \frac{(1+\beta\mu)^{-\alpha(n+1)}}{\alpha(n+1)}\right)\Bigg\}\end{aligned} \tag{25}$$

and

$$\begin{aligned}\delta_2(M) = \frac{1}{1-e^{-\lambda}} \Bigg\{ &M\left(2e^{-\lambda(1+\beta M)^{-\alpha}} - e^{-\alpha} - 1\right) \\ &+ \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{n+1}}{(n+1)!(1-(n+1)\alpha)} \\ &- \frac{2\alpha}{\beta}\sum_{n=0}^{\infty}\frac{\lambda^{n+1}(-1)^n}{n!}\Bigg(\frac{(1+\beta M)^{1-\alpha(n+1)}}{1-\alpha(n+1)} \\ &+ \frac{(1+\beta M)^{-\alpha(n+1)}}{\alpha(n+1)}\Bigg)\Bigg\}\end{aligned} \tag{26}$$

***Proof*.** The proof follows by plugging the density function of the PLD into equation (23) and working out the integration I, where

$$I = \int_{\mu}^{\infty} xg(x)dx = \frac{\alpha\beta\lambda}{1-e^{-\lambda}}\int_{\mu}^{\infty} x(1+\beta x)^{-(\alpha+1)}e^{-\lambda(1+\beta x)^{-\alpha}}dx$$

Setting $y = 1+\beta x$, so $dy = \beta dx$ and using the expansion $e^x = \sum_{n=0}^{\infty} x^n/n!$, yields

$$I = \frac{-\alpha}{\beta(1-e^{-\lambda})}\sum_{n=0}^{\infty}\frac{\lambda^{n+1}(-1)^n}{n!}\left(\frac{(1+\beta\mu)^{1-\alpha(n+1)}}{1-\alpha(n+1)} + \frac{(1+\beta\mu)^{-\alpha(n+1)}}{\alpha(n+1)}\right)$$

Substituting I into relation (23) and manipulating the other terms gives directly the desired result. Similarly, the measure $\delta_2(M)$ can be obtained. ■

4. Estimation

In this section we consider maximum likelihood estimation (MLE) to estimate the involved parameters. Asymptotic distribution of $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ are obtained using the elements of the inverse Fisher information matrix.

4.1. Maximum Likelihood Estimation

The idea behind the maximum likelihood parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. For this purpose, let $X_1, X_2, \cdots, X_n$ is be random sample from $X \sim PLD(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\alpha, \beta, \lambda)$. Then the likelihood function of the observed sample is given by

$$\begin{aligned} L(\boldsymbol{\theta}; x) &= \prod_{i=1}^{n} f(x_i; \boldsymbol{\theta}) \\ &= \prod_{i=1}^{n} \frac{\lambda\alpha\beta(1+\beta x_i)^{-(\alpha+1)}\ e^{-\lambda(1+\beta x_i)^{-\alpha}}}{(1-e^{-\lambda})} \\ &= \frac{(\lambda\alpha\beta)^n}{(1-e^{-\lambda})^n} \prod_{i=1}^{n} (1+\beta x_i)^{-(\alpha+1)}\ e^{-\lambda\sum_{i=1}^{n}(1+\beta x_i)^{-\alpha}} \end{aligned} \tag{27}$$

The log-likelihood function is given by

$$\begin{aligned} \ell(x; \alpha, \beta, \lambda) &= n\ln(\alpha) + n\ln(\beta) + n\ln(\lambda) - (\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i) \\ &\quad -\lambda\sum_{i=1}^{n}(1+\beta x_i)^{-\alpha} - n\ln(1-e^{-\lambda}) \end{aligned} \tag{28}$$

The MLEs of α, β and λ say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$, respectively, can be worked out by the solutions of the system of equations obtained by letting the first partial derivatives of the total log-likelihood equal to zero with respect to $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$. Therefore, the system of equations is as follows:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\ln(1+\beta x_i) + \lambda\sum_{i=1}^{n}(1+\beta x_i)^{-\alpha}\ln(1+\beta x_i) = 0$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i} + \alpha\lambda\sum_{i=1}^{n}x_i(1+\beta x_i)^{-(\alpha+1)} = 0$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}(1+\beta x_i)^{-\alpha} - \frac{n}{(e^{\lambda}-1)} = 0$$

For simplicity, we define A_i to be as $A_i = 1 + \beta x_i$. Thus, we have

$$\hat{\alpha} = n\left[\sum_{i=1}^{n}\ln(A_i)\ (1-\lambda A_i^{-\alpha})\right]^{-1} \tag{29}$$

$$\hat{\beta} = n\left[\sum_{i=1}^{n}\frac{x_i}{A_i}\ (\alpha+1-\alpha\lambda A_i^{-\alpha})\right]^{-1} \tag{30}$$

$$\hat{\lambda} = n\left[\sum_{i=1}^{n}A_i^{-\alpha} + \frac{n}{e^{\lambda}-1}\right]^{-1} \tag{31}$$

The solutions of nonlinear equations (29), (30) and (31) are complicated to obtain, therefore an iterative procedure is applied to solve these equations numerically.

4.2. Asymptotic Distribution

We obtain the asymptotic distribution of $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$. The asymptotic variances of MLEs are given by the elements of the inverse of the Fisher information matrix. The Fisher information matrix of $\boldsymbol{\theta}$, denoted by $\boldsymbol{J}(\boldsymbol{\theta}) = \boldsymbol{E}(\boldsymbol{I}, \boldsymbol{\theta})$, where $\boldsymbol{I}_{ij}$, $i, j = 1, 2, 3$ is the observed information matrix. The second partial derivatives of the maximum likelihood function are given as the following:

$$
\begin{aligned}
I_{11} &= -\frac{n}{\alpha^2} - \lambda \sum_{i=1}^{n} (1+\beta x_i)^{-\alpha} [\ln(1+\beta x_i)]^2 \\
&= -\frac{n}{\alpha^2} - \lambda \sum_{i=1}^{n} A_i^{-\alpha} [\ln(A_i)]^2 \\
I_{12} = I_{21} &= \sum_{i=1}^{n} \frac{-x_i}{(1+\beta x_i)} + \lambda \sum_{i=1}^{n} x_i (1+\beta x_i)^{-(\alpha+1)} \left[1 - \alpha \ln(1+\beta x_i)\right] \\
&= \sum_{i=1}^{n} \left[\frac{x_i}{A_i} \left(-1 - \lambda \alpha A_i^{-\alpha} \ln(A_i) + \lambda A_i^{-\alpha}\right) \right] \\
I_{13} = I_{31} &= \sum_{i=1}^{n} (1+\beta x_i)^{-\alpha} \ln(1+\beta x_i) = \sum_{i=1}^{n} A_i^{-\alpha} \ln(A_i) \\
I_{22} &= -\frac{n}{\beta^2} + (\alpha+1) \sum_{i=1}^{n} \frac{x_i^2}{(1+\beta x_i)^2} - \lambda \alpha (\alpha+1) \sum_{i=1}^{n} \frac{x_i^2 (1+\beta x_i)^{-\alpha}}{(1+\beta x_i)^2} \\
&= -\frac{n}{\beta^2} + (\alpha+1) \sum_{i=1}^{n} \left(\frac{x_i}{A_i}\right)^2 \left(1 - \lambda \alpha A_i^{-\alpha}\right) \\
I_{23} = I_{32} &= \alpha \sum_{i=1}^{n} x_i (1+\beta x_i)^{-(\alpha+1)} = \alpha \sum_{i=1}^{n} x_i A_i^{-(\alpha+1)} \\
I_{33} &= -\frac{n}{\lambda^2} + \frac{n e^{\lambda}}{(e^{\lambda}-1)^2}
\end{aligned}
$$

The exact mathematical expressions for $\boldsymbol{J}(\boldsymbol{\theta}) = \boldsymbol{E}(\boldsymbol{I}, \boldsymbol{\theta})$ are complicated to obtain. Therefore, the observed Fisher information matrix can be used instead of the Fisher information matrix. The variance-covariance matrix may be approximated as $\boldsymbol{V}_{ij} = \boldsymbol{I}_{ij}^{-1}$. The asymptotic distribution of the maximum likelihood can be written as follows (see Miller 1981).

$$
\left[(\hat{\alpha} - \alpha), (\hat{\beta} - \beta), (\hat{\lambda} - \lambda)\right] \sim N_3\left(\mathbf{0}, \boldsymbol{V}\right) \tag{32}
$$

Since $\boldsymbol{V}$ involves the parameters α, β and λ, we replace the parameters by the corresponding MLEs in order to obtain an estimate of $\boldsymbol{V}$, which is denoted by $\hat{\boldsymbol{V}}$. By using (32), approximate $100(1-\vartheta)\%$ confidence intervals for α, β and λ are determined, respectively, as

$$
\hat{\alpha} \pm Z_{\vartheta/2} \sqrt{\hat{\boldsymbol{V}}_{11}}, \quad \hat{\beta} \pm Z_{\vartheta/2} \sqrt{\hat{\boldsymbol{V}}_{22}}, \quad \hat{\lambda} \pm Z_{\vartheta/2} \sqrt{\hat{\boldsymbol{V}}_{33}},
$$

where Z_{ϑ} is the upper 100ϑ-th percentile of the standard normal distribution.

In the order to numerically illustrate the estimation of the involved parameters, we have simulated the ML estimators for different sample sizes. The calculation of the estimation is based on 10,000 simulated samples from the standard PLD. Table 4 shows the MLEs, mean squared errors (MSE) and 95% confidence limits (LCL & UCL) for the parameters α, β, and λ. The true values of the parameters used for simulation were $\alpha = 1, \beta = 1$, and $\lambda = 2$. It is observed that when the sample size n increases, the MLE of α and λ decrease to approach the true one while the MLEs of the parameters β increase.

Table 4: Simulation study: parameter values used for simulation (TRUE) $\alpha = 1, \beta = 1, \lambda = 2$, MLEs, mean squared errors (MSE) and 95% confidence limits (LCL & UCL) for the parameters.

				95% Confi. Limits	
Parameters	n	Estimates	MSE	LCL	UCL
α	20	1.10868	0.05159	-2.00901	4.22637
	30	1.08199	0.03129	-1.12927	3.29326
	40	1.06866	0.02202	-0.62073	2.75807
	50	1.06119	0.01762	-1.43696	3.55935
	60	1.05224	0.01431	0.10648	1.99800
	70	1.04646	0.01203	0.15111	1.94181
	80	1.04378	0.01034	0.01529	2.07227
	90	1.03915	0.00871	0.18454	1.89376
	100	1.03811	0.00791	0.21745	1.85878
	200	1.02512	0.00375	0.30619	1.74405
β	20	0.94360	0.05699	0.52240	1.36480
	30	0.94997	0.03854	0.60608	1.29387
	40	0.95472	0.03019	0.65637	1.25308
	50	0.96011	0.02421	0.69225	1.22797
	60	0.96078	0.02043	0.71629	1.20527
	70	0.96329	0.01748	0.73662	1.18997
	80	0.96387	0.01600	0.75180	1.17594
	90	0.96371	0.01401	0.76389	1.16353
	100	0.97031	0.01216	0.77951	1.16110
	200	0.97528	0.00683	0.83990	1.11065
λ	20	2.07641	0.05612	0.38236	3.77045
	30	2.05300	0.03373	0.67893	3.42706
	40	2.03975	0.02294	0.85301	3.22649
	50	2.03150	0.01773	0.97162	3.09137
	60	2.02744	0.01478	1.06066	2.99422
	70	2.02349	0.01221	1.12896	2.91801
	80	2.02025	0.01077	1.18387	2.85662
	90	2.01885	0.00929	1.23050	2.80719
	100	2.01723	0.00845	1.26951	2.76495
	200	2.00944	0.00388	1.48125	2.53762

5. Application

We have considered a dataset corresponding to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee & Wang (2003). The data are given as follows: 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. We have fitted the Poisson-Lomax distribution to the dataset using MLE, and compared the proposed PLD with Lomax, extended Lomax and Lomax-Logarithmic distributions.

The model selection is carried out using the AIC (Akaike information criterion), the BIC (Bayesian information criterion), the CAIC (consistent Akaike information criteria) and the HQIC (Hannan-Quinn information criterion).

$$\left.\begin{array}{l} \mathrm{AIC} = -2l(\hat{\theta}) + 2q, \\ \mathrm{BIC} = -2l(\hat{\theta}) + q\log(n), \\ \mathrm{HQIC} = -2l(\hat{\theta}) + 2q\log(\log(n)), \\ \mathrm{CAIC} = -2l(\hat{\theta}) + \frac{2qn}{n-q-1} \end{array}\right\} \tag{33}$$

where $l(\hat{\theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, q is the number of parameters, and n is the sample size. Here we let $\boldsymbol{\theta}$ denote the parameters, i.e., $\boldsymbol{\theta} = (\alpha, \beta, \lambda)$. An iterative procedure is applied to solve equations (29), (30) and (31) and consequently obtain $\hat{\boldsymbol{\theta}} = (\hat{\alpha} = 2.8737, \hat{\beta} = 8.2711, \hat{p} = 3.3515)$. At these values we calculate the log-likelihood function given by (28) and apply relation (33). The model with minimum AIC (or BIC, CAIC and HQIC) value is chosen as the best model to fit the data. From Table 5, we conclude that the PLD is best comparable to the Lomax, extended Lomax and Lomax-Logarithmic models.

Table 5: MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC and CAIC.

	Estimates				Measures			
Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	AIC	BIC	HQIC	CAIC
Lomax	13.9384	121.0222			831.67	837.37	833.98	831.76
	(15.3837)	(142.6940)						
MOEL	23.7437	2.0487	2.2818		825.08	833.64	828.56	825.27
	(35.8106)	(2.5891)	(0.5551)					
PLD	2.8737	8.2711		3.3515	824.77	833.33	828.25	824.96
	(0.8869)	(4.8795)		(1.0302)				

For an ordered random sample, $X_1, X_2, \dots, X_n$, from $PLD(\alpha, \beta, \lambda)$, where the parameters α, β and λ are unknown, the Kolmogorov-Smirnov D_n, Cramér-von Mises W_n^2, Anderson and Darling A_n^2, Watson U_n^2 and Liao-Shimokawa L_n^2 tests statistics are given as follows: (For details see e.g. Al-Zahrani (2012) and references therein).

$$D_n = \max_i \left[\frac{i}{n} - G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda}), G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda}) - \frac{i-1}{n} \right]$$

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left[G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda}) - \frac{2i-1}{2n} \right]^2$$

$$A_n^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left[\log(G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda})) + \log(1 - G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda})) \right]^2$$

$$U_n^2 = W_n^2 + \sum_{i=1}^{n} \left[\frac{G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda})}{n} - \frac{1}{2} \right]^2$$

$$L_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\max_i \left[\frac{i}{n} - G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda}), G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda}) - \frac{i-1}{n} \right]}{\sqrt{G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda})[1 - G_{PL}(x_i, \hat{\alpha}, \hat{\beta}, \hat{\lambda})]}}.$$

Table 6 indicates that the test statistics D_n, W_n^2, A_n^2, U_n^2 and L_n have the smallest values for the data set under PLD model with regard to the other models. The proposed model offers an attractive alternative to the Lomax, Lomax-Logarithmic and extended Lomax models. Figure 3 displays the empirical and fitted densities for the data. Estimated survivals for data are shown in Figure 4. The Poisson-Lomax distribution approximately provides an adequate fit for the data. The quantile-quantile or Q-Q plot is used to check the validity of the distributional assumption for the data. Figure 5 shows that the data seems to follow a PLD reasonably well, except some points on extreme.

Table 6: Goodness-of-fit tests.

	Statistics				
Distribution	D_n	W_n^2	A_n^2	U_n^2	L_n
Lomax	0.0967	0.2126	1.3768	31.7017	1.0594
MOEL	0.0302	0.0151	0.0926	31.5177	0.3728
LLD	0.0821	0.1274	0.8739	31.6200	0.8491
PLD	0.0281	0.0134	0.0835	31.5164	0.3567

6. Concluding Remarks

In this paper we have proposed a new distribution, referred to as the PLD. A mathematical treatment of the proposed distribution including explicit formulas for the density and hazard functions, moments, order statistics, and mean and median deviations have been provided. The estimation of the parameters has been approached by maximum likelihood. Also, the asymptotic variance-covariance matrix of the estimates has been obtained. Finally, a real data set was analyzed to show the potential of the proposed PLD. The result indicates that the PLD may be used for a wider range of statistical applications. Further study can be conducted on the proposed distribution. Here, we mention some of possible directions which are still open for further works. The problem of parameter estimation can be studied using e.g. Bayesian approach and making future prediction. The parameters of the proposed distribution can be estimated based on censored data. Some recurrence relations can be established for the single moments and product moments of order statistics.

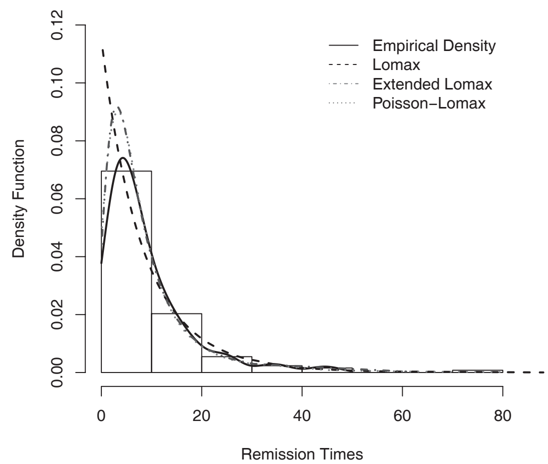

Figure 3: Estimated densities for bladder cancer data.

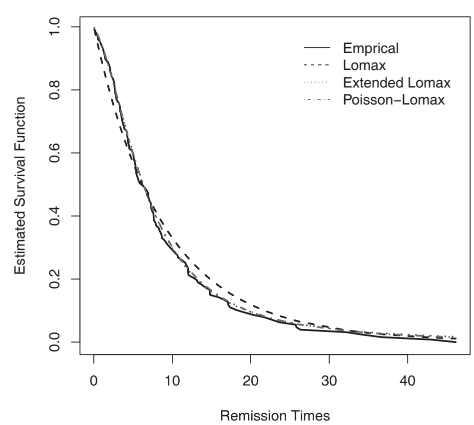

Figure 4: Estimated survivals for bladder cancer data.

Acknowledgments

The authors are grateful to the Editor and the anonymous referees for their valuable comments and suggestions that improved the presentation of the paper. This study is a part of the Master Thesis of the second named author whose work was supervised by the first named author.

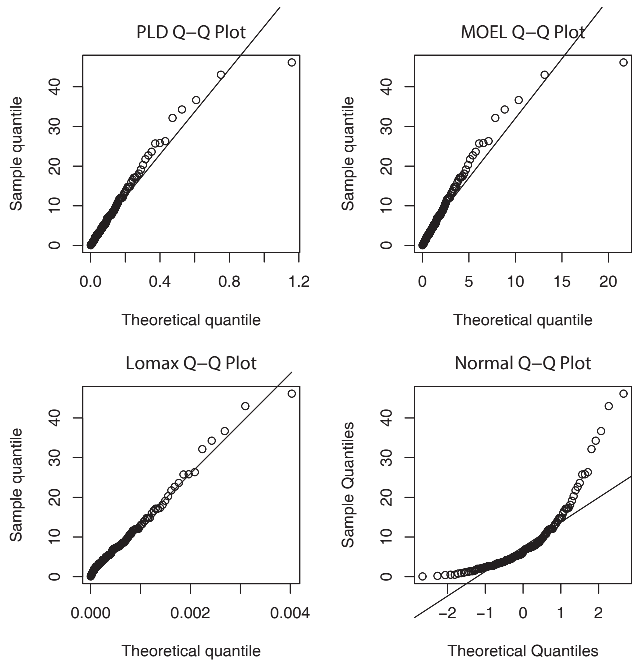

Figure 5: The Q-Q plot for bladder cancer data.

[Recibido: noviembre de 2013 — Aceptado: abril de 2014]

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