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HIGH ORDER NECESSARY OPTIMALITY CONDITIONS

Abstract.
In this paper we present a method for determining some variations of a singular trajectory of an affine control system. These variations provide necessary optimality conditions which may distinguish between maximizing and minimizing problems. The generalized Legendre-Clebsch conditions are an example of these type of conditions.

1. Introduction

The variational approach to Majer minimization control problems can be roughly summarized in the following way: let x^* be a solution on the interval $[t_i, t_e]$ relative to the control u^*; if the pair (x^*, u^*) is optimal, then the cone of tangent vectors to the reachable set at $x^*(t_e)$ is contained in the subspace where the cost increase. If there are constraints on the end-points, then the condition is no more necessary; nevertheless in [1] it has been proved that particular subcones of tangent vectors, the regular tangent cones, have to be contained in a cone which depends on the cost and on the constraints. Tangent vectors whose collection is a regular tangent cone are named good trajectory variations, see [8].

The aim of this paper is to construct good trajectory variations of a trajectory of an affine control process which contains singular arcs, i.e. arcs of trajectory relative to the drift term of the process. It is known, [2], that the optimal trajectory of an affine control process may be of this type; however the pair $(x^*, 0)$ may satisfy the Pontrjagin Maximum Principle without being optimal. Therefore it is of interest in order to single out a smaller number of candidates to the optimum, to know as many good trajectory variations as we can.

In [5] good trajectory variations of the pair $(x^*, 0)$ have been constructed by using the relations in the Lie algebra associated to the system at the points of the trajectory. The variations constructed in that paper are of bilateral type, i.e. both the directions $+v$ and $-v$ are good variations. In this paper I am going to find conditions which single out unilateral variations, i.e. only one direction need to be a variation. Unilateral variations are of great interest because, contrary to the bilateral ones, they distinguish between maximizing and minimizing problems.

2. Notations and preliminary results

To each family $\mathbf{f} = (f_0, f_1, \ldots, f_m)$ of C^∞ vector fields on a finite dimensional manifold M we associate the affine control process $\Sigma_\mathbf{f}$ on M

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \qquad |u_i| \leq \alpha \tag{1}$$

where the control $u = (u_1, \ldots, u_m)$ is a piecewise constant map whose values belong to the hypercube $|u_i| \le \alpha$. We will denote by $S_{\mathbf{f}}(t, t_0, y, u)$ the value at time t of the solution of $\Sigma_{\mathbf{f}}$ relative to the control u, which at time t_0 is equal to y. We will omit t_0 if it is equal to 0, so that $S_{\mathbf{f}}(t, y, u) = S_{\mathbf{f}}(t, 0, y, u)$; we will also use the exponential notation for constant control map, for example $\exp\, t f_0 \cdot y = S_{\mathbf{f}}(t, y, 0)$.

We want to construct some variations of the trajectory $t \mapsto x_{\mathbf{f}}(t) = \exp t f_0 \cdot x_0$, $t \in [t_0, t_1]$ at time $\tau \in [t_0, t_1]$. We will consider trajectory variations produced by needle-like control variations concentrated at τ. The definition is the following:

DEFINITION 1. *A vector $v \in T_{x_{\mathbf{f}}(\tau)} M$ is a right (left) trajectory variation of $x_{\mathbf{f}}$ at τ if for each $\epsilon \in [0, \overline{\epsilon}]$ there exists a control map $u(\epsilon)$ defined on the interval $[0, a(\epsilon)]$, $\lim_{\epsilon \to 0^+} a(\epsilon) = 0$, such that $u(\epsilon)$ depends continuously on ϵ in the L^1 topology and the map $\epsilon \mapsto \exp\,(-a(\epsilon)\, f_0) \cdot S_{\mathbf{f}}(a(\epsilon), x_{\mathbf{f}}(\tau), u(\epsilon))$, ($\epsilon \mapsto S_{\mathbf{f}}(a(\epsilon), x_{\mathbf{f}}(\tau - a(\epsilon)), u(\epsilon))$) has v as tangent vector at $\epsilon = 0$.*

The variations at τ indicates the controllable directions of the reference trajectory from $x_{\mathbf{f}}(\tau)$; they are local objects at $x_{\mathbf{f}}(\tau)$ and in any chart at $x_{\mathbf{f}}(\tau)$ they are characterized by the property

$$S_{\mathbf{f}}(a(\epsilon), x_{\mathbf{f}}(\tau), u(\epsilon)) = x_{\mathbf{f}}(\tau + a(\epsilon)) + \epsilon\, v + o(\epsilon) \in R(\tau + a(\epsilon), x_{\mathbf{f}}(t_0)) \tag{2}$$

where $R(t, x)$ is the set of points reachable in time t from x.

The transport along the reference flow generated by the 0 control from time τ to time t_1 of a variation at τ is a tangent vector to the reachable set at time t_1 in the point $x_{\mathbf{f}}(t_1)$. The transport of particular trajectory variations, the good ones, gives rise to tangent vectors whose collection is a regular tangent cone. The definition of good variations is the following:

DEFINITION 2. *A vector $v \in T_{x^*(\tau)} M$ is a good* right variation (left variation) at τ of order *k if there exists positive numbers $\overline{c}$, $\overline{\epsilon}$ and for each $\epsilon \in [0, \overline{\epsilon}]$ a family of admissible control maps, $u_\epsilon(c)$, $c \in [0, \overline{c}]$ with the following properties:*

1. *$u_\epsilon(c)$ is defined on the interval $[0, a\epsilon^k]$*
2. *for each ϵ, $c \mapsto u_\epsilon(c)$ is continuous in the L^1 topology*
3. *$\exp[-(1+a)\epsilon^k] f_0 \cdot S_{\mathbf{f}}(a\epsilon^k, x_{\mathbf{f}}(\tau + \epsilon^k), u_\epsilon(c)) = x_{\mathbf{f}}(\tau) + \epsilon c\, v + o(\epsilon)$ ($\exp \epsilon^k f_0 \cdot S_{\mathbf{f}}(a\epsilon^k, x_{\mathbf{f}}(\tau - (1+a)\epsilon^k), u_\epsilon(c)) = x_{\mathbf{f}}(\tau) + \epsilon c\, v + o(\epsilon)$) uniformly w.r.t. c.*

The good trajectory variations will be simpler named g-variations. Standing the definitions, the variations of a trajectory are more easily found than its g-variations. However a property proved in [8] allows to find g-variations as limit points of trajectory ones. More precisely the following Proposition holds:

PROPOSITION 1. *Let I be an interval contained in $[t_0, t_1]$ and let $g \in L^1(I)$ be such that $g(t)$ is a right (left) trajectory variation at t for each t in the set L^+, (L^-), of right (left) Lebesgue points of g. For each $t \in L^+$, ($t \in L^-$), let $[0, a_t(\epsilon)]$ be the interval as in Definition 1 relatively to the variation $g(t)$. If there exists positive numbers N and s such that for each $\tau \in L^+$, ($\tau \in L^-$), $0 < a_\tau(\epsilon) \le (N\epsilon)^s$, then for each $t \in L^+$ $g(t)$ is a right variation, (for each $t \in L^-$, $g(t)$ is a left variation), at t of order s.*

Let τ be fixed; to study the variations at τ we can suppose without loss of generality that M is an open neighborhood of $0 \in \mathbb{R}^n$. Moreover by Corollary 3.3 in [5], we can substitute to the family **f** the family ϕ where ϕ_i is the Taylor polynomial of f_i of order sufficiently large. We can therefore suppose that **f** is an analytic family of vector fields on $\mathbb{R}^n$.

Let me recall some properties of analytic family of vectors fields. Let $\mathbf{X} = \{X_0, X_1, \dots, X_m\}$ be $(m+1)$ indeterminate; $L(\mathbf{X})$ is the Lie algebra generated by $\mathbf{X}$ with Lie bracket defined by

$$[S, T] = ST - TS .$$

$\hat{L}(\mathbf{X})$ denotes the set of all formal series, $\sum_{k=1}^{\infty} P_k$, each P_k homogeneous Lie polynomial of degree k. For each $S \in \hat{L}(\mathbf{X})$ we set

$$\exp S = \sum_{k=0}^{\infty} \frac{S^k}{k!}$$

and

$$\log(Id + Z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} Z^k}{k} .$$

The following identities hold

$$\exp(\log Z) = Z \qquad\qquad \log(\exp S) = S .$$

Formula di Campbell-Hausdorff [9]
For each $P,\ Q \in \hat{L}(\mathbf{X})$ there exists an unique $Z \in \hat{L}(\mathbf{X})$ such that

$$\exp P \cdot \exp Q = \exp Z$$

and Z is given by

$$Z = P + Q + \frac{1}{2}[P, Q] + \cdots$$

Let u be an admissible piecewise constant control defined on the interval $[0, T(u)]$; by the Campbell Hausdorff formula we can associate to u an element of $\hat{L}(\mathbf{X})$, $\log u$, in the following way: if $u(t) = (\omega_1^i, \dots, \omega_m^i)$ in the interval (t_{i-1}, t_i) then

$$\exp\log u = \exp(T(u) - t_{k-1}) \left(X_0 + \sum_{i=1}^{m} \omega_i^k X_i \right) \cdot \cdots \cdot \exp t_1 \left(X_0 + \sum_{i=1}^{m} \omega_i^1 X_i \right)$$

If **f** is an analytic family, then $\log u$ is linked to $S_{\mathbf{f}}(T(u), y, u)$ by the following proposition [6]

PROPOSITION 2. *If* **f** *is an analytic family of vector fields on an analytic manifold M then for each compact $K \subset M$ there exist T such that, if $\log_{\mathbf{f}} u$ denotes the serie of vector fields obtained by substituting in $\log u$, f_i for X_i, then $\forall y \in K$ and $\forall u$, $T(u) < T$, the serie $\exp\log_{\mathbf{f}} u \cdot y$ converges uniformly to $S_{\mathbf{f}}(T(u), y, u)$.*

In the sequel we will deal only with right variations. The same ideas can be used to construct left variations.

To study the right trajectory variations it is useful to introduce $\operatorname{Log} u$ defined by

$$\exp(\operatorname{Log} u) = \exp -T(u) X_0 \cdot \exp(\log u) .$$

By definition it follows that if y belongs to a compact set and $T(u)$ is sufficiently small, then $\exp(-T(u))\, f_0 \cdot S_{\mathbf{f}}(T(u), y, u)$ is defined and it is the sum of the serie $\exp(\operatorname{Log}_{\mathbf{f}} u) y$; notice that

$\exp(\mathrm{Log}_{\mathbf{f}} u)y$ is the value at time $T(u)$ of the solution of the pullback system introduced in [4] starting at y.

Let $u(\epsilon)$ be a family of controls which depend continuously on ϵ and such that $T(u(\epsilon)) = o(1)$. Such a family will be named control variation if

$$\mathrm{Log}\, u(\epsilon) = \sum \epsilon^{j_i} Y^i \tag{3}$$

with $Y^i \in \mathrm{Lie}\,\mathbf{X}$ and $j_i < j_{i+1}$. Let j_i be the smallest integer for which $Y^s_{\mathbf{f}}(x_{\mathbf{f}}(\tau)) \neq 0$; Y^i is named **f**-*leading term* of the control variation at τ because it depends on the family **f** and on the time τ.

The definition of exp and Proposition 2 imply that if Y_i is an **f**-leading term of a control variation, then

$$\exp(-T(u(\epsilon)) \cdot S_{\mathbf{f}}(T(u(\epsilon)),\ x_{\mathbf{f}}(\tau),\ u(\epsilon)) = x_{\mathbf{f}}(\tau) + \epsilon^{j_i} Y^i_{\mathbf{f}}(x_{\mathbf{f}}(\tau)) + o(\epsilon^{j_i})\,;$$

therefore by Definition 1, $Y^j_{\mathbf{f}}(x(\tau))$ is a variation of $x_{\mathbf{f}}$ at τ of order $1/j_i$. Since the set of variation is a cone, we have:

PROPOSITION 3. *Let Θ be an element of* Lie **X***; if a positive multiple of Θ is the* **f***-leading element at τ of a control variation, then $\Theta_{\mathbf{f}}(x_{\mathbf{f}}(\tau))$ is a variation at τ.*

3. General Result

The results of the previous section can be improved by using the relations in Lie **f** at $x_{\mathbf{f}}(\tau)$. The idea is that these relations allow to modify the leading term of a given control variation and therefore one can obtain more than one trajectory variation from a control variation.

Let us recall some definitions given by Susmann, [6], [7].

DEFINITION 3. *An* admissible weight *for the process (1) is a set of positive numbers, $\mathbf{l} = (l_0,\ l_1,\ \ldots, l_m)$, which verify the relations $l_0 \leq l_i$, $\forall i$.*

By means of an admissible weight, one can give a weight to each bracket in Lie **X**, [6]. Let Λ be a bracket in the indeterminate $X_i' s$; $|\Lambda|_i$ is the number of times that X_i appears in Λ.

DEFINITION 4. *Let $\mathbf{l} = \{l_0, l_1, \ldots, l_m\}$ be an admissible weight, the* **l***-weight of a bracket Φ is given by*

$$\|\Phi\|_{\mathbf{l}} = \sum_{i=0}^{m} l_i |\Phi|_i\,.$$

An element $\Theta \in$ Lie **X** *is said* **l***-homogeneous if it is a linear combination of brackets with the same* **l***-weight, which we name the* **l***-weight of the element.*

The weight of a bracket, Φ, with respect to the standard weight $\mathbf{l} = \{1, 1, \ldots, 1\}$ coincides with its length and it is denoted by $\|\Phi\|$.

The weight introduce a partial order relation in Lie **X**.

DEFINITION 5. *Let $\Theta \in$* Lie **X***; following Susmann [7] we say that Θ is* **l f**-neutralized *at a point y if the value at y of $\Theta_{\mathbf{f}}$ is a linear combination of the values of brackets with less* **l***-weight, i.e. $\Theta_{\mathbf{f}}(y) = \sum \alpha_j\, \Phi^j_{\mathbf{f}}(y)$, $\|\Phi^j\|_{\mathbf{l}} < \|\Theta\|_{\mathbf{l}}$. The number* $\max \|\Phi_j\|$ *is the* order of the neutralization.

Let N be a positive integer; with S_N we denote the subspace of $\mathrm{Lie}\,\mathbf{X}$ spanned by the brackets whose length is not greater than N and with Q_N we denote the subspace spanned by the brackets whose length is greater than N. $\mathrm{Lie}\,\mathbf{X}$ is direct sum of S_N and Q_N.

DEFINITION 6. *Let u be any control;* $\log_N u$ *and* $\mathrm{Log}_N u$ *are the projections of* $\log u$ *and* $\mathrm{Log}\, u$ *respectively, on S_N.*

DEFINITION 7. *An element $\Phi \in S_N$ is a N-*good *element if there exists a neighborhood V of* 0 *in S_N and a C^1 map $u : V \to L^1$, such that $u(V)$ is contained in the set of admissible controls and*

$$\mathrm{Log}_N\, u(\Theta) = \Phi + \Theta\,.$$

Notice that there exist N-good elements whatever is the natural N.

We are going to present a general result.

THEOREM 1. *Let Z be an N-good element and let $\mathbf{l}$ be an admissible weight. $Z = \sum Y^i$, Y^i $\mathbf{l}$-homogeneous element such that if $b_i = \|Y^i\|_{\mathbf{l}}$, then $b_i \leq b_j$ if $i < j$. If there exists j such that for each $i < j$, Y^i is $\mathbf{l}\,\mathbf{f}$-neutralized at τ with order not greater than N and $b_j < b_{j+1}$, then*

1. *$Y^j_{\mathbf{f}}(x_{\mathbf{f}}(\tau))$ is a variation at τ of order $\|Y^j\|_{\mathbf{l}}$;*
2. *if Φ is a bracket contained in S_N, $\|\Phi\|_{\mathbf{l}} < b_j$, then $\pm\Phi_{\mathbf{f}}(x_{\mathbf{f}}(\tau))$ is a variation at τ of order $\|\Phi\|_{\mathbf{l}}$.*

Proof. We are going to provide the proof in the case in which there is only one element which is $\mathbf{l}\,\mathbf{f}$-neutralized at τ. The proof of the general case is analogous. By hypothesis there exist $\mathbf{l}$-homogeneous elements W^j, $c_j = \|W^j\|_{\mathbf{l}} < \|Y^1\|_{\mathbf{l}}$, such that:

$$Y^1_{\mathbf{f}}(x_{\mathbf{f}}(\tau)) = \sum \alpha_j W^j_{\mathbf{f}}(x_{\mathbf{f}}(\tau))\,. \tag{4}$$

Let u be an admissible control; the control defined in $[0, \epsilon^{l_0} T(u)]$ by

$$\delta_\epsilon\, u(t) = (\epsilon^{l_1 - l_0} u_1(t/\epsilon^{l_0}), \ldots, \epsilon^{l_m - l_0} u_m(t/\epsilon^{l_0}))$$

is an admissible control; such control will be denoted by $\delta_\epsilon u$. The map $\epsilon \mapsto \delta_\epsilon u$ is continuous in the L^1 topology and $T(\delta_\epsilon u)$ goes to 0 with ϵ.

Let Y be any element of $\hat{L}(\mathbf{X})$; $\delta_\epsilon(Y)$ is the element obtained by multiplying each indeterminate X_i in Y by ϵ^{l_i}. The definition of $\delta_\epsilon u$ implies:

$$\mathrm{Log}\,\delta_\epsilon u = \delta_\epsilon\,\mathrm{Log}\, u\,.$$

$\delta_\epsilon Y^1 = \epsilon^{b_1} Y_1$ and $\delta_\epsilon(\sum \alpha_j W^j) = \sum \alpha_j \epsilon^{c_j} W^j$; therefore

$$\delta_\epsilon (Y^1 - \sum \alpha_j \epsilon^{(b_1 - c_j)} W^j)_{\mathbf{f}}$$

vanishes at $x_{\mathbf{f}}(\tau)$. By hypothesis there exists a neighborhood V of $0 \in S_N$ and a continuous map $u : V \to L^1$ such that

$$\mathrm{Log}_N\, u(\Phi) = Z + \Phi\,.$$

Set $\Theta(\epsilon) = -\sum \alpha_j \epsilon^{(b_1 - c_j)} W^j$; $\Theta(\epsilon)$ depends continuously from ϵ and since $(b_1 - c_j) < 0$, $\Theta(\epsilon) \in V$ if ϵ is sufficiently small. Therefore the control variation $\delta_\epsilon u(\Theta(\epsilon))$ proves the first assertion.

Let Φ satisfies the hypothesis; if σ and ϵ are sufficiently small $\sigma\Phi + \Theta(\epsilon) \in V$ and

$$\delta_\epsilon u(\Theta(\epsilon) + \sigma\Phi)$$

ia a control variation which has **f**-leading term equal to $\delta\Phi$. The second assertion is proved.

□

For the previous result to be applicable, we need to know how the N-good controls are made. The symmetries of the system give some information on this subject.

Let me recall some definitions introduced in [6] and in [4].

DEFINITION 8. *The* bad brackets *are the brackets in* Lie **X** *which contain X_0 an odd number of times and each X_i an even number of times. Let $\mathcal{B}$ be the set of bad brackets*

$$\mathcal{B} = \{\Lambda\ ,\ |\Lambda|_0 \textit{ is odd } |\Lambda|_i \textit{ is even } i = 1, \ldots, m\}\,.$$

The set of the obstructions *is the set*

$$\mathcal{B}^* = \mathrm{Lie}\,(X_0, \mathcal{B}) \setminus \{aX_0\ ,\ a \in \mathbb{R}\}\,.$$

PROPOSITION 4. *For each integer N there exists a N-good element which belongs to $\mathcal{B}^*$.*

Proof. In [6] Sussmann has proved that there exists an element $\Psi \in \mathcal{B}$ and a C^1 map, $\overline{u}$, from a neighborhood of $0 \in S_N$ in L^1 such that the image of $\overline{u}$ is contained in the set of admissible controls and

$$\log_N \overline{u}(\Theta) = \Psi + \Theta\,.$$

This result is obtained by using the symmetries of the process. Standard arguments imply that it is possible to construct a C^1 map u from a neighborhood of $0 \in S_N$ to L^1 such that the image of u is contained in the set of admissible controls and

$$\mathrm{Log}_N\, u(\Theta) = \Xi + \Theta$$

with $\Xi \in \mathcal{B}^*$.

□

The previous proposition together with Theorem 1 imply that the trajectory variations are linked to the neutralization of the obstruction.

Theorem 1 can be used to find g-variations if the **f l**-neutralization holds on an interval containing τ.

4. Explicit optimality conditions for the single input case

In this section I am going to construct g-variations of a trajectory of an affine control process at any point of an interval in which the reference control is constantly equal to 0. It is known that if x^* is a solution of a sufficiently regular control process such that

$$g_0(x^*(t_1)) = \min_{y \in R(t_1, x^*(t_0)) \cap S} g_0(y)\,,$$

then there exists an adjoint variable $\lambda(t)$ satisfying the Pontrjagin Maximum Principle and such that for each τ and for each g-variation, v, of x^* at τ

$$\lambda(\tau)v \le 0\,.$$

Therefore the g-variations I will obtain, provide necessary conditions of optimality for the singular trajectory.

For simplicity sake I will limit myself to consider an affine single input control process

$$\dot{x} = f_0(x) + u f_1(x)$$

and I suppose that the control which generate the reference trajectory, x^*, which we want to test, is constantly equal to 0 on an interval I containing τ so that $x^*(t) = x_{\mathbf{f}}(t)$, $\forall t \in I$. The $\mathbf{f}\,\mathbf{l}$-neutralization of the obstructions on I provides g-variations at τ. In [4] it has been proved that if each bad bracket, Θ, $\|\Theta\|_{\mathbf{l}} \leq p$ is $\mathbf{f}\,\mathbf{l}$-neutralized on $x_{\mathbf{f}}(I)$, then all the obstructions whose $\mathbf{l}$-weight is not greater than p are $\mathbf{f}\,\mathbf{l}$-neutralized on $x_{\mathbf{f}}(I)$. Moreover if Φ is a bracket which is $\mathbf{f}\,\mathbf{l}$-neutralized on I, then $[X_0, \Phi]$ is $\mathbf{f}\,\mathbf{l}$-neutralized on I. Therefore to know which obstructions are $\mathbf{f}\,\mathbf{l}$-neutralized on I it is sufficient to test those bad brackets whose first element is equal to X_1. Let $\mathbf{l}$ be an admissible weight; $\mathbf{l}$ induces an increasing filtration in Lie $\mathbf{X}$

$$\{0\} = Y_{\mathbf{l}}^0 \subset Y_{\mathbf{l}}^1 \subset \cdots \subset Y_{\mathbf{l}}^n \subset \ldots$$

$Y^i = \operatorname{span}\{\Phi : \|\Phi\|_{\mathbf{l}} \leq p_i\}$, $p_i < p_{i+1}$. Let p_j be such that each bad bracket whose weight is less than or equal to p_j is $\mathbf{f}\,\mathbf{l}$-neutralized on an interval containing τ. We know that $Y_{\mathbf{f}}^j(x_{\mathbf{f}}(\tau))$ is a subspace of g-variation at τ, which are obviously bilateral variations. Unilateral g-variation can be contained in the set of $\mathbf{l}$-homogeneous elements belonging to $Y_{\mathbf{f}}^{j+1}(x_{\mathbf{f}}(\tau))$. Notice that each subspace $Y_{\mathbf{f}}^i(x_{\mathbf{f}}(\tau))$ is finite dimensional and that the sequence $\{Y_{\mathbf{f}}^i(x_{\mathbf{f}}(\tau))\}$ become stationary for i sufficiently large. Therefore we are interested only in the elements of S_N with N sufficiently large. Let N be such that each $Y_{\mathbf{f}}^i(x_{\mathbf{f}}(\tau))$ is spanned by brackets whose length is less than N. The following Lemma proves that it is possible to modify the weight $\mathbf{l}$ in order to obtain a weight $\bar{\mathbf{l}}$ with the following properties:

1. each bracket which is $\mathbf{f}\,\mathbf{l}$-neutralized at τ is $\mathbf{f}\,\bar{\mathbf{l}}$-neutralized at τ;
2. the $\bar{\mathbf{l}}$-homogeneous elements are linear combination of brackets which contain the same numbers both of X_0 than of X_1.

LEMMA 1. *Let $\mathbf{l}$ be an admissible weight; for each integer N there exists an admissible weight $\bar{\mathbf{l}}$ with the following properties: if Φ and Θ are brackets whose length is not greater than N, then*

1. $\|\Phi\|_{\mathbf{l}} < \|\Theta\|_{\mathbf{l}}$ *implies* $\|\Phi\|_{\bar{\mathbf{l}}} < \|\Theta\|_{\bar{\mathbf{l}}}$
2. $\|\Phi\|_{\mathbf{l}} = \|\Theta\|_{\mathbf{l}}$ *and* $\|\Phi\|_1 < \|\Theta\|_1$, *implies* $\|\Phi\|_{\bar{\mathbf{l}}} < \|\Theta\|_{\bar{\mathbf{l}}}$
3. $\|\Phi\|_{\mathbf{l}} = \|\Theta\|_{\mathbf{l}}$, $\|\Phi\|_1 = \|\Theta\|_1$ *and* $\|\Phi\|_0 < \|\Theta\|_0$, *implies* $\|\Phi\|_{\bar{\mathbf{l}}} < \|\Theta\|_{\bar{\mathbf{l}}}$

Proof. The set of brackets whose weight is not greater than N is a finite set therefore if ϵ_0, ϵ_1 are positive numbers sufficiently small, then $\bar{\mathbf{l}} = \{l_0 - \epsilon_0,\, l_1 + \epsilon_1\}$ is an admissible weight for which the properties 1), 2) and 3) hold.

□

In order to simplify the notation, we will use the multiplicative notation for the brackets:

$$XY = [X, Y], \quad ZXY = [Z, XY], \quad \text{etc.}$$

It is known, [10], that each bracket, Φ, in Lie $\mathbf{X}$ is linear combination of brackets right normed, i.e. of the following type:

$$X_0^{i_1} X_1^{i_2} \dots X_1^{i_s}\,, \quad i_j \in \{0, 1, \dots\}\,,$$

which contains both X_0 than X_1 the same number of times of Φ; therefore it is sufficient to test the neutralization of right normed bad brackets.

Any N-good element, Z, of Lie $\mathbf{X}$ can be written as:

$$Z = \sum a_i \Phi_i$$

Φ_i right normed bracket; we name a_i *coefficient* of Φ_i in Z.

LEMMA 2. *Let $N > 2n + 3$. The coefficient of $X_1 X_0^{2n+1} X_1$ in any N-good element is positive if n is even and negative if n is odd; the coefficient of $X_1^{2n-1} X_0 X_1$ is always positive.*

Proof. Let Z be a N-good element and let us consider the control process

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dots \\ \dot{x}_{n+1} = x_n \\ \dot{x}_{n+2} = x_{n+1}^2\,. \end{cases}$$

Take as reference trajectory $x_{\mathbf{f}}(t) \equiv 0$. The reachable set $R(t, 0)$ is contained, for any positive t, in the half space $x_{n+1} \geq 0$ and hence $-\frac{\partial}{\partial x_{n+1}}$ cannot be a variation at any time. The only elements in Lie $\mathbf{f}$ which are different from 0 in 0 are:

$$\begin{aligned} (X_1)_{\mathbf{f}} &= \frac{\partial}{\partial x_1}\,, \\ (X_0^i X_1)_{\mathbf{f}} &= (-1)^i \frac{\partial}{\partial x_{i+1}}\,,\ i = 1, \dots, n \\ (X_1 X_0^{2n+1} X_1)_{\mathbf{f}} &= (-1)^n 2 \frac{\partial}{x_{n+1}}\,. \end{aligned}$$

If the coefficient of $X_1 X_0^{2n+1} X_1$ in Z were equal to 0, then 0 will be locally controllable [6], which is an absurd. Therefore it is different from 0; its sign has to be equal to $(-1)^n$ because otherwise $-\frac{\partial}{\partial x_{n+1}}$ would be a variation. The first assertion is proved.

The second assertion is proved by using similar arguments applied to the system:

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^{2n} \end{cases}$$

$\square$

Let us now compute explicitly some g-variations. We recall that

$$\mathrm{ad}_X Y = [X, Y]\,, \quad \mathrm{ad}_X^{n+1} Y = [X, \mathrm{ad}_X^n Y]\,.$$

THEOREM 2. *If there exists an admissible weight* $\mathbf{l}$ *such that each bad bracket of* $\mathbf{l}$*-weight less than* $(2n+1)l_0 + 2l_1$ *is* $\mathbf{f}\,\mathbf{l}$*-neutralized on an interval containing* τ*, then*

$$(-1)^n [X_1, \mathrm{ad}_{X_0}^{2n+1} X_1]_{\mathbf{f}}(x^*(\tau))$$

is a g-variation at τ.

Proof. We can suppose that the weight $\mathbf{l}$ has the properties 1), 2) and 3) as in Lemma 1. Therefore the brackets of $\mathbf{l}$-weight equal to $2l_1 + (2n+1)l_0$ contains 2 X_1 and (2n+1) X_0. The brackets which have as first element X_0 are the adjoint with respect to X_0 of brackets which by hypothesis are $\mathbf{f}\,\mathbf{l}$-neutralized on the interval I and therefore are $\mathbf{f}\,\mathbf{l}$-neutralized. Since the only bad bracket of $\mathbf{l}$-weight $(2n+1)l_0 + 2l_1$ which has as first element X_1, is $X_1 X_0^{2n+1} X_1$ the theorem is a consequence of Theorem 1 and of Lemma 2.

□

Notice that the theorem contains as particular case the well known generalized Legendre-Clebsch conditions. In fact it is possible to choose σ such that each bracket which is $\mathbf{f}$-neutralized on I with respect to the weight $(0, 1)$ is $\mathbf{f}$-neutralized with respect to the weight $\mathbf{l} = (\sigma, 1)$; moreover bearing in mind that only a finite set of brackets are to be considered, we can suppose that if two brackets contain a different number of X_1, then the one which contains less X_1 has less $\mathbf{l}$-weight and that two brackets have the same $\mathbf{l}$-weight if and only if they contain the same number both of X_0 than of X_1. Since each bad bracket contains at least two X_1, then the only bad bracket whose weight is less than $(2n+1)\sigma + 2$ contains two X_1 and $(2i+1)$ X_0, $i = 0, \ldots, (n-1)$; among these the only ones which we have to consider are $X_1 X_0^{2i+1} X_1$. Set

$$S^i = \mathrm{span}\{\Phi;\ \text{which contains } i \text{ times } X_1\}\,.$$

If

$$(X_1 X_0^{2i+1} X_1)_{\mathbf{f}}(x_{\mathbf{f}}(I)) \in S^1_{\mathbf{f}}(x_{\mathbf{f}}(I))\,, \quad i = (1, \ldots, (n-1))$$

Theorem 2 implies that $(-1)^n (X_1 X_0^{2n+1} X_1)_{\mathbf{f}}(x_{\mathbf{f}}(\tau))$ is a g-variation of $x_{\mathbf{f}}$ at τ; therefore if x^* is optimal, then the adjoint variable can be chosen in such a way that

$$(-1)^n \lambda(t)(X_1 X_0^{2i+1} X_1)_{\mathbf{f}}(x^*(\tau)) \leq 0\,, \qquad t \in I$$

condition which is known as generalized Legendre-Clebsch condition.

The following example shows that by using Theorem 2 one can obtain further necessary conditions which can be added to the Legendre-Clebsch ones.

EXAMPLE 1. Let:

$$\begin{aligned} f_0 &= \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} + \left(\frac{x_3^2}{2} - \frac{x_2^3}{6}\right) \frac{\partial}{\partial x_5} + \frac{x_4^2}{2} \frac{\partial}{\partial x_6} \\ f_1 &= \frac{\partial}{\partial x_2}\,. \end{aligned}$$

The generalized Legendre-Clebsch condition implies that $-(X_1 X_0^3 X_1)_{\mathbf{f}}(x_{\mathbf{f}}(\tau)) = \frac{\partial}{\partial x_5}$ is a g-variation at τ. Let us apply Theorem 2 with the weight $\mathbf{l} = (1, 1)$; the bad brackets of $\mathbf{l}$-weight less than 7 are:

$$X_1 X_0 X_1,\ X_1 X_0^3 X_1,\ X_1^3 X_0 X_1,\ X_1^2 X_0^2 X_1 X_0 X_1,\ X_1^2 X_0 X_1 X_0^2 X_1,$$

the only one different from 0 along the trajectory is $(X_1 X_0^3 X_1)_{\mathbf{f}}$ which is at $x_{\mathbf{f}}(I)$ a multiple $(X_1^2 X_0 X_1)_{\mathbf{f}}(x_{\mathbf{f}}(I))$. Therefore it is $\mathbf{f}\,\mathbf{l}$-neutralized. Theorem 1 implies that $\pm(X_1 X_0^3 X_1)_{\mathbf{f}}(x_{\mathbf{f}}(\tau)) = \pm\frac{\partial}{\partial x_5}$, and $(X_1 X_0^5 X_1)_{\mathbf{f}}(x_{\mathbf{f}}(\tau)) = \frac{\partial}{\partial x_6}$ are g-variations.

Another necessary optimality condition can be deduced from Theorem 1 and Lemma 2.

THEOREM 3. *If there exists an admissible weight* $\mathbf{l}$ *such that all the bad brackets whose* $\mathbf{l}$*-weight is less than* $l_0 + 2n\, l_1$ *are* $\mathbf{f}\,\mathbf{l}$*-neutralized on an interval containing* τ*, then*

$$(X_1^{2n-1} X_0 X_1)_{\mathbf{f}}(x_{\mathbf{f}}(\tau))$$

is a g-variation at τ*.*

Proof. We can suppose that the weight $\mathbf{l}$ is such that the brackets with the same $\mathbf{l}$-weight contain the same number both of X_1 than of X_0. Since there is only one bracket, $X_1^{2n-1} X_0 X_1$ which contain $2n$ X_1 and 1 X_0, the theorem is a consequence of Theorem 1 and of Lemma 2.

□

Notice that this condition is active also in the case in which the degree of singularity is $+\infty$.

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