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Turin Fortnight Lectures on Nonlinear Analysis

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Preface

The Turin Fortnight on Nonlinear Analysis is a conference that, every two years, gathers at the Department of Mathematics of the University of Torino a number of researchers from Italy and abroad with the aim of discussing recent developments in nonlinear differential equations (both ordinary and partial). The conference consists of seminars and short courses, providing an up-to-date introduction to contemporary research. This special issue of the Rendiconti del Seminario Matematico (Universita' e Politecnico di Torino) collects the lecture notes from three of the short courses of the III Turin Fortnight, which was held in September 2001. The lecturers were F. Bethuel (Universite' Paris VI-Pierre et Marie Curie), C. Rebelo (CMAF-Lisbon) and F. Zanolin (University of Udine). A fourth course was given by V. Benci (University of Pisa); for technical reasons we are not able to include in this special issue the related notes, which we hope to publish in this journal soon.
We wish that these lecture notes, which represent a short introduction to relevant contemporary themes, will be of interest to young people starting their research activity. On the other hand, mature mathematicians will find here recent results not yet published or dispersed in several research papers.
We thank the lecturers for accepting our invitation to give the course and to write these notes (in collaboration with some participants to the conference). We also thank all the speakers and the participants to the III Turin Fortnight, who made it interesting and successful. Finally, we thank the institutions that gave a financial support: the Department of Mathematics (University of Torino) and the GNAMPA-INDAM (National Group for Mathematical Analysis, Probability and its Applications of the National Institute for High Mathematics).

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PARAMETRIC SURFACES WITH PRESCRIBED MEAN CURVATURE

Abstract. This article contains an overview on some old and new problems concerning two-dimensional parametric surfaces in $\mathbb{R}^3$ with prescribed mean curvature. Part of this exposition has constituted the subject of a series of lectures held by the first author at the Department of Mathematics of the University of Torino, during the Third Turin Fortnight on Nonlinear Analysis (September 23-28, 2001).

1. Introduction

The main focus of this article is the following problem: given a smooth, real function H in $\mathbb{R}^3$, find surfaces M having exactly mean curvature $H(p)$ at any point p belonging to M.

In order to get some intuition in the geometric and analytical aspects of this question, we believe that it might be of interest to consider first its two dimensional analog, where most concepts become rather elementary. Therefore, in this introductory part we will first discuss the following questions:

(Q_0) Given a smooth, real function κ on the plane $\mathbb{R}^2$, *find a closed curve* $\mathcal{C}$, such that for any point p in $\mathcal{C}$ the curvature of the curve at this point is exactly $\kappa(p)$ (we may possibly impose furthermore that $\mathcal{C}$ has no self intersection: $\mathcal{C}$ is then topologically a circle).

(Q_1) [*Planar Plateau problem*] Given two points a and b in the plane, and a smooth, real function κ on $\mathbb{R}^2$, *find a curve* $\mathcal{C}$ with $\partial\mathcal{C} = \{a, b\}$, such that for any point p in $\mathcal{C}$ the curvature of the curve at p is exactly $\kappa(p)$.

1.1. Parametrization

In order to provide an analytical formulation of these problems, the most natural approach is to introduce a parametrization of the curve $\mathcal{C}$, i.e., a map $u\colon I \to \mathbb{R}^2$, such that $|\dot{u}| = 1$, $u(I) = \mathcal{C}$, where I represents some compact interval of $\mathbb{R}$, and the notation $\dot{u} = \frac{du}{ds}$ is used. Notice that, nevertheless there are possible alternative approaches to parametrization: we will discuss this for surfaces in the next sections.

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Then, questions (Q_0) and (Q_1) can be formulated in terms of ordinary differential equations. More precisely, the fact that $\mathcal{C}$ has curvature $\kappa(u(s))$ at every point $u(s)$ belonging to $\mathcal{C}$ reads

$$\ddot{u} = i\kappa(u)\dot{u} \quad \text{on } I, \tag{1}$$

where i denotes the rotation by $\frac{\pi}{2}$. Note that the sign of the term of the r.h.s. depends on a choice of orientation, and the curvature might therefore take negative values.

The constraint $|\dot{u}| = 1$ might raise difficulties in order to find solutions to (Q_0) and (Q_1). It implies in particular that $|I| =$ length of $\mathcal{C}$, and this quantity is not know a priori. This difficulty can be removed if we consider instead of (1) the following equivalent formulation

$$\frac{|I|^{\frac{1}{2}}}{(\int_I |\dot{u}|^2\, ds)^{\frac{1}{2}}}\ddot{u} = i\kappa(u)\dot{u} \quad \text{on } I. \tag{2}$$

To see that (2) is an equivalent formulation of (1), note first that any solution u to (2) verifies

$$\frac{1}{2}\frac{d}{ds}(|\dot{u}|^2) = \ddot{u}\cdot\dot{u} = \frac{(\int_I |\dot{u}|^2\, ds)^{\frac{1}{2}}}{|I|^{\frac{1}{2}}}\kappa(u)i\dot{u}\cdot\dot{u} = 0,$$

so that $|\dot{u}| = C_0 =$ *const*. and, introducing the new parametrization $v(s) = u(s/C_0)$, we see that $|\dot{v}| = 1$, and v solves (1).

Hence, an important advantage of formulation (2) is that we do not have to impose any auxiliary condition on the parametrization since equation (2) is independent of the interval I. Thus, we may choose $I = [0, 1]$ and (2) reduces to

$$\ddot{u} = iL(u)\kappa(u)\dot{u} \quad \text{on } [0, 1]\,, \tag{3}$$

where

$$L(u) := \left(\int_I |\dot{u}|^2\, ds\right)^{\frac{1}{2}}\,.$$

Each of the questions (Q_0) and (Q_1) has then to be supplemented with appropriate boundary conditions:

$$u(0) = u(1),\ \dot{u}(0) = \dot{u}(1) \quad \text{for } (Q_0)$$

(or alternatively, to consider $\mathbb{R}/\mathbb{Z}$ instead of $[0, 1]$), and

$$u(0) = a,\ u(1) = b, \quad \text{for } (Q_1). \tag{4}$$

1.2. The case of constant curvature

We begin the discussion of these two questions with the simplest case, namely when the function κ is a constant $\kappa_0 > 0$. It is then easily seen that the *only* solutions to equations (1) (or (3)) are portions of circles of radius $R_0 = \frac{1}{\kappa_0}$. Therefore, for (Q_0) we obtain the simple answer: the solutions are circles of radius $1/\kappa_0$.

For question (Q_1) a short discussion is necessary: we have to compare the distance $l_0 := |a - b|$ with the diameter $D_0 = 2R_0$. Three different possibilities may occur:

(i) $l_0 > D_0$, i.e., $\frac{1}{2} l_0 \kappa_0 > 1$. In this case there is no circle of diameter D_0 containing simultaneously a and b, and therefore problem (Q_1) has no solution.

(ii) $l_0 = D_0$, i.e., $\frac{1}{2} l_0 \kappa_0 = 1$. There is exactly *one* circle of diameter D_0 containing simultaneously a and b. Therefore (Q_1) has exactly *two* solutions, each of the half-circles joining a to b.

(iii) $l_0 < D_0$, i.e., $\frac{1}{2} l_0 \kappa_0 < 1$. There are exactly *two* circles of diameter D_0 containing simultaneously a and b. These circles are actually symmetric with respect to the axis ab. Therefore (Q_1) has exactly *four* solutions: two *small* solutions, symmetric with respect to the axis ab, which are arcs of circles of angle strictly smaller than π, and two *large* solutions, symmetric with respect to the axis ab, which are arcs of circles of angle strictly larger than π. Notice that the length of the small solutions is $2 \arccos(\frac{1}{2} l_0 \kappa_0)) \kappa_0^{-1}$, whereas the length of the large solutions is $2(\pi - \arccos(\frac{1}{2} l_0 \kappa_0)) \kappa_0^{-1}$, so that the sum is the length of the circle of radius R_0.

As the above discussion shows, the problem can be settled using very elementary arguments of geometric nature.

We end this subsection with a few remarks concerning the parametric formulation, and its analytical background: these remarks will be useful when we will turn to the general case.

Firstly, we observe that equation (3) in the case $\kappa \equiv \kappa_0$ is variational: its solutions are critical points of the functional

$$F_{\kappa_0}(v) = L(v) - \kappa_0 S(v)$$

where $L(v)$ has been defined above and

$$S(v) := \frac{1}{2} \int_0^1 i v \cdot \dot{v} \, ds \, .$$

The functional space for (Q_0) is the Hilbert space

$$H_{\text{per}} := \{ v \in H^1([0,1], \mathbb{R}^2) \mid v(0) = v(1) \} \, ,$$

whereas the functional space for (Q_1) is the affine space

$$H_{a,b} := \{ v \in H^1([0,1], \mathbb{R}^2) \mid v(0) = a, \ v(1) = b \} \, .$$

The functional $S(v)$ have a nice geometric interpretation. Indeed, for v belonging to the space H_{per}, $S(v)$ represents the (signed) *area* of the (inner) domain bounded by the curve $\mathcal{C}(v) = v([0,1])$. Whereas, for v in H_{per} or $H_{a,b}$, the quantity $L(v)$ is less or equal to the length of $\mathcal{C}(v)$ and equality holds if and only $|\dot{v}|$ is constant. In particular, for v in H_{per}, we have the inequality

$$4\pi |S(v)| \leq L^2(v),$$

which is the analytical form of the isoperimetric inequality in dimension two. Therefore solutions of (Q_0), with $\kappa \equiv \kappa_0$ are also solutions to the isoperimetric problem

$$\sup\{S(v) \mid v \in H_{\mathrm{per}}\ ,\ L(v) = 2\pi\kappa_0^{-1}\}\ .$$

This, of course, is a well known fact.

Finally, we notice that the small solutions to (Q_0), in case (iii) are local minimizers of F. More precisely, it can be proved that they minimize F on the set $\{v \in H_{a,b} \mid \|v\|_\infty \leq \kappa_0^{-1}\}$ (in this definition, the origin is taken as the middle point of ab). In this context, the large solution can then also be analyzed (and obtained) variationally, as a mountain pass solution. We will not go into details, since the arguments will be developed in the frame of H-surfaces (here however they are somewhat simpler, since we have less troubles with the Palais-Smale condition).

1.3. The general case of variable curvature

In the general case when the prescribed curvature $\kappa(p)$ depends on the point p, there are presumably no elementary geometric arguments which could lead directly to the solution of (Q_0) and (Q_1). In that situation, the parametric formulation offers a natural approach to the problems.

In this subsection we will leave aside (Q_0), since it is probably more involved and we will concentrate on question (Q_1). We will see in particular, that we are able to extend (at least partially) some of the results of the previous subsection to the case considered here using analytical tools.

We begin with the important remark that (3) is variational, even in the nonconstant case: solutions of (3) and (4) are critical points on $H_{a,b}$ of the functional

$$F_\kappa(v) = L(v) - S_\kappa(v)\ ,$$

where

$$S_\kappa(v) = \int_0^1 i\,Q(v)\cdot \dot{v}\,ds,$$

for any vector field $Q\colon \mathbb{R}^2 \to \mathbb{R}^2$ verifying the relation div $Q(w) = \kappa(w)$ for $w = (w_1, w_2) \in \mathbb{R}^2$. A possible choice for such as a vector field is

$$Q(w_1, w_2) = \frac{1}{2}\Big(\int_0^{w_1} \kappa(s, w_2)\,ds, \int_0^{w_2} \kappa(w_1, s)\,ds\Big)\ .$$

Notice that in the case $\kappa \equiv \kappa_0$ is constant, the previous choice of Q yields $Q(w) = \frac{1}{2}w$, and we recover the functional F_{κ_0}, as written in the previous subsection.

The existence of “small” solutions to (Q_1) can be established as follows.

PROPOSITION 1. *Assume that $l_0 > 0$ and $\kappa \in C^1(\mathbb{R}^2)$ verify the condition*

$$\frac{1}{2} l_0 \|\kappa\|_\infty < 1.$$

Then equation (3) *possesses a solution $\underline{u}$, which minimizes F_κ on the set*

$$M_0 := \{v \in H_{a,b} \mid \|v\|_\infty \leq \|\kappa\|_\infty^{-1}\} .$$

In the context of H-surfaces, this type of result has been established first by S. Hildebrandt [30], and we will explain in details his proof in section 4. The proof of Proposition 1 is essentially the same and therefore we will omit it. Note that, in view of the corresponding results for the constant case, i.e., case (iii) of the discussion in the previous subsection, Proposition 1 seems rather optimal.

We next turn to the existence problem for “large” solutions. It is presumably more difficult to obtain a general existence result, in the same spirit as in the previous proposition (i.e., involving only some norms of the function κ). We leave to the reader to figure out some possible counterexamples. We believe that the best one should be able to prove is a perturbative result, i.e., to prove existence of the large solution for functions κ that are close, in some norm, to a constant. In this direction, we may prove the following result.

PROPOSITION 2. *Let $l_0, \kappa_0 > 0$, and assume that*

$$\frac{1}{2} l_0 \kappa_0 < 1 .$$

Then, there exists $\varepsilon > 0$ (depending only on the number $l_0\kappa_0$), such that, for every function $\kappa \in C^1(\mathbb{R}^2)$ verifying

$$\|\kappa - \kappa_0\|_{C^1} < \varepsilon ,$$

equation (3) *has four different solutions $\underline{u}_1$, $\underline{u}_2$, $\overline{u}_1$ and $\overline{u}_2$, where one of the small solutions $\underline{u}_1$ and $\underline{u}_2$ corresponds to the minimal solution given by proposition 1.*

The new solutions $\overline{u}_1$ and $\overline{u}_2$ provided by proposition 2 correspond to the large solutions of the problem: one can actually prove that they converge, as $\|\kappa - \kappa_0\|_{C^1}$ goes to zero, to the large portion of the two circles of radius κ_0^{-1}, joining a to b, given in case (iii) of the previous subsection.

Proof. A simple proof of Proposition 2 can be provided using the implicit function theorem. Indeed, consider the affine space

$$C^2_{a,b} := \{v \in C^2([0,1], \mathbb{R}^2) \mid v(0) = a,\ v(1) = b\} ,$$

and the map $\Phi \colon C^2_{a,b} \times \mathbb{R} \to C^0 := C^0([0,1], \mathbb{R}^2)$ defined by

$$\Phi(v, t) = -\ddot{v} + i(\kappa_0 + t(\kappa(v) - \kappa_0))L(v)\dot{v} .$$

Clearly Φ is of class C^1 and for $w \in C^2_{0,0}$ one has:

$$\begin{aligned}\partial_v \Phi(v,t)(w) &= -\ddot{w} + iL(v)((\kappa_0 + t(\kappa(v) - \kappa_0))\dot{w} + t\kappa'(v)\cdot\dot{v}\, w) \\ &\quad + i(\kappa_0 + t(\kappa(v)-\kappa_0))L(v)^{-1}\dot{v}\int_0^1 \dot{v}\cdot\dot{w}\, ds\,.\end{aligned}$$

Let u_0 be one of the four solutions for κ_0. Notice that for an appropriate choice of orthonormal coordinates in the plane, u_0 is given by the explicit formula

$$u_0(s) = \kappa_0^{-1}\exp(iL_0\kappa_0 s),$$

where $L_0 = L(u_0)$ (recall that $L_0 = 2\kappa_0^{-1}\arccos(\frac{1}{2}l_0\kappa_0))$ for small solutions, or $L_0 = 2\kappa_0^{-1}(\pi - \arccos(\frac{1}{2}l_0\kappa_0))$ for large solutions). We compute the derivative at the point $(u_0, 0)$:

$$\partial_v\Phi(u_0,0)(w) = -\ddot{w} + iL_0\kappa_0\dot{w} + i\kappa_0 L_0^{-1}\dot{u}_0\int_0^1 \dot{u}_0\cdot\dot{w}\, ds\,,$$

It remains merely to prove that $\partial_v\Phi(u_0, 0)$ is invertible, i.e., by Fredholm theory, that $\ker \partial_v\Phi(u_0, 0) = \{0\}$. If $w \in \ker \partial_v\Phi(u_0, 0)$, then

$$\ddot{w} = iL_0\kappa_0\dot{w} - \alpha(w)L_0\kappa_0\exp(iL_0\kappa_0 s), \tag{5}$$

where $\alpha(w) = L_0^{-1}\int_0^1 \dot{u}_0\cdot\dot{w}\, ds$. Taking α as a parameter, equation (5) can be solved explicitly and its solution is given by:

$$w(x) = C_1 + C_2\exp(iL_0\kappa_0 s) + i\alpha s\exp(iL_0\kappa_0 s)$$

where C_1 and C_2 are some (complex-valued) constants. The boundary conditions $w(0) = w(1) = 0$ determine C_1 and C_2 as functions of α. In view of the definition of α, one deduces an equation for α. After computations, since $\frac{1}{2}l_0\kappa_0 < 1$, it turns out that the only solution is $\alpha = 0$, and then $w = 0$. Thus the result follows by an application of the implicit function theorem. □

The result stated in proposition 2 can be improved if one uses instead a variational approach based on the mountain pass theorem. More precisely, one may replace the C^1 norm there, by the L^∞ norm, i.e., prove that if, for some small $\varepsilon > 0$, depending only on the value $l_0\kappa_0$ one has

$$\|\kappa - \kappa_0\|_\infty < \varepsilon,$$

then a large solution exists, for the problem (Q_1) corresponding to the curvature function κ. The analog of this result for surfaces will be discussed in Section 6, and it is one of the important aspects of the question we want to stress.

At this point, we will leave the planar problem for curves, and we turn to its version for surfaces in the three dimensional space $\mathbb{R}^3$. It is of course only for one dimensional objects that the curvature could be expressed by a simple scalar function. For higher dimensional submanifolds, one needs to make use of a tensor (in the context of surfaces, the second fundamental form). However, some "curvature" functions, deduced from this tensor are of great geometric interest. For surfaces in $\mathbb{R}^3$ the Gaussian curvature and the mean curvature in particular are involved in many questions.

2. Some geometric aspects of the mean curvature

In this section, we will introduce the main definitions and some natural problems involving the notion of *curvature*. Although this notion is important in arbitrary dimension and arbitrary codimension, we will mainly restrict ourselves to *two-dimensional surfaces embedded in* $\mathbb{R}^3$. More precisely, our main goal is to introduce some problems of *prescribed mean curvature*, and their links to *isoperimetric problems*.

We remark that mean curvature concerns problems in *extrinsic* geometry, since it deals with the way objects are embedded in the ambient space. In contrast, problems in *intrinsic* geometry do not depend on the embedding and for this kind of problems one considers the Gaussian curvature.

Let us start by recalling some geometric background.

2.1. Basic definitions

Let M be a two-dimensional regular surface in $\mathbb{R}^3$. Fixed $p_0 \in M$, let us consider near p_0 a parametrization of M, that is a map $u\colon \mathcal{O} \to M$ with $\mathcal{O}$ open neighborhood of 0 in $\mathbb{R}^2$, $u(0) = p_0$, and u diffeomorphism of $\mathcal{O}$ onto an open neighborhood of p_0 in M. Note that, denoting by $\wedge$ the exterior product in $\mathbb{R}^3$, one has $u_x \wedge u_y \neq 0$ on $\mathcal{O}$, and

$$\vec{n} = \frac{u_x \wedge u_y}{|u_x \wedge u_y|} \tag{6}$$

(evaluated at $(x, y) \in \mathcal{O}$) defines a unit normal vector at $u(x, y)$.

The metric on N is given by the **first fundamental form**

$$g_{ij}\; du^i du^j = E\;(dx)^2 + 2F\;dx\;dy + G\;(dy)^2$$

where

$$E = |u_x|^2, \quad F = u_x \cdot u_y, \quad G = |u_y|^2.$$

The notion of curvature can be expressed in terms of the second fundamental form. More precisely, let $\gamma\colon (-1, 1) \to M$ be a parametric curve on M of the form $\gamma(t) = u(x(t), y(t))$, with $x(0) = y(0) = 0$. Thus $\gamma(0) = p_0$.

Since $\frac{d\gamma}{dt}$ and $\vec{n}$ are orthogonal, one has

$$\frac{d^2\gamma}{dt^2} \cdot \vec{n} = u_{xx} \cdot \vec{n} \left(\frac{dx}{dt}\right)^2 + 2\,u_{xy} \cdot \vec{n}\;\frac{dx}{dt}\frac{dy}{dt} + u_{yy} \cdot \vec{n}\left(\frac{dy}{dt}\right)^2. \tag{7}$$

Setting

$$L = u_{xx} \cdot \vec{n}\,, \quad M = u_{xy} \cdot \vec{n}\,, \quad N = u_{yy} \cdot \vec{n}\,,$$

the right hand side of (7), evaluated at $(x, y) = (0, 0)$,

$$L\;(dx)^2 + 2M\;dx\;dy + N\;(dy)^2$$

defines the **second fundamental form**. By standard linear algebra, there is a basis (e_1, e_2) in $\mathbb{R}^2$ (depending on p_0) such that the quadratic forms

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad Q = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

can be simultaneously diagonalized; in particular $du(e_1)$ and $du(e_2)$ are orthogonal. The unit vectors

$$\nu_1 = \frac{du(e_1)}{|du(e_1)|}, \quad \nu_2 = \frac{du(e_2)}{|du(e_2)|}$$

are called **principal directions** at p_0, while the **principal curvatures** at p_0 are the values

$$\kappa_1 = \left\langle \frac{d^2\gamma_1}{dt^2}, \vec{n} \right\rangle, \quad \kappa_2 = \left\langle \frac{d^2\gamma_2}{dt^2}, \vec{n} \right\rangle$$

for curves $\gamma_i : (-1, 1) \to M$ such that $\gamma_i(0) = p_0$ and $\gamma_i'(0) = \nu_i$ $(i = 1, 2)$.

The **mean curvature** at p_0 is defined by

$$\boxed{H = \tfrac{1}{2}(\kappa_1 + \kappa_2)}$$

(homogeneous to the inverse of a length), whereas the **Gaussian curvature** is

$$K = \kappa_1\kappa_2.$$

Notice that H and K *do not depend* on the choice of the parametrization.

In terms of the first and second fundamental forms, we have

$$\boxed{2H = \tfrac{1}{EG-F^2}\,(GL - 2FM + EN) = \operatorname{tr}\left(A^{-1}Q\right)\ .} \tag{8}$$

REMARK 1. Suppose that M can be represented as a *graph*, i.e. M has a parametrization of the form

$$u(x, y) = (x, y,\ f(x, y))$$

with $f \in C^1(\mathcal{O}, \mathbb{R})$. Using the formula (8) for H, a computation shows that

$$2H = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right), \tag{9}$$

whereas the Gaussian curvature is

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{1 + |\nabla f|^2}\ .$$

Let us note that every regular surface admits locally a parametrization as a graph. Moreover, if $p_0 = (x_0, y_0,\ f(x_0, y_0))$, by a suitable choice of orthonormal coordinates one may also impose that $\nabla f(x_0, y_0) = 0$.

2.2. Conformal parametrizations and the H-system

In problems concerning mean curvature, it is convenient to use conformal parametrizations, since this leads to an equation for the mean curvature that can be handle with powerful tools in functional analysis.

DEFINITION 1. Let M be a two-dimensional regular surface in $\mathbb{R}^3$ and let $u\colon \mathcal{O} \to M$ be a (local) parametrization, $\mathcal{O}$ being a connected open set in $\mathbb{R}^2$. The parametrization u is said to be **conformal** if and only if for every $z \in \mathcal{O}$ the linear map $du(z)\colon \mathbb{R}^2 \to T_{u(z)}M$ preserves angles (and consequently multiplies lengths by a constant factor), that is there exists $\lambda(z) > 0$ such that

$$\langle du(z)h, du(z)k\rangle_{\mathbb{R}^3} = \lambda(z)\langle h, k\rangle_{\mathbb{R}^2} \quad \text{for every } h, k \in \mathbb{R}^2\,. \tag{10}$$

In other words, u is conformal if and only if for every $z \in \mathcal{O}$ $du(z)$ is the product of an isometry and a homothety from $\mathbb{R}^2$ into $\mathbb{R}^3$. Note also that the condition of conformality (10) can be equivalently written as:

$$|u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y \tag{11}$$

at every point $z \in \mathcal{O}$. In what will follow, an important role is played by the *Hopf differential*, which is the complex-valued function:

$$\omega = \left(|u_x|^2 - |u_y|^2\right) - 2i u_x \cdot u_y.$$

In particular, u is conformal if and only if $\omega = 0$.

REMARK 2. If the target space of a conformal map u has dimension two, then u is analytical. This follows by the fact that, given a domain $\mathcal{O}$ in $\mathbb{R}^2$, a mapping $u \in C^1(\mathcal{O}, \mathbb{R}^2)$ is conformal if and only if u is holomorphic or anti-holomorphic (we identify $\mathbb{R}^2$ with the complex field $\mathbb{C}$). However for conformal maps $u\colon \mathcal{O} \to \mathbb{R}^k$ with $k \geq 3$ there is no such as regularity result.

We turn now to the expression of H for conformal parametrizations. If u is conformal, then

$$\begin{cases} E = |u_x|^2 = \left|u_y\right|^2 = G \\ F = u_x \cdot u_y = 0, \end{cases}$$

so that

$$2H(u) = \frac{\Delta u \cdot \overrightarrow{n}}{|u_x|^2} \quad \text{on } \mathcal{O}\,. \tag{12}$$

On the other hand, deriving conformality conditions (11) with respect to x and y, we can deduce that Δu is orthogonal both to u_x and to u_y. Hence, recalling the expression (6) of the normal vector $\overrightarrow{n}$, we infer that Δu and $\overrightarrow{n}$ are parallel. Moreover, by (11), $|u_x \wedge u_y| = |u_x|^2 = |u_y|^2$, and then, from (12) it follows that

$$\boxed{\Delta u = 2H(u) u_x \wedge u_y \quad \text{on } \mathcal{O}\,.} \tag{13}$$

Let us emphasize that (13) is a system of equations, often called **H-system**, or also **H-equation**, and for this system the scalar coefficient $H(u)$ has the geometric meaning of mean curvature for the surface M parametrized by u at the point $u(z)$ provided that u is conformal and $u(z)$ is a regular point, i.e., $u_x(z) \wedge u_y(z) \neq 0$.

2.3. Some geometric problems involving the H-equation

Equation (13) is the main focus of this article. In order to justify its importance let us list some related geometric problems.

It is useful to recall that the area of a two-dimensional regular surface M parametrized by some mapping $u\colon \mathcal{O} \to \mathbb{R}^3$ is given by the integral

$$A(u) = \int_{\mathcal{O}} |u_x \wedge u_y| \ .$$

In particular, if u is conformal, the area functional equals the Dirichlet integral:

$$E_0(u) = \frac{1}{2} \int_{\mathcal{O}} |\nabla u|^2 \tag{14}$$

One of the most famous geometric problems is that of **minimal surfaces**.

DEFINITION 2. A two-dimensional regular surface in $\mathbb{R}^3$ is said to be minimal if and only if it admits a parametrization u which is a critical point for the area functional, that is, $\frac{dA}{ds}(u + s\varphi)\big|_{s=0} = 0$ for every $\varphi \in C_c^\infty(\mathcal{O}, \mathbb{R}^3)$.

An important fact about minimal surfaces is given by the following statement.

PROPOSITION 3. *A two-dimensional regular surface M in $\mathbb{R}^3$ is minimal if and only if $H \equiv 0$ on M.*

Proof. Fixing a point p_0 in the interior of M, without loss of generality, we may assume that a neighborhood M_0 of p_0 in M is parametrized as a graph, namely there exist a neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^2$ and a function $f \in C^1(\mathcal{O}, \mathbb{R})$ such that M_0 is the image of $u(x, y) = (x, y, f(x, y))$ as $(x, y) \in \mathcal{O}$. In terms of f, the area functional (restricted to M_0) is given by

$$A_0(f) = \int_{\mathcal{O}} \sqrt{1 + |\nabla f|^2}$$

and then

$$\frac{dA_0}{ds}(f + s\psi)\bigg|_{s=0} = -\int_{\mathcal{O}} \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right)\psi$$

for every $\psi \in C_c^\infty(\mathcal{O}, \mathbb{R})$. Hence, keeping into account of (9), the thesis follows. □

Another famous geometric problem is given by the so-called **isoperimetric problem** that we state in the following form. Given any two-dimensional regular *compact* surface M *without boundary*, let $V(M)$ be the volume enclosed by M. The general principle says that:

> *Surfaces which are critical for the area, among surfaces enclosing a prescribed volume, (i.e., solutions of isoperimetric problems) verify $H \equiv const.$*

REMARK 3. Consider for instance the standard isoperimetric problem:

> *Fixing $\lambda > 0$, minimize the area of M among compact surfaces M without boundary such that $V(M) = \lambda$.*

It is well known that this problem admits a unique solution, corresponding to the sphere of radius $\sqrt[3]{\frac{3\lambda}{4\pi}}$. This result agrees with the previous general principle since the sphere has constant mean curvature. Nevertheless, there are many variants for the isoperimetric problem, in which one may add some constrains (on the topological type of the surfaces, or boundary conditions, etc.).

In general, the isoperimetric problem can be phrased in analytical language as follows: consider any surface M admitting a conformal parametrization $u\colon \mathcal{O} \to \mathbb{R}^3$, where $\mathcal{O}$ is a standard reference surface, determined by the topological type of M (for instance the sphere $\mathbb{S}^2$, the torus $\mathbb{T}^2$, etc.). For the sake of simplicity, suppose that M is parametrized by the sphere $\mathbb{S}^2$ that can be identified with the (compactified) plane $\mathbb{R}^2$ through stereographic projection. Hence, if $u\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a conformal parametrization of M, the area of M is given by (14), whereas the (algebraic) volume of M is given by

$$V(u) = \frac{1}{3}\int_{\mathbb{R}^2} u \cdot u_x \wedge u_y \; .$$

In this way, the above isoperimetric problem can be written as follows:

> *Fixing $\lambda > 0$, minimize $\int_{\mathbb{R}^2} |\nabla u|^2$ with respect to the class of conformal mappings $u\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $\int_{\mathbb{R}^2} u \cdot u_x \wedge u_y = 3\lambda$.*

One can recognize that if u solves this minimization problem, or also if u is a critical point for the Dirichlet integral satisfying the volume constraint, then, by the Lagrange multipliers Theorem, u solves an H-equation with H constant.

As a last remarkable example, let us consider the **prescribed mean curvature problem**: given a mapping $H\colon \mathbb{R}^3 \to \mathbb{R}$ study existence and possibly multiplicity of two-dimensional surfaces M such that for all $p \in M$ the mean curvature of p at M equals $H(p)$. Usually the surface M is asked to satisfy also some geometric or topological side conditions.

This kind of problem is a generalization of the previous ones and it appears in various physical and geometric contexts. For instance, it is known that in some evolution problems, interfaces surfaces move according to mean curvature law. Again, nonconstant mean curvature arises in capillarity theory.

3. The Plateau problem: the method of Douglas-Radó

In this section we consider the classical *Plateau problem* for minimal surfaces. Let γ be a Jordan curve in $\mathbb{R}^3$, that is γ is the support of a smooth mapping $g\colon \mathbb{S}^1 \to \mathbb{R}^3$ with no self-intersection. The question is:

$$\boxed{\begin{array}{c}\textit{Is there any surface } M \textit{ minimizing (or critical for)}\\ \textit{the area, among all surfaces with boundary } \gamma\ ?\end{array}}$$

In view of our previous discussions, the Plateau problem becomes:

$$(P_0) \qquad \boxed{\begin{array}{c}\textit{Find a surface } M \textit{ such that } \partial M = \gamma \textit{ and having}\\ \textit{zero mean curvature at all points.}\end{array}}$$

Note that in general, this problem may admit more than one solution.

We will discuss this problem by following the method of Douglas-Radó, but we point out that many methods have been successfully proposed to solve the Plateau problem. Here is a nonexhaustive list of some of them.

1. When γ is a *graph*, try to find M as a graph. More precisely suppose γ to be close to a plane curve γ_0. Note that for γ_0 the obvious solution is the planar region bounded by γ_0 itself. Let $g\colon \mathbb{S}^1 \to \mathbb{R}$ be such that $\gamma = \widetilde{g}(\mathbb{S}^1)$ where $\widetilde{g}(z) = (z, g(z))$ as $z = (x, y) \in \mathbb{S}^1$. If g is "small", we may use *perturbation techniques* (Schauder method) to solve the *nonlinear* problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0 & \text{in } D^2 \\ f = g & \text{on } \partial D^2 = \mathbb{S}^1 \end{cases}$$

   where D^2 is the open unit disc in $\mathbb{R}^2$ (compare with (9), being now $H = 0$).

2. Given a Jordan curve γ, find a surface M spanning γ, with M parametrized in conformal coordinates. This is the Douglas-Radó method that we will develop in more details. Here we just note that, differently from the previous case, now the conformal parametrization u of M solves the *linear* equation $\Delta u = 0$.

3. Use the tools from *geometric measure theory* ([21], [39], [40]), especially designed for that purpose. The advantage of this method is that it is free from conformality equations, and it is very good for minimization problems, but it needs a lot of work to recover regularity of the solutions. Actually, this method is not very useful to handle with saddle critical points.

4. Use *singular limit problems*:

$$E_\varepsilon(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{\varepsilon^2}\int \left(1 - |u|^2\right)^2 .$$

   As the previous one, this method does not use any parametrization.

Let us turn now to the Douglas-Radó method. Looking for a conformal parametrization of M, by (13), the Plateau problem is reduced to the following form:

$$(P_0) \qquad \boxed{\begin{array}{l} \textit{Find } u \in C^0\left(\overline{D^2}, \mathbb{R}^3\right) \cap C^2(D^2, \mathbb{R}^3) \textit{ such that} \\ \begin{cases} \Delta u = 0 & \textit{in } D^2 \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \textit{in } D^2 \\ u\big|_{\partial D^2} \textit{ monotone parametrization of } \gamma \ . \end{cases} \end{array}}$$

On one hand the Laplace equation is completely standard. On the other hand, the boundary condition is less usual than the Dirichlet one and, besides, one has to deal with the conformality conditions.

The first step in the Douglas-Radó approach consists in translating problem (P_0) into a minimization problem. To this aim let us introduce the Sobolev space $H^1 = H^1(D^2, \mathbb{R}^3)$ and the set

$$W = \{v \in H^1 \ : \ v\big|_{\partial D^2} \text{ continuous, monotone, parametrization of } \gamma\} \tag{15}$$

and for every $v \in W$ let us denote by $E_0(v)$ the Dirichlet integral of v on D^2, as in (14). Recall that if v is conformal then $E_0(v)$ gives the area of the surface parametrized by v.

LEMMA 1. *If $u \in W$ minimizes E_0 on W, then u is a solution of the Plateau problem (P_0).*

The most surprising result in this statement is that the conformality conditions come out as part of the Euler-Lagrange equation.

Proof. Since u minimizes the Dirichlet integral for all H^1 maps with the same boundary value, u is a weak solution to $\Delta u = 0$ in D^2. In fact, from regularity theory, $u \in C^\infty$. Now, let

$$\omega = |u_x|^2 - |u_y|^2 - 2i u_x \cdot u_y$$

be the Hopf differential associated to u. Since u solves the Laplace equation, it is easy to verify that $\frac{\partial \omega}{\partial \bar{z}} = 0$, and then ω is constant. In order to prove that $\omega \equiv 0$, i.e., u is conformal, the idea is to use *variations of the domain*. More precisely, let $\overrightarrow{X}$ be an arbitrary vector field on D^2 such that $\overrightarrow{X} \cdot \overrightarrow{n} = 0$ on ∂D^2, and let $\phi(t, z)$ be the flow generated by $\overrightarrow{X}$, i.e.

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \overrightarrow{X}(\phi) \\ \\ \phi(0, z) = z \ . \end{cases}$$

Then $\phi(t, z) = z + t\overrightarrow{X}(z) + o\left(t^2\right)$ and $\phi_t := \phi(t, \cdot) \colon D^2 \to D^2$ is a diffeomorphism for every $t \geq 0$. If we set $u_t = u \circ \phi_t$ then $u \in W$ implies $u_t \in W$ for every $t \geq 0$ and therefore, by the minimality of u,

$$\frac{d}{dt}E_0(u_t) = 0,$$

i.e.

$$\frac{d}{dt}\int_{D^2}\left|\nabla u\left(z + t\overrightarrow{X}(z)\right)\right|^2 = 0. \tag{16}$$

After few computations, (16) can be rewritten as

$$\int_{D^2}\omega\cdot\frac{\partial\overrightarrow{X}}{\partial\overline{z}} = 0$$

which holds true for every $\overrightarrow{X} \in C^\infty\left(D^2, \mathbb{R}^2\right)$ such that $\overrightarrow{X}\cdot\overrightarrow{n} = 0$ on ∂D^2. This implies $\omega \equiv 0$, that is the thesis.

□

Thanks to Lemma 1, a solution to problem (P_0) can be found by solving the following minimization problem:

$$(Q_0) \qquad \boxed{\textit{Find } u \in W \textit{ such that } E_0(u) = \inf_{v\in W} E_0(v)}$$

where $E_0(v)$ is the Dirichlet integral of v and W is defined in (15).

Conformal invariance

The greatest difficulty in the study of problem (Q_0) is that minimizing sequences are not necessarily compact in W, because of the *conformal invariance* of the problem. Let us consider the group G of all conformal diffeomorphisms of D^2:

$$G = \{\phi \in C^1(D^2, D^2) \quad : \quad \phi \text{ one to one and orientation preserving,} \\ |\phi_x|^2 - |\phi_y|^2 = 0 = \phi_x\cdot\phi_y\}.$$

It is easy to verify that, given any $v \in W$ and $\phi \in G$ one has $|\nabla(v\circ\phi)| = \lambda|(\nabla v)\circ\phi|$ where $\lambda = |\phi_x| = |\phi_y|$. Since $\lambda^2 = |\text{Jac}\,\phi|$, one obtains

$$\int_{D^2}|\nabla(v\circ\phi)|^2 = \int_{D^2}|\nabla v|^2$$

that is

$$E_0(v\circ\phi) = E_0(v),$$

the energy is invariant under a conformal change on D^2. Note also that

$$u \in W,\ \phi \in G \ \Rightarrow\ u\circ\phi \in W$$

because if $\phi \in G$ then $\phi\big|_{\partial D^2} : \partial D^2 \to \partial D^2$ is monotone.

As a consequence of conformal invariance, we are going to see that W *is not sequentially weakly closed* in H^1. In order to do that, let us first describe G. As already mentioned in remark 2, conformal maps $\phi \in G$ are holomorphic or antiholomorphic; by a choice of orientation, we can restrict ourselves to holomorphic diffeomorphisms. It is then a (not so easy) exercise in complex analysis to prove that

$$G = \left\{\phi \in C^1(D^2, \mathbb{C}) : \exists a \in \mathbb{C},\ |a| < 1,\ \exists\theta \in [0, 2\pi)\ s.t.\ \phi = \phi_{\theta,a}\right\}$$

where

$$\phi_{\theta,a}(z) = \frac{z+a}{1-\overline{a}z} e^{i\theta} \qquad \left(z \in D^2\right).$$

Hence G is parametrized by $D^2 \times \mathbb{S}^1$, a noncompact three-dimensional manifold with boundary.

Note now that, given $v \in W \cap C(\overline{D^2}, \mathbb{R}^3)$ and $(a_n) \subset D^2$, if $a_n \to a \in \partial D^2$ then $v \circ \phi_{0,a_n} \to v(a)$ pointwise and weakly in H^1 (but not strongly), and the weak limit in general does not belong to W which does not contain any constant.

The three points condition

In order to remove conformal invariance, we have to "fix a gauge", choosing for every $v \in W$ a special element in the orbit $\{v \circ \phi\}_{\phi \in G}$. For this purpose, let us fix a monotone parametrization $g \in C(\mathbb{S}^1, \gamma)$ of γ and then, let us introduce the class

$$W^* = \left\{v \in W\ :\ v\big(e^{\frac{2ik\pi}{3}}\big) = g\big(e^{\frac{2ik\pi}{3}}\big),\ k = 1, 2, 3\right\}.$$

Since $W^* \subset W$ and for every $v \in W$ there exists $\varphi \in G$ such that $v \circ \varphi \in W^*$, one has that:

LEMMA 2. $\inf_{v \in W^*} E_0(v) = \inf_{v \in W} E_0(v)$.

Hence, in order to find a solution to the Plateau problem (P_0), it is sufficient to solve the minimization problem defined by $\inf_{v \in W^*} E_0(v)$. This can be accomplished by using the following result.

LEMMA 3 (COURANT-LEBESGUE). *W^* is sequentially weakly closed in H^1.*

Proof. We limit ourselves to sketch the proof. To every $v \in W^*$, one associates (in a unique way) a continuous mapping $\varphi \colon [0, 2\pi] \to [0, 2\pi]$ such that

$$v\big(e^{i\theta}\big) = g\big(e^{i\varphi(\theta)}\big),\ \ \varphi(0) = 0. \tag{17}$$

The function φ turns out to be increasing and satisfying

$$\varphi\left(\frac{2k\pi}{3}\right) = \frac{2k\pi}{3} \text{ for } k = 0, \ldots, 3. \tag{18}$$

Take a sequence $(v_n) \subset W^*$ converging to some v weakly in H^1. Let $(\varphi_n) \in C([0, 2\pi])$ be the corresponding sequence, defined according to (17). Since every φ_n

is increasing and satisfies (18), for a subsequence, $\varphi_n \to \varphi$ almost everywhere, being φ an increasing function on $[0, 2\pi]$ satisfying (18). One can show that φ is continuous on $[0, 2\pi]$, this is the hard step in the proof. Then, from monotonicity, $\varphi_n \to \varphi$ uniformly on $[0, 2\pi]$. By continuity of g, from (17) it follows that $u\big|_{\partial D^2}$ is a continuous monotone parametrization of γ and then u belongs to W^*.

$\square$

Hence, apart from regularity at the boundary, we proved that the Plateau problem (P_0) admits at least a solution, characterized as a minimum.

4. The Plateau problem for H-surfaces (the small solution)

A natural extension of the previous Plateau problem (P_0) is to look for surfaces with prescribed mean curvature bounding a given Jordan curve γ, that is

(P_H) | *Find a surface M such that $\partial M = \gamma$ and the mean curvature of M at p equals $H(p)$, for all $p \in M$.*

where $H : \mathbb{R}^3 \to \mathbb{R}$ is a given function (take for instance a constant).

Some restrictions on the function H or on γ are rather natural. This can be seen even for the equivalent version of problem (P_H) in lower dimension. Indeed, a curve in the plane with constant curvature $K_0 > 0$ is a portion of a circle with radius $R_0 = 1/K_0$. Therefore, fixing the end points $a, b \in \mathbb{R}^2$, such as a curve joining a and b exists provided that $|a - b| \leq 2R_0$. Choosing the origin in the middle of the segment $\overline{ab}$, this condition becomes $\sup\{|a|, |b|\}K_0 \leq 1$.

The necessity of some smallness condition on H or on γ is confirmed by the following *nonexistence result* proved by E. Heinz in 1969 [26]:

THEOREM 1. *Let γ be a circle in $\mathbb{R}^3$ of radius R. If $H_0 > 1/R$ then there exists no surface of constant mean curvature H_0 bounding γ.*

Hence we are led to assume a condition like $\|H\|_\infty \|\gamma\|_\infty \leq 1$. Under this condition, in 1969 S. Hildebrandt [30] proved the next *existence result*:

THEOREM 2. *Let γ be a Jordan curve in $\mathbb{R}^3$ and let $H : \mathbb{R}^3 \to \mathbb{R}$ be such that*

$$\|H\|_\infty \|\gamma\|_\infty \leq 1.$$

Then there exists a surface of prescribed mean curvature H, bounding γ.

We will give some ideas of the proof of the Hildebrandt theorem. Firstly, by virtue of what discussed in section 2, problem (P_H) can be expressed analytically as follows:

$$(P_H) \qquad \boxed{\begin{array}{c} \textit{Find a (regular) } u\colon \overline{D^2} \to \mathbb{R}^3 \textit{ such that} \\ \begin{cases} \Delta u = 2H(u) u_x \wedge u_y & \textit{in } D^2 \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \textit{in } D^2 \\ u\big|_{\partial D^2} \textit{ monotone parametrization of } \gamma \ . \end{cases} \end{array}}$$

The partial differential equation for u is now *nonlinear* and this is of course the main difference with the Plateau problem (P_0) for minimal surfaces. The solution of (P_H) found by Hildebrandt is characterized as a minimum, and it is often called **small solution**. In fact, under suitable assumptions, one can find also a second solution to (P_H) which does not correspond to a minimum point but to a saddle critical point, the so-called **large solution** (see section 6).

The conformality condition can be handled as in the Douglas-Radó approach (three-point condition). In doing that, we are led to consider the more standard Dirichlet problem

$$(D_H) \qquad \begin{cases} \Delta u = 2H(u) u_x \wedge u_y & \text{in } D^2 \\ u = g & \text{on } \partial D^2 \ , \end{cases}$$

where g is a fixed continuous, monotone parametrization of γ.

The main point in Hildebrandt's proof is the existence of solutions to the problem (D_H), that is:

THEOREM 3. *Let $g \in H^{1/2}(\mathbb{S}^1, \mathbb{R}^3) \cap C^0$ and let $H\colon \mathbb{R}^3 \to \mathbb{R}$ be such that*

$$\|g\|_\infty \|H\|_\infty \leq 1.$$

Then problem (D_H) admits a solution.

Proof. Let us show this result in case the strict inequality $\|g\|_\infty \|H\|_\infty < 1$ holds. We will split the proof in some steps.

Step 1: Variational formulation of problem (D_H).

Problem (D_H) is variational, that is, solutions to (D_H) can be detected as critical points of a suitable energy functional, defined as follows. Let $Q_H\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field such that

$$\operatorname{div} Q_H(u) = H(u) \text{ for all } u \in \mathbb{R}^3.$$

For instance, take

$$Q_H(u) = \frac{1}{3}\left(\int_0^{u_1} H(t, u_2, u_3)\, dt, \int_0^{u_2} H(u_1, t, u_3)\, dt, \int_0^{u_3} H(u_1, u_2, t)\, dt \right).$$

Then, denote

$$H^1_g = \{u \in H^1(D^2, \mathbb{R}^3) : u\big|_{\partial D^2} = g\}$$

and

$$E_H(u) = \frac{1}{2}\int_{D^2} |\nabla u|^2 + 2\int_{D^2} Q_H(u) \cdot u_x \wedge u_y \ .$$

Note that in case of constant mean curvature $H(u) \equiv H_0$ one can take $Q_{H_0}(u) = \frac{1}{3}H_0 u$ and E_H turns out to be the sum of the Dirichlet integral with the volume integral.

One can check that critical points of E_H on H^1_g correspond to (weak) solutions to problem (D_H). Actually, as far as concerns the regularity of E_H on the space H^1_g some assumptions on H are needed. For instance, E_H is of class C^1 if $H \in C^0(\mathbb{R}^3)$ and $H(u)$ is constant for $|u|$ large. A reduction to this case will be done in the next step.

Step 2: Truncation on H and study of a minimization problem.

By scaling, we may assume $h = \|H\|_\infty < 1$ and $\|g\|_\infty \leq 1$. Then, let $h' \in (h, 1)$ and $\widetilde{H}: \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that

$$\widetilde{H}(u) = \begin{cases} H(u) & \text{as } |u| \leq 1 \ , \\ 0 & \text{as } |u| \geq \frac{1}{h'} \ , \end{cases}$$

and with $\|\widetilde{H}\|_\infty < h'$. Let us denote by $Q_{\widetilde{H}}$ and $E_{\widetilde{H}}$ the functions corresponding to $\widetilde{H}$. Since $|Q_{\widetilde{H}}(u)| \leq 1$ for all $u \in \mathbb{R}^3$, we obtain

$$\frac{1}{3}E_0(u) \leq E_{\widetilde{H}}(u) \leq \frac{5}{3}E_0(u) \text{ for all } u \in H^1_g \ .$$

Moreover, $E_{\widetilde{H}}$ turns out to be weakly lower semicontinuous on H^1_g. Therefore

$$\inf_{v \in H^1_g} F_{\widetilde{H}}(v)$$

is achieved by some function $u \in H^1_g$. By standard arguments, u is a critical point of $E_{\widetilde{H}}$ and thus, a (weak) solution of

$$(D_{\widetilde{H}}) \qquad \begin{cases} \Delta u = 2\widetilde{H}(u) u_x \wedge u_y & \text{in } D^2 \\ u = g & \text{on } \partial D^2 \ . \end{cases}$$

Step 3: Application of the maximum principle.

In order to prove that u is solution to the original problem (D_H), one shows that $\|u\|_\infty \leq 1$. One has that (in a weak sense)

$$-\Delta |u|^2 = -2\left(|\nabla u|^2 + u \cdot \Delta u\right) \leq -2|\nabla u|^2 \big(1 - |u|\big|\widetilde{H}(u)\big|\big) \leq 0.$$

Hence $|u|^2$ is subharmonic and the maximum principle yields

$$\|u\|_{L^\infty(D^2)} \leq \|u\|_{L^\infty(\partial D^2)} = \|g\|_\infty \leq 1.$$

Since $\widetilde{H}(u) = H(u)$ as $|u| \leq 1$, u turns out to solve (D_H).

$\square$

REMARK 4. 1.The implementation of the Douglas-Radó method passing from the Dirichlet problem (D_H) to the Plateau problem (P_H) is made possible by the fact that the functional E_H is conformally invariant. Actually, note that the volume functional

$$V_H(u) = \int_{D^2} Q_H(u) \cdot u_x \wedge u_y$$

is invariant with respect to the (larger) group of the orientation preserving diffeomorphisms of D^2 into itself.
2. When H is constant (e.g. $H \equiv 1$) and $u \in H^1_g$ is regular, the functional $V_H(u)$ has a natural geometric interpretation as a (signed) volume of the region bounded by the surface parametrized by u and a fixed surface given by the portion of cone with vertex at the origin and spanning g. When H is nonconstant a similar interpretation holds, considering $\mathbb{R}^3$ endowed with an H-weighted metric (see Steffen [39]).
3. Although the condition $\|\gamma\|_\infty \|H\|_\infty \le 1$ is natural and sufficient for existence of solutions to problem (P_H), it is not necessary. Think for instance of long and narrow "strips". In this direction there are some existence results (by Heinz [25], Wente [47], and K. Steffen [40]) both for the Dirichlet problem (D_H) and for the Plateau problem (P_H) where a solution characterized as a minimum is found assuming that

$$\|H\|_\infty \sqrt{A_\gamma} \le C_0$$

where A_γ denotes the minimal area bounding γ and C_0 is some explicit positive constant.
4. In case of constant mean curvature $H(u) \equiv H_0 > 0$, if γ is a curve lying on a sphere of radius $R_0 = 1/H_0$, the solution given by the above Hildebrandt theorem corresponds to the smaller part of the sphere spanning γ (*small solution*). In this special case, the larger part of the sphere is also a solution to (P_H), the *large solution*. We will see below that this kind of multiplicity result holds true for more general γ and H, but it does not happen, in general, for minimal surfaces.
5. There are also conformal solutions of the H-equation which define compact surfaces (this is impossible for minimal surfaces). A typical example is the sphere $\mathbb{S}^2$. More surprisingly, Wente in [49] constructed also immersed tori of constant mean curvature.

5. Analytical aspects of the H-equation

In this section we will study properties of solutions of the H-equation (13). More precisely, we will study:

(i) the *regularity theory* as well as some aspects of the energy functional E_H (Wente's result [47] and its extensions by Heinz [27], [28], Bethuel and Ghidaglia [8], [9], Bethuel [7]),

(ii) *a priori bounds* of solutions of problem (P_H) (or also (D_H)),

(iii) *isoperimetric inequalities*.

Clearly, questions (i) and (ii) are elementary for the minimal surface equation $\Delta u = 0$. For the H-equation (13), they are rather involved, because the nonlinearity is "critical".

5.1. Regularity theory

Here we consider *weak solutions* of the equation

$$\Delta u = 2H(u)u_x \wedge u_y \text{ on } \mathcal{O} \tag{19}$$

where $\mathcal{O}$ is any domain in $\mathbb{R}^2$. Owing to the nonlinearity $2H(u)u_x \wedge u_y$ as well as to the variational formulation discussed in the previous section, it is natural to consider solutions of (19) which are in the space $H^1(\mathcal{O}, \mathbb{R}^3)$.

The first regularity result for (19) was given by H. Wente [47], for H constant.

THEOREM 4. *If H is constant, then any $u \in H^1(\mathcal{O}, \mathbb{R}^3)$ solution of* (19) *is smooth, i.e., $u \in C^\infty(\mathcal{O})$.*

Nowadays, this result is a special case of a more general theorem (see Theorem 5 below) that will be discussed in the sequel. In any case, we point out that the proof of Theorem 4 relies on the special structure of the nonlinearity:

$$u_x \wedge u_y = \begin{pmatrix} u_x^2 u_y^3 - u_x^3 u_y^2 \\ u_x^3 u_y^1 - u_x^1 u_y^3 \\ u_x^1 u_y^2 - u_x^2 u_y^1 \end{pmatrix} = \begin{pmatrix} \{u^2, u^3\} \\ \{u^3, u^1\} \\ \{u^1, u^2\} \end{pmatrix}.$$

Here we have made use of the notation

$$\{f, g\} = f_x g_y - f_y g_x$$

which represents the Jacobian of the map $(x, y) \mapsto (f(x, y), g(x, y))$. Thus, considering the equation (19) with H constant, we are led to study the more general *linear* equation

$$\Delta\phi = \{f, g\} \text{ in } \mathcal{O}$$

where f, g satisfy $\int_{\mathcal{O}} |\nabla f|^2 < +\infty$ and $\int_{\mathcal{O}} |\nabla g|^2 < +\infty$. Obviously $\{f, g\} \in L^1(\mathcal{O})$ but, in dimension two, $\Delta\phi \in L^1(\mathcal{O})$ implies $\phi \in W^{1,p}_{loc}(\mathcal{O})$ only for $p < 2$, while the embedding $W^{1,p} \hookrightarrow L^\infty$ holds true only as $p > 2$. However, $\{f, g\}$ has a special structure of divergence form, and precisely

$$\{f, g\} = \frac{\partial}{\partial x}\left(f g_y\right) - \frac{\partial}{\partial y}\left(f g_x\right),$$

and this can be employed to prove what stated in the following lemmata, which have been used in various forms since the pioneering work by Wente [47].

LEMMA 4. *Let $\phi \in W^{1,1}_{loc}(\mathbb{R}^2)$ be the solution of*

$$\begin{cases} -\Delta\phi = \{f, g\} & \text{on } \mathbb{R}^2 \\ \phi(z) \to 0 & \text{as } |z| \to +\infty. \end{cases}$$

Then

$$\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2} \leq C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}.$$

Proof. Let $-\frac{1}{2\pi}\ln|z|$ be the fundamental solution of $-\Delta$. Since the problem is invariant under translations, it suffices to estimate $\phi(0)$. We have

$$\phi(0) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|z|\,\{f,g\}\,dz.$$

In polar coordinates, one has

$$\{f,g\} = \frac{1}{r}\frac{\partial}{\partial\theta}(fg_r) - \frac{\partial}{\partial r}(fg_\theta).$$

Hence, integrating by parts, we obtain

$$\begin{aligned}
\phi(0) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{r}fg_\theta\,dz \\
&= \frac{1}{2\pi}\int_0^{+\infty}\frac{dr}{r}\left(\int_{|z|=r} fg_\theta\,d\theta\right).
\end{aligned}$$

Setting $\overline{f} = \frac{1}{2\pi r}\int_{|z|=r} f\,d\theta$, then, using Cauchy-Schwartz and Poincaré inequality,

$$\begin{aligned}
\left|\int_{|z|=r} fg_\theta\,d\theta\right| &= \left|\int_{|z|=r}\left(f-\overline{f}\right)g_\theta\,d\theta\right| \\
&\leq \left(\int_{|z|=r}\left|f-\overline{f}\right|^2\,d\theta\right)^{\frac{1}{2}}\left(\int_{|z|=r}|g_\theta|^2\,d\theta\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{|z|=r}|f_\theta|^2\,d\theta\right)^{\frac{1}{2}}\left(\int_{|z|=r}|g_\theta|^2\,d\theta\right)^{\frac{1}{2}} \\
&\leq Cr^2\left(\int_{|z|=r}|\nabla f|^2\,d\theta\right)^{\frac{1}{2}}\left(\int_{|z|=r}|\nabla g|^2\,d\theta\right)^{\frac{1}{2}}.
\end{aligned}$$

Going back to $\phi(0)$, using again Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|\phi(0)| &\leq C\int_0^{+\infty}\left(r\int_{|z|=r}|\nabla f|^2\,d\theta\right)^{\frac{1}{2}}\left(r\int_{|z|=r}|\nabla g|^2\,d\theta\right)^{\frac{1}{2}}dr \\
&\leq C\left(\int_0^{+\infty}\int_{|z|=r}|\nabla f|^2\,d\theta\,r\,dr\right)^{\frac{1}{2}}\left(\int_0^{+\infty}\int_{|z|=r}|\nabla g|^2\,d\theta\,r\,dr\right)^{\frac{1}{2}} \\
&= C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}.
\end{aligned}$$

Hence

$$\|\phi\|_{L^\infty} \leq C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}.$$

Finally, multiplying the equation $-\Delta\phi = \{f, g\}$ by ϕ and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla\phi|^2 &\leq \|\{f, g\}\|_{L^1}\|\phi\|_{L^\infty} \\ &\leq 2\|\phi\|_{L^\infty}\|\nabla f\|_{L^2}\|\nabla g\|_{L^2} \\ &\leq C\|\nabla f\|^2_{L^2}\|\nabla g\|^2_{L^2}\ .\end{aligned}$$

□

Using the maximum principle, it is possible to derive similarly (as obtained by H. Brezis and J.M. Coron [13]) the following analogous result:

LEMMA 5. *Assume $f, g \in H^1(D^2, \mathbb{R})$ and let $\phi \in W^{1,1}_0(D^2, \mathbb{R})$ be the solution of*

$$\begin{cases} -\Delta\phi = \{f, g\} & \text{on } D^2 \\ \phi = 0 & \text{on } \partial D^2. \end{cases}$$

Then

$$\|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^2} \leq C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}.$$

Another proof of the above lemmas can be obtained by using tools of harmonic analysis. It has been proved (Coifman-Lions-Meyer-Semmes [19]) that if $f, g \in H^1(\mathbb{R}^2)$ then $\{f, g\}$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$, a strict subspace of $L^1(\mathbb{R}^2)$, defined as follows:

$$\mathcal{H}^1(\mathbb{R}^2) = \{u \in L^1(\mathbb{R}^2) : \mathrm{K}_j u \in L^1 \text{ for } j = 1, 2\},$$

where $\mathrm{K}_j = \partial/\partial x_j(-\Delta)^{1/2}$. As a consequence, since any Riesz transform $\mathrm{R}_j = \partial/\partial x_j(-\Delta)^{-1/2}$ maps $\mathcal{H}^1(\mathbb{R}^2)$ into itself, one has that if $-\Delta\phi = \{f, g\}$ on $\mathbb{R}^2$ then

$$-\frac{\partial^2\phi}{\partial x_i \partial x_j} = \mathrm{R}_i\mathrm{R}_j(-\Delta\phi) \in \mathcal{H}^1(\mathbb{R}^2) \quad for\ i, j = 1, 2$$

and hence $\phi \in W^{2,1}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$. This argument holds similarly true in the situation of lemma 5 and can be pushed further to obtain the desired estimate, exploiting the fact that the fundamental solution (on $\mathbb{R}^2$) to the Laplace equation belongs to BMO($\mathbb{R}^2$), the dual of $\mathcal{H}^1(\mathbb{R}^2)$.

We now turn to the case of variable H. Regularity of (weak) H^1-solutions has been established under various assumptions on the function H. For instance, $H \in C^\infty(\mathbb{R}^3)$

and

$\sup_{y\in\mathbb{R}^3} \vert H(y)\vert(1+\vert y\vert) \le \alpha < 1$	(Heinz, [27])
$\Vert H\Vert_\infty < +\infty,\ H(y) = H(y_1, y_2)$	(Bethuel-Ghidaglia, [8])
$\Vert H\Vert_\infty < +\infty,\ \sup_{y\in\mathbb{R}^3} \vert\nabla H(y)\vert(1+\vert y\vert) < +\infty$	(Heinz, [28])
$\Vert H\Vert_{L^\infty} < +\infty,\ \sup_{y\in\mathbb{R}^3} \frac{\partial H}{\partial y_3}(y)(1+\vert y_3\vert) \le C$	(Bethuel-Ghidaglia, [9]).

However we will describe another regularity theorem, due to F. Bethuel [7].

THEOREM 5. *If $H \in C^\infty(\mathbb{R}^3)$ satisfies*

$$\|H\|_{L^\infty} + \|\nabla H\|_{L^\infty} < +\infty \tag{20}$$

then any solution $u \in H^1(D^2, \mathbb{R}^3)$ to $\Delta u = 2H(u)u_x \wedge u_y$ on D^2 is smooth, i.e., $u \in C^\infty(D^2)$.

The proof of this theorem involves the use of Lorentz spaces, which are borderline for Sobolev injections, and relies on some preliminary results. Thus we are going to recall some background on the subject, noting that the interest for Lorentz spaces, in our context, was pointed out by F. Hélein [29], who used them before for harmonic maps.

If Ω is a domain in $\mathbb{R}^N$ and μ denotes the Lebesgue measure, we define $L^{2,\infty}(\Omega)$ as the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that the *weak $L^{2,\infty}$-norm*

$$\|f\|_{L^{2,\infty}} = \sup_{t>0}\{t^{\frac{1}{2}}\mu(\{x \in \Omega : f(x) > t\})\}$$

is finite. If $L^{2,1}(\Omega)$ denotes the dual space of $L^{2,\infty}(\Omega)$, one has $L^{2,1}(\Omega) \subset L^2(\Omega) \subset L^{2,\infty}(\Omega)$, the last inclusion being strict since, for instance, $1/r \in L^{2,\infty}(D^2)$ but $1/r \notin L^2(D^2)$. Moreover, if Ω is bounded, then $L^{2,\infty}(\Omega) \subset L^p(\Omega)$ for every $p < 2$. See [50] for thorough details.

Denoting by $B_r = B_r(z_0)$ the disc of radius $r > 0$ and center $z_0 \in \mathbb{R}^2$, let now $\phi \in W_0^{1,1}(B_r)$ be the solution of

$$\begin{cases} -\Delta\phi = \{f, g\} & \text{in } B_r \\ \phi = 0 & \text{on } \partial B_r \end{cases}$$

where $f, g \in H^1(B_r)$; recalling lemma 5, one has

$$\|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^2} + \|\nabla\phi\|_{L^{2,1}(B_{r/2})} \le C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}. \tag{21}$$

The estimate of $L^{2,1}$-norm of the gradient was obtained by L. Tartar [45] using interpolation methods, but can also be recovered as a consequence of the embedding $W^{1,1} \hookrightarrow L^{2,1}$ due to H. Brezis (since, as we have already mentioned, the fact that

$\{f, g\}$ belongs to the Hardy space $\mathcal{H}^1$ implies that $\phi \in W^{2,1}$). Moreover, if g is constant on ∂B_r, then it can be proved (see [7]) that

$$\|\nabla\phi\|_{L^2} \leq C\|\nabla f\|_{L^2}\|\nabla g\|_{L^{2,\infty}}. \tag{22}$$

Finally, we recall the following classical result: if $h \in L^1(B_r)$, then the solution $\phi \in W_0^{1,1}(B_r)$ to

$$\begin{cases} -\Delta\phi = h & \text{in } B_r \\ \phi = 0 & \text{on } \partial B_r \end{cases}$$

verifies

$$\|\nabla\phi\|_{L^{2,\infty}(B_{r/2})} \leq C\|h\|_{L^1}. \tag{23}$$

Proof of Theorem 5. At first we note that the hypothesis (20) grants that $|\nabla H(u)| \leq C|\nabla u|$ and $H(u) \in H^1$. The proof is then divided in some steps.

Step 1: Rewriting equation (19).

Let $B_{2r}(z_0) \subset D^2$ and $\{H(u), u\} = (\{H(u), u^1\}, \{H(u), u^2\}, \{H(u), u^2\})$. The idea is to introduce a (Hodge) decomposition of $2H(u)\nabla u$ in B_{2r}:

$$2H(u)\nabla u = \nabla A + \nabla^{\perp}\beta \qquad where\ \nabla^{\perp} = \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right).$$

Since

$$\frac{\partial}{\partial x}(2H(u)u_y) + \frac{\partial}{\partial y}(-2H(u)u_x) = 2\{H(u), u\},$$

the solution $\beta \in W_0^{1,1}(B_{2r}, \mathbb{R}^3)$ to

$$\begin{cases} -\Delta\beta = \{H(u), u\} & \text{in } B_{2r} \\ \beta = 0 & \text{on } \partial B_{2r} \end{cases}$$

belongs, by lemma 5, to $H^1(B_{2r}, \mathbb{R}^3)$ and satisfies

$$\frac{\partial}{\partial x}(2H(u)u_y + \beta_x) + \frac{\partial}{\partial y}(-2H(u)u_x + \beta_y) = 0.$$

Hence, there exists $A \in H^1(B_{2r}, \mathbb{R}^3)$ such that

$$A_x = 2H(u)u_x - \beta_y, \quad A_y = 2H(u)u_y + \beta_x \tag{24}$$

and equation (19), on B_{2r}, rewrites:

$$\Delta u = A_x \wedge u_y + \beta_y \wedge u_y. \tag{25}$$

Step 2: "Morrey type" inequality for the $L^{2,\infty}$ norm.

Since regularity is a local property, as we may reduce the radius r we can assume without loss of generality that $\|\nabla u\|_{L^2(B_r)} < \varepsilon < 1$. We are now going to show that there exists $\theta \in (0,1)$ such that

$$\|\nabla u\|_{L^{2,\infty}(B_{\theta r})} \leq \frac{1}{2}\|\nabla u\|_{L^{2,\infty}(B_r)}. \tag{26}$$

This is the main step of the proof. Let us consider $B_{r_0} \subset B_{r/2}$ and let $\widetilde{u}$ be the harmonic extension to B_{r_0} of $u_{|\partial B_{r_0}}$. Note that the radius r_0 can be chosen such that $\|\nabla\widetilde{u}\|_{L^2(B_{r_0})} \leq C\|\nabla u\|_{L^{2,\infty}(B_r)}$ (see [7] for details). In B_{r_0}, using (25), we may write

$$u = \widetilde{u} + \psi_1 + \psi_2 + \psi_3$$

where the functions ψ_1, ψ_2, ψ_3 are defined by

$$\begin{gathered}\Delta\psi_1 = A_x \wedge (u-\widetilde{u})_y,\ \Delta\psi_2 = A_x \wedge \widetilde{u}_y,\ \Delta\psi_3 = \beta_y \wedge u_y \quad in\ \ B_{r_0}\\ \psi_1 = \psi_2 = \psi_3 = 0 \quad on\ \partial B_{r_0}.\end{gathered}$$

Note that, using (24), (21), (20) and the fact that $\varepsilon < 1$, computations give

$$\|\nabla A\|_{L^2(B_r)} \leq C\|\nabla u\|_{L^2(B_r)}.$$

By (22), we have

$$\begin{aligned}\|\nabla\psi_1\|_{L^2(B_{r_0})} &\leq C\|\nabla A\|_{L^2(B_r)}\|\nabla(u-\widetilde{u})\|_{L^{2,\infty}(B_{r_0})}\\ &\leq C\|\nabla u\|_{L^2(B_r)}\|\nabla u\|_{L^{2,\infty}(B_r)} \leq C\varepsilon\|\nabla u\|_{L^{2,\infty}(B_r)}\end{aligned} \tag{27}$$

and, using (21), we obtain

$$\begin{aligned}\|\nabla\psi_2\|_{L^2(B_{r_0})} &\leq C\|\nabla A\|_{L^2(B_{r_0})}\|\nabla\widetilde{u}\|_{L^2(B_{r_0})}\\ &\leq C\|\nabla u\|_{L^2(B_r)}\|\nabla\widetilde{u}\|_{L^2(B_{r_0})} \leq C\varepsilon\|\nabla u\|_{L^{2,\infty}(B_r)}.\end{aligned} \tag{28}$$

Using the duality of $L^{2,1}$ and $L^{2,\infty}$, (23) and (21) yield

$$\begin{aligned}\|\nabla\psi_3\|_{L^{2,\infty}(B_{r_0/2})} &\leq C\|\nabla\beta\|_{L^{2,1}(B_{r/2})}\|\nabla u\|_{L^{2,\infty}(B_{r/2})}\\ &\leq C\varepsilon^2\|\nabla u\|_{L^{2,\infty}(B_r)} \leq C\varepsilon\|\nabla u\|_{L^{2,\infty}(B_r)}.\end{aligned} \tag{29}$$

By the properties of harmonic functions, one has that

$$\forall\alpha \in (0,1) \qquad \|\nabla\widetilde{u}\|_{L^2(B_{\alpha r_0})} \leq C\alpha\|\nabla\widetilde{u}\|_{L^2(B_{r_0})} \leq C\alpha\|\nabla u\|_{L^{2,\infty}(B_r)}. \tag{30}$$

Combining (27)–(30) and recalling the decomposition of u in B_{r_0}, we finally deduce that

$$\forall\alpha \in (0,1) \qquad \|\nabla u\|_{L^{2,\infty}(B_{\alpha r_0})} \leq C(\varepsilon+\alpha)\|\nabla u\|_{L^{2,\infty}(B_r)}$$

and, by a suitable choice of ε and α, (26) follows.

Step 3: Hölder continuity.

From the last result, by iteration, we deduce that there exists $\mu \in (0,1)$ such that $\|\nabla u\|_{L^{2,\infty}(B_r(z_0))} \leq Cr^{\mu}$ for every disc $B_{2r}(z_0) \subset D^2$ and, thanks to a theorem of C. Morrey (see [22] for example), this yields that $u \in C^{0,\alpha}$ for every $\alpha \in (0,\mu)$. Higher regularity can be derived by standard arguments.

□

As a first consequence of regularity, we will now prove a result which shows that, for solutions not supposed to be a priori conformal, however the defect of conformality can be "controlled".

THEOREM 6. *If $u \in H^2(\mathcal{O},\mathbb{R}^3)$ is a solution to* (19), *then its Hopf differential $\omega = (|u_x|^2 - |u_y|^2) - 2iu_x \cdot u_y$ satisfies (in the weak sense) $\frac{\partial \omega}{\partial \bar{z}} = 0$ in $\mathcal{O}$.*

Proof. Let $X \in C_c^{\infty}(\mathcal{O},\mathbb{R}^2)$ be a vector field on $\mathcal{O}$ and let $\varphi = X_1 u_x + X_2 u_y$. Since we have assumed that $u \in H^2$, we deduce that $\varphi \in H_0^1$ and therefore we may take φ as a test function for (19). Being $H(u)u_x \wedge u_y \cdot \varphi = 0$, one has

$$0 = \Delta u \cdot \varphi = X_1(u_{xx} \cdot u_x + u_{yy} \cdot u_x) + X_2(u_{xx} \cdot u_y + u_{yy} \cdot u_y)$$

which yields directly the result.

□

REMARK 5. Note that the argument would fail for H^1-solutions, but it holds still true for smooth solutions and, moreover, ω turns out to be holomorphic.

5.2. L^{∞}-bounds for the H-equation

The a priori bounds on solutions to the H-equation we are going to describe are basic in the context of the analytical approach to the following geometric problem. Let us consider a Jordan curve γ in $\mathbb{R}^3$ and a surface $M \subset \mathbb{R}^3$ of mean curvature H and such that $\partial M = \gamma$. The question is:

Is it possible to bound $\sup_{p \in M} |H(p)|$
by a function of $\|\gamma\|_{L^\infty}$ and the area of M?

Although a direct approach to this problem is probably possible, the analytical one (based on ideas of M. Grüter [23] and rephrased by F. Bethuel and O. Rey [11]) relies on the following estimates, which play a central role also in the variational setting of the H-problem.

THEOREM 7. *Let u be a smooth solution to problem (D_H). Assume u conformal and H bounded. Then*

$$\|u\|_{L^\infty} \leq C\left(\|g\|_{L^\infty} + \|H\|_{L^\infty}\int_{D^2}|\nabla u|^2 + \left(\int_{D^2}|\nabla u|^2\right)^{\frac{1}{2}}\right). \tag{31}$$

Proof. The proof is based on the introduction, for $z_0 \in D^2$ and $r > 0$ such that $\mathrm{dist}(u(z_0), \gamma) > r$, of the following sets and functions:

$$W(r) = u^{-1}(B_r(u(z_0))), \quad V(r) = \partial W(r)$$

$$\phi(r) = \int_{W(r)} |\nabla u|^2, \quad \psi(r) = \int_{V(r)} \left| \frac{\partial |u|}{\partial \nu} \right|$$

where ν is the outward normal to $V(r)$. Obviously, $B_r(u(z_0)) \cap \gamma = \emptyset$. We limit ourselves to describe briefly the steps which lead to the conclusion.

Step 1. Using the conformality condition, we have

$$\frac{d}{dr}\phi(r) \geq 2\psi(r). \tag{32}$$

In fact, assuming (without loss of generality) $u(z_0) = 0$ and noting that

$$|\nabla u|^2 = 2 \left| \frac{\partial u}{\partial \nu} \right|^2 \geq 2 \left| \frac{\partial |u|}{\partial \nu} \right|^2,$$

we obtain

$$\frac{d}{dr}\phi(r) \geq 2\frac{d}{dr} \int_{W(r)} \left| \frac{\partial |u|}{\partial \nu} \right|^2 = 2\frac{d}{dr} \int_{W(r)} |\nabla |u|\,|^2 = 2\psi(r),$$

where the last equality can be deduced from the coarea formula of Federer [21].

Step 2. Again by conformality, it is possible to prove that

$$\limsup_{r \to 0} \frac{\phi(r)}{r^2} \geq 2\pi \ , \quad \text{assuming } |\nabla u(z_0)| \neq 0 \ . \tag{33}$$

The idea is the following. As $r \to 0$, the image of u becomes locally flat, so that the area A_r of the image of u in $B_r(u(z_0))$ is close to πr^2. On the other hand, $\phi(r) = 2A_r$.

Step 3. Using the H-equation and (32), we have

$$2\phi(r) - r\frac{d}{dr}\phi(r) \leq 2H_0 r\phi(r). \tag{34}$$

In fact, integrating by parts, we obtain

$$\begin{aligned}
\phi(r) &= \int_{W(r)} |\nabla u|^2 = \int_{W(r)} -\Delta u \cdot u + \int_{V(r)} u \cdot \frac{\partial u}{\partial \nu} \\
&\leq H_0 \int_{W(r)} |u|\, |\nabla u|^2 + r \int_{V(r)} \left| \frac{\partial |u|}{\partial \nu} \right| \\
&\leq H_0 \int_{W(r)} |u|\, |\nabla u|^2 + r\psi(r) \\
&\leq H_0 r\phi(r) + \frac{1}{2} r\frac{d}{dr}\phi(r).
\end{aligned}$$

Step 4. Combining (32), (33) and (34), it is possible to prove that

$$\phi(r) \geq \frac{2\pi}{e} r^2 \tag{35}$$

for every $0 < r \leq \frac{1}{2H_0}$.

Step 5. Combining the estimate (35) with a covering argument, the proof of the theorem can be completed.

□

A relevant fact is that the conformality assumption of theorem 7 can be removed. More precisely, we have:

THEOREM 8. *Let u be a smooth solution to the problem (D_H). If H is smooth and bounded, then*

$$\|u\|_{L^\infty} \leq C\left(\|g\|_{L^\infty} + \|H\|_{L^\infty}\left(1 + \int_{D^2} |\nabla u|^2\right)\right). \tag{36}$$

Proof. Let us note that, if u were conformal, for the theorem 7 it would satisfy the inequality (31), which would directly yield (36). When u is not conformal, an adaptation of an argument of R. Shoen [38] allows a reduction to the conformal case. This procedure is based on the following construction. It is possible to determine a function $\psi\colon D^2 \to \mathbb{C}$ such that

$$\frac{\partial \psi}{\partial z} = -\frac{1}{4}\omega \quad \text{and} \quad \frac{\partial \psi}{\partial \bar{z}} = 0 \tag{37}$$

where $\omega = |u_x|^2 - |u_y|^2 - 2i u_x \cdot u_y$ is holomorphic (see remark 5). Then, defining

$$v = v_1 + i v_2 = \bar{z} + \psi + \alpha \tag{38}$$

where the constant $\alpha \in \mathbb{C}$ is to be chosen later, we have

$$\Delta v = 0 \tag{39}$$

and

$$-\frac{1}{4}\omega = \left\langle \frac{\partial v}{\partial z}, \frac{\partial v}{\partial \bar{z}} \right\rangle_{\mathbb{C}} = \frac{1}{4}\left(|v_x|^2 - |v_y|^2 - 2i\Re e\langle v_x, v_y\rangle_{\mathbb{C}}\right). \tag{40}$$

If we set

$$U = (u, v_1, v_2) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R},$$

then, by (37) and (40), we have $|U_x|^2 - |U_y|^2 - 2i U_x \cdot U_y = 0$ and, by (39) and the H-equation, we obtain $|\Delta U| \leq H_0 |\nabla U|^2$. Now, one may apply to U a generalized version of theorem 7, the proof being essentially the same. See [11] for thorough details.

□

Turning back to the geometric problem mentioned at the beginning of this subsection and as an application of the previous estimates, we quote the following result, again from [11].

THEOREM 9. *Let M be a compact surface in $\mathbb{R}^3$, diffeomorphic to $\mathbb{S}^2$ and of mean curvature H. Then*

$$\max_{p\in M}|H(p)| \geq C\frac{\operatorname{diam}(M)}{\operatorname{area}(M)}$$

where $\operatorname{diam}(M) = \max_{p,q\in M}|p-q|$.

5.3. Isoperimetric inequalities

We conclude this section recalling some central results of the work of Wente [48]. Considering the Plateau problem for H-surfaces in the case of constant H under a variational point of view, he observed that the volume functional

$$V(u) = \frac{1}{3}\int_{D^2} u \cdot u_x \wedge u_y,$$

whose existence needs u bounded, could instead be well defined by continuous extension for any $u \in H^1$ with bounded trace $u_{|_{\partial D^2}}$. To define this extension, he used the decomposition $u = h + \phi$ where $\phi \in H_0^1$ and h is the bounded harmonic part of u (i.e., the minimizer for Dirichlet integral on $u + H_0^1$). Then, the classical isoperimetric inequality can be applied to ϕ provided that it is regular enough and, since the area functional $A(\phi)$ does not exceed the Dirichlet integral $E_0(\phi) = \frac{1}{2}\int_{D^2}|\nabla u|^2$, one has that $|V(\phi)| \leq (1/\sqrt{36\pi})A(\phi)^{3/2} \leq (1/\sqrt{36\pi})E_0(\phi)^{3/2}$ (see Bononcini [12]). From the fact $V(\phi)$ is a cubic form in ϕ, Wente deduced that V can be continuously extended on H_0^1 with the same inequality:

THEOREM 10. *Let $u \in H_0^1(D^2, \mathbb{R}^3)$. Then*

$$\left|\int_{D^2} u \cdot u_x \wedge u_y\right| \leq \frac{1}{\sqrt{32\pi}}\left(\int_{D^2}|\nabla u|^2\right)^{3/2}.$$

Moving from this result and in order to achieve the extension to whole H^1, Wente also obtained that, for any $u \in H^1$ with bounded trace, the integral

$$\int_{D^2} \varphi \cdot u_x \wedge u_y$$

defines a continuous functional of $\varphi \in H_0^1$. This fact is of great importance in the variational setting of the H-problem, for constant H.

As far as the case of variable H is concerned, we just note that K. Steffen in [39] pointed out the intimate connection between isoperimetric inequalities and the Plateau problem with prescribed mean curvature. In particular, using the theory of integer currents, he proved the following version of isoperimetric inequality for the generalized volume functional

$$V_H(u) = \int_{D^2} Q_H(u) \cdot u_x \wedge u_y \ ,$$

where $Q_H : \mathbb{R}^3 \to \mathbb{R}^3$ is such that div $Q_H = H$.

THEOREM 11. *If $H \in L^\infty(\mathbb{R}^3)$, then there exists a constant C_H (depending only on $\|H\|_\infty$) such that*

$$|V_H(u)| \le C_H \left(\int_{D^2} |\nabla u|^2 \right)^{3/2} \quad \textit{for every } u \in H_0^1 \cap L^\infty \ .$$

Moreover the functional V_H admits a unique continuous extension on H_0^1, and it satisfies the above inequality for every $u \in H_0^1$.

6. The large solution to the H-problem (Rellich's conjecture)

As we noticed in section 4, remark 4, if $H_0 > 0$ and γ is a perfect circle lying on a sphere of radius $R_0 = 1/H_0$, the solution given by the Hildebrandt's theorem 2 corresponds to the smaller part of the sphere spanning γ, the small solution. However also the larger part of the sphere is a solution to the same Plateau problem, the so-called large solution.

This example has lead to conjecture that in case of constant mean curvature $H_0 \neq 0$, if γ is Jordan curve such that $\|\gamma\|_\infty |H_0| < 1$, then there exists a pair of parametric surfaces spanning γ (*Rellich's conjecture*).

In 1984 H. Brezis and J.-M. Coron [13] proved this conjecture. Independently, also M. Struwe [42] obtained essentially the same result.

Technically, the main difficulty in showing the Rellich's conjecture is to prove that the Dirichlet problem

$$(D_{H_0}) \qquad \begin{cases} \Delta u = 2H_0 u_x \wedge u_y & \text{in } D^2 \\ u = g & \text{on } \partial D^2 \end{cases}$$

admits two different solutions. Here $g : \mathbb{S}^1 \to \gamma$ is a regular, monotone parametrization of γ. In this section we will discuss the following multiplicity result, proved by Brezis and Coron in [13].

THEOREM 12. *Let $g \in H^{1/2} \cap C^0(\partial D^2, \mathbb{R}^3)$ and let $H_0 \neq 0$ be such that*

$$\|g\|_{L^\infty} |H_0| < 1.$$

If g is nonconstant, then the problem (D_{H_0}) admits at least two solutions.

The existence of a first solution $\underline{u}$ (the small solution) is assured by theorem 2. Brezis and Coron proved the existence of a second solution $\overline{u} \neq \underline{u}$. As a consequence, even the corresponding Plateau problem has a second solution; we will not discuss this matter, we just limit ourselves to say that the proof can be deduced from the Dirichlet problem, using the usual tools (e.g. the three points condition) discussed in section 3.

We prefer to focus the discussion on the proof of a second solution to (D_{H_0}), in which the main difficulty is the behavior of the Palais-Smale sequences of the functional involved in its variational formulation. It is a typical example of a variational problem with *lack of compactness*, the overcoming of which moved on from the breakthrough analysis of Sacks and Uhlenbeck [37], and Aubin [5]. Let us notice that this kind of matters appears in many conformally invariant problems, such as harmonic maps (in dimension 2), Yamabe problem and prescribed scalar curvature problem, elliptic problems with critical exponent, Yang-Mills equations.

In the next subsections 6.1, 6.2 and 6.4 we will give an outline of the proof of theorem 12. We always assume all the hypotheses given in the statement of the theorem. Moreover, we will denote by $\underline{u}$ the small solution to (D_{H_0}) given by theorem 2.

6.1. The mountain-pass structure

Let us recall that the problem (D_{H_0}) has a variational structure (see the proof of theorem 2), i.e. its (weak) solutions are critical points of the functional

$$E_{H_0}(u) = \frac{1}{2}\int_{D^2} |\nabla u|^2 + \frac{2H_0}{3}\int_{D^2} u \cdot u_x \wedge u_y \tag{41}$$

on

$$H^1_g = \{u \in H^1(D^2, \mathbb{R}^3) : u\big|_{\partial D^2} = g\}.$$

Now, we are going to point out that the functional E_{H_0} has, essentially, a mountain pass geometry. Let us first recall the classical mountain pass lemma, stated by A. Ambrosetti and P. Rabinowitz in 1973 [4].

THEOREM 13 (MOUNTAIN PASS LEMMA). *Let X be a real Banach space and let $F : X \to \mathbb{R}$ be a functional of class C^1. Assume that*

(**mp₁**) *there exists $\rho > 0$ such that $\inf_{\|x\|=\rho} F(x) > F(0)$,*

(**mp₂**) *there exists $x_1 \in X$ such that $\|x_1\| > \rho$ and $F(x_1) \le F(0)$.*

Then, setting $\mathcal{P} = \{p \in C^0([0,1], X) : p(0) = 0,\ p(1) = x_1\}$, the value

$$c = \inf_{p\in\mathcal{P}} \max_{s\in[0,1]} F(p(s)) \tag{42}$$

is a generalized critical value, i.e., there exists a sequence (x_n) in X such that $F(x_n) \to c$ and $dF(x_n) \to 0$ in X'.

REMARK 6. 1. In the situation of the theorem 13, since $\|x_1\| > \rho$, by the hypothesis (**mp₁**), it is clearly $\max_{s\in[0,1]} F(p(s)) \ge \alpha$ for all $p \in \mathcal{P}$, being $\alpha = \inf_{\|x\|=\rho} F(x)$. Hence, $c \ge \alpha > F(0)$.

2. A sequence $(x_n) \subset X$ satisfying $F(x_n) \to c$ and $dF(x_n) \to 0$ in X' is known as a Palais-Smale sequence for the functional F at level c.
3. Recall that a functional $F \in C^1(X, \mathbb{R})$ is said to satisfy the Palais-Smale condition if any Palais-Smale sequence for F is relatively compact, i.e., it admits a strongly convergent subsequence. Hence, if in the above theorem, the functional F satisfies the Palais-Smale condition (at level c) then it admits a critical point at level c, i.e, c is a critical value.

Coming back to our functional E_{H_0}, the possibility to apply the mountain-pass lemma is granted by the following properties.

LEMMA 6. *The functional E_{H_0} is of class C^2 on H^1_g and for all $u \in H^1_g$ one has*

$$dE_{H_0}(u) = -\Delta u + 2H_0 u_x \wedge u_y. \tag{43}$$

Here the fact that $u_x \wedge u_y \in H^{-1}$, which is implied by Wente's result given in theorem 10, is of fundamental importance, since it clearly yields $dE_{H_0}(u) \in H^{-1}$ for any $u \in H^1_g$ and hence that E_{H_0} is differentiable. We also remark that for variable H it is no longer clear and rather presumably false that $H(u)u_x \wedge u_y \in H^{-1}$ for every $u \in H^1_g$.

LEMMA 7. *The second derivative of E_{H_0} at $\underline{u}$ is coercive, i.e., there exists $\delta > 0$ such that*

$$d^2 E_{H_0}(\underline{u})(\varphi, \varphi) = \int_{D^2} \left(|\nabla\varphi|^2 + 4H_0 \underline{u} \cdot \varphi_x \wedge \varphi_y \right) \geq \delta \int_{D^2} |\nabla\varphi|^2$$

for all $\varphi \in H^1_0(D^2, \mathbb{R}^3)$.

A proof of this lemma is given in [13].

Finally, since the volume term $V_{H_0}(u) = \frac{2H_0}{3} \int_{D^2} u \cdot u_x \wedge u_y$ is cubic, whereas the Dirichlet integral is quadratic, the next result immediately follows.

LEMMA 8. $\inf_{u \in H^1_g} E_{H_0}(u) = -\infty$.

Proof. Let $v \in H^1_0$ be such that $V_{H_0}(v) \neq 0$. Taking $-v$ instead of v, if necessary, we may assume $V_{H_0}(v) < 0$. The thesis follows by noting that

$$E_{H_0}(tv + \underline{u}) = 2t^3 V_{H_0}(v) + O(t^2)$$

as $t \to +\infty$.

$\square$

Now we apply the mountain pass lemma to the functional $F \colon H^1_0 \to \mathbb{R}$ defined by

$$F(v) = E_{H_0}(v + \underline{u}) - E_{H_0}(\underline{u}) \ . \tag{44}$$

The regularity of F is assured by lemma 6, since $\underline{u} \in H^1_g = \underline{u} + H^1_0$ and

(45) $$dF(v) = dE_{H_0}(v + \underline{u}) \; .$$

The condition ($\mathbf{mp_1}$) is granted by lemma 7. The condition ($\mathbf{mp_2}$) follows immediately from lemma 8. Hence, by theorem 13, the functional F admits a Palais-Smale sequence $(v_n) \subset H^1_0$ at a level $c > 0$. By (44) and (45), setting $u_n = v_n + \underline{u}$, we obtain a Palais-Smale sequence in H^1_g for the functional E_{H_0} at level $c + E_{H_0}(\underline{u})$.

Owing to the conformal invariance of the problem, the functional E_{H_0} is not expected to verify the Palais-Smale condition, and a deeper analysis of the Palais-Smale sequences for E_{H_0} is needed.

6.2. Palais-Smale sequences for E_{H_0}

Recalling remark 6, by (41) and (43), a Palais-Smale sequence for the functional E_{H_0} is a sequence $(u^n) \subset H^1_g$ such that

(46) $$E_{H_0}(u_n) \to \bar{c}$$

(47) $$\Delta u^n = 2H_0 u^n_x \wedge u^n_y + f_n \text{ in } D^2, \text{ with } f_n \to 0 \text{ in } H^{-1}$$

for some $\bar{c} \in \mathbb{R}$.

As a first fact, we have the following result.

LEMMA 9. *Any Palais-Smale sequence $(u_n) \subset H^1_g$ for E_{H_0} is bounded in H^1.*

Proof. Since $(u_n) \subset H^1_g$ it is enough to prove that $\sup \|\nabla u_n\|_2 < +\infty$. Setting $\varphi_n = u_n - \underline{u}$, and keeping into account that $dE_{H_0}(\underline{u}) = 0$, one has

$$\begin{aligned} E_{H_0}(u_n) &= E_{H_0}(\underline{u}) + \frac{1}{2} d^2 E_{H_0}(\underline{u})(\varphi_n, \varphi_n) + 2V_{H_0}(\varphi_n) \\ dE_{H_0}(u_n)\varphi_n &= d^2 E_{H_0}(\underline{u})(\varphi_n, \varphi_n) + 6V_{H_0}(\varphi_n). \end{aligned}$$

Hence, subtracting, one obtains

$$3E_{H_0}(u_n) = E_{H_0}(\underline{u}) + \frac{1}{2} d^2 E_{H_0}(\underline{u})(\varphi_n, \varphi_n) + dE_{H_0}(u_n)\varphi_n.$$

Using Lemma 7, one gets

$$\begin{aligned} \delta \|\nabla \varphi_n\|_2^2 &\le d^2 E_{H_0}(\underline{u})(\varphi_n, \varphi_n) \\ &= 6(E_{H_0}(u_n) - E_{H_0}(\underline{u})) - 2dE_{H_0}(u_n)\varphi_n \\ &\le C + \|dE_{H_0}(u_n)\| \, \|\nabla \varphi_n\|_2. \end{aligned}$$

By (46) and (47) one infers that (φ_n) is bounded in H^1_0 and then the thesis follows. □

In the case of variable H, it is not clear whether the lemma holds or not. A method to overcome this kind of difficulty can be found in Struwe [42].

From the previous lemma we can deduce that all Palais-Smale sequences for E_{H_0} are relatively weakly compact. The next result states that the weak limit is a solution to (D_{H_0}).

LEMMA 10. *Let $(u_n) \subset H^1_g$ be a Palais-Smale sequence for E_{H_0} converging weakly in H^1 to some $\bar{u} \in H^1_g$. Then $dE_{H_0}(\bar{u}) = 0$, i.e., $\bar{u}$ is a (weak) solution to (D_{H_0}).*

Proof. Fix an arbitrary $\varphi \in C^\infty_c(D^2, \mathbb{R}^3)$. By (47), one has

$$\int_{D^2} \nabla u_n \cdot \nabla\varphi + 2H_0 L(u_n, \varphi) \to 0$$

where we set

$$L(u, \varphi) = \int_{D^2} \varphi \cdot u_x \wedge u_y .$$

By weak convergence $\int_{D^2} \nabla u_n \cdot \nabla\varphi \to \int_{D^2} \nabla\bar{u} \cdot \nabla\varphi$. Moreover, using the divergence expression $2u_x \wedge u_y = (u \wedge u_y)_x + (u_x \wedge u)_y$, one has that

$$2L(u, \varphi) = -\int_{D^2} (\varphi_x \cdot u \wedge u_y + \varphi_y \cdot u_x \wedge u) .$$

Hence $L(u_n, \varphi) \to L(\bar{u}, \varphi)$, since $u_n \to \bar{u}$ strongly in L^2 and weakly in H^1. In conclusion, one gets

$$\int_{D^2} \nabla\bar{u} \cdot \nabla\varphi + 2H_0 \int_{D^2} \varphi \cdot \bar{u}_x \wedge \bar{u}_y = 0$$

that is the thesis.

□

However, the Palais-Smale sequences for E_{H_0} are not necessarily relatively strongly compact in H^1. In the spirit of Aubin [5] and Sacks-Uhlenbeck [37], and inspired by the concentration-compactness principle by P.-L. Lions [35], Brezis and Coron in [14] have precisely analyzed the possible defect of strong convergence, as the following theorem states.

THEOREM 14. *Suppose that $(u_n) \in H^1_g$ is a Palais-Smale sequence for E_{H_0}. Then there exist*

(i) *$\overline{u} \in H^1_g$ solving $\Delta\overline{u} = 2H_0\overline{u}_x \wedge \overline{u}_y$ in D^2,*

(ii) *a finite number $p \in \mathbb{N} \cup \{0\}$ of nonconstant solutions $v^1, \dots, v^p$ to $\Delta u = 2H_0 u_x \wedge u_y$ on $\mathbb{R}^2$,*

(iii) *p sequences $(a^1_n), \dots, (a^p_n)$ in D^2*

(iv) *p sequences* $(\varepsilon_n^1), \ldots, (\varepsilon_n^p)$ *in* $\mathbb{R}_+$ *with* $\lim_{n\to+\infty} \varepsilon_i^n = 0$ *for any* $i = 1, \ldots, p$

such that, up to a subsequence, we have

$$\left\| u_n - \overline{u} - \sum_{i=1}^{p} v^i \left(\frac{\cdot - a_n^i}{\varepsilon_n^i} \right) \right\|_{H^1} \to 0$$

$$\int_{D^2} |\nabla u_n|^2 = \int_{D^2} |\nabla \overline{u}|^2 + \sum_{i=1}^{p} \int_{\mathbb{R}^2} |\nabla v^i|^2 + o(1)$$

$$E_{H_0}(u_n) = E_{H_0}(\overline{u}) + \sum_{i=1}^{p} \bar{E}_{H_0}(v^i) + o(1)\ ,$$

where in general $\bar{E}_{H_0}(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 + \frac{2H_0}{3}\int_{\mathbb{R}^2} v \cdot v_x \wedge v_y$. *In case* $p = 0$ *any sum* $\sum_{i=1}^{p}$ *is zero and* $u_n \to \bar{u}$ *strongly in* H^1.

REMARK 7. The conformal invariance is reflected in the concentrated maps $v^i\left(\frac{\cdot - a_n^i}{\varepsilon_n^i}\right)$. This theorem also emphasizes the role of solutions of the H_0-equation on whole $\mathbb{R}^2$, which are completely known (see below).

6.3. Characterization of solutions on $\mathbb{R}^2$

The solutions to the H_0-equation on the whole plane $\mathbb{R}^2$ are completely classified in the next theorem. It basically asserts that all solutions of the problem

$$\begin{cases} \Delta u = 2H_0 u_x \wedge u_y \quad \text{on } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |\nabla u|^2 < +\infty \end{cases} \tag{48}$$

are conformal parametrizations of the sphere of radius $R_0 = 1/|H_0|$.

Note first that, if u is a solution to (48), defining $\omega = |u_x|^2 - |u_y|^2 - 2i u_x \cdot u_y$ the usual defect of conformality for u, it holds that $\frac{\partial \omega}{\partial \bar{z}} = 0$ (by the equation), and $\int_{\mathbb{R}^2} |\omega| < +\infty$ (by the summability condition on ∇u). Hence $\omega \equiv 0$, that is, u is conformal.

Pushing a little further the analysis, Brezis and Coron obtained the following result (see [14]).

THEOREM 15. *Let* $u \in L^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$ *be a solution to* (48) *with* $H_0 \neq 0$. *Then* u *has the form*

$$u(z) = \frac{1}{H_0} \Pi \left(\frac{P(z)}{Q(z)} \right) + C,$$

where C *is a constant vector in* $\mathbb{R}^3$, P *and* Q *are (irreducible) polynomials (in the complex variable* $z = (x, y) = x + iy$*) and* $\Pi \colon \mathbb{C} \to \mathbb{S}^2$ *is the stereographic projection.*

Moreover

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u|^2 &= \frac{8\pi k}{H_0^2},\\ \bar{E}_{H_0}(u) &= \frac{4\pi k}{3H_0^2},\end{aligned}$$

where $k = \max\{\deg P, \deg Q\}$ *is the number of coverings of the sphere* $\mathbb{S}^2$ *by the parametrization* u.

We point out that problem (48) is invariant with respect to the conformal group. For instance, if u is a solution to (48), then $u_\lambda(z) = u(\lambda z)$ is also a solution. Note that $u_\lambda \to const$ as $\lambda \to +\infty$, or as $\lambda \to 0$.

6.4. Existence of the large solution

In this subsection, taking advantage from the results stated in the previous subsections, we will sketch the conclusion of the proof of theorem 12.

Let us recall that the functional F defined by (44) admits a mountain pass level $c > 0$. In view of the result on the Palais-Smale sequences stated in Theorem 14, it is useful also an upper bound for c, and precisely:

LEMMA 11. $c < \frac{4\pi}{3H_0^2}$.

This estimate is obtained by evaluating the functional E_{H_0} along an explicit mountain pass path which, roughly speaking, is constructed by attaching in a suitable way a sphere to the small solution.

Let now $(u_n) \subset H^1_g$ be the Palais-Smale sequence for E_{H_0} introduced at the end of the subsection 6.1. We have already seen that, up to a subsequence, (u_n) converges weakly to a solution $\bar{u}$ to (D_{H_0}). If $u_n \to \bar{u}$ strongly in H^1 then

$$E_{H_0}(\overline{u}) = E_{H_0}(\underline{u}) + c > E_{H_0}(\underline{u}) \tag{49}$$

because $c > 0$.

On the contrary, if no subsequence of (u_n) converges strongly in H^1, then we use theorem 14 on the characterization of Palais smale sequences. In particular, with the same notation of theorem 14, we have $p \geq 1$ and, denoting by $\mathcal{S}$ the set of all nonconstant solutions to (48),

$$
\begin{aligned}
E_{H_0}(\overline{u}) &= E_{H_0}(\underline{u}) + c - \sum_{i=1}^{p} \bar{E}_{H_0}(v_i) \\
&\leq E_{H_0}(\underline{u}) + c - p \inf_{v \in \mathcal{S}} \bar{E}_{H_0}(v) \\
&\leq E_{H_0}(\underline{u}) + c - \inf_{\omega \in \mathcal{S}} \bar{E}_{H_0}(\omega) \\
&\leq E_{H_0}(\underline{u}) + c - \frac{4\pi}{3H_0^2} \\
&< E_{H_0}(\underline{u}) \qquad (50)
\end{aligned}
$$

according to (46), theorem 15 and lemma 11.

Thus, either from (49) or from (50), it follows that $\overline{u} \neq \underline{u}$ and the conclusion of theorem 12 is achieved.

6.5. The second solution for variable H

In the previous sections, we have seen how Brezis and Coron proved the existence of a second solution (different from the small one) to the problem (D_H), for constant H. Unfortunately, in the attempt of extending their proof to the case of variable H, lot of the main arguments fail. In view to overcome such obstacle, Struwe introduced in [44] a perturbed functional, which brings some compactness into the problem, and he succeeded to prove existence of a large solution for a class of curvature functions H, which is a dense subset in a small neighborhood of a nonzero constant, for some strong norm involving, in particular, a weighted C^1 norm. His results were then slightly improved by Wang in [46].

Here we present a result by Bethuel and Rey [11] (see also [10]), more general than the above mentioned results by Struwe and Wang, which extends theorem 12 for variable H, in a perturbative setting. A similar result is contained in [33] (see also [34]).

THEOREM 16. *Let $g \in H^{1/2} \cap C^0(\partial D^2, \mathbb{R}^3)$ be nonconstant and let $H_0 \neq 0$ be such that $\|g\|_{L^\infty}|H_0| < 1$. Then there exists $\alpha > 0$ such that for any $H \in C^1(\mathbb{R}^3)$ satisfying*

$$\|H - H_0\|_{L^\infty} < \alpha$$

the problem (D_H) admits at least two solutions.

The proof is developed by a direct variational approach (see [11]). Fundamental tools in the proof are: a careful analysis of the Palais-Smale sequences (which is more delicate than in the case of constant H); the *a priori* bound on solutions given in theorem 7, which permits the truncation on H outside a suitable ball. Indeed, replacing the original H by a function $\widetilde{H}$ such that $\widetilde{H}(u) = H(u)$ as $|u| \leq R$, $\widetilde{H}(u) = H_0$ as $|u| \geq 2R$, and solving the problem with $\widetilde{H}$, the *a priori* bound yields that the solution found to the truncated problem is also a solution to the original problem.

7. H-bubbles

In this section we deal with $\mathbb{S}^2$-type parametric surfaces in $\mathbb{R}^3$ with prescribed mean curvature H, briefly *H-bubbles*. On this subject, which might have some applications to physical problems (e.g., capillarity phenomena, see [24]), we discuss here some very recent results obtained in a series of papers by P. Caldiroli and R. Musina (see [15]–[18]).

Let us make some preliminary remarks, useful in the sequel. First, we observe that the "H-bubble problem":

> *Given a (smooth) function $H\colon \mathbb{R}^3 \to \mathbb{R}$, find an $\mathbb{S}^2$-type surface M such that the mean curvature of M at p equals $H(p)$, for all $p \in M$,*

after the identification of $\mathbb{S}^2$ with the compactified plane $\mathbb{R}^2 \cup \{\infty\}$, via stereographic projection, and using conformal coordinates, admits the following analytical formulation:

> *Find a nonconstant, conformal function $u\colon \mathbb{R}^2 \to \mathbb{R}^3$, smooth as a map on $\mathbb{S}^2$, satisfying*

$$(B)_H \qquad \begin{cases} \Delta u = 2H(u)u_x \wedge u_y & \text{on } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |\nabla u|^2 < +\infty. \end{cases}$$

In principle, the two formulations of the H-bubble problem are not exactly equivalent, since in the analytical version one cannot exclude a priori the presence of branch points (i.e., self-intersection points, or points $p = u(z)$ where $\nabla u(z) = 0$). We do not enter in this aspect of geometric regularity and, from now on, we just study the analytical version $(B)_H$ of the H-bubble problem.

Observe that if $H \equiv 0$, clearly the only solutions of $(B)_H$ are the constants. Moreover, as we saw in the previous section, when the prescribed mean curvature is a nonzero constant $H(u) \equiv H_0$, Brezis and Coron in [14] completely characterized the set of solutions of (B_H) (see Theorem 15).

REMARK 8. 1. We point out that it is enough to look for weak solutions of $(B)_H$. Indeed, by regularity theory for H-systems (see Section 5), if H is smooth, then also any solution of $(B)_H$ is so. In particular, if $H \in C^1$, then any solution of $(B)_H$ turns out to be of class $C^{3,\alpha}$.
2. If u solves $(B)_H$, then u is conformal for free. Indeed, by Theorem 6, its Hopf differential is constant on $\mathbb{R}^2$, and actually, by the summability condition $\int_{\mathbb{R}^2} |\nabla u|^2 < +\infty$, it is zero, namely u is conformal. The deep reason of this rests on the fact that problem $(B)_H$ contains no boundary condition and it is invariant under the action of the conformal group of $\mathbb{S}^2 \approx \mathbb{R}^2 \cup \{\infty\}$. This invariance means that in fact we deal with a problem on the image of the unknown u, rather than on the mapping u itself.

Problem $(B)_H$ can be tackled by using variational methods. In particular, one can detect solutions of (B_H) as critical points of the energy functional

$$E_H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + 2\int_{\mathbb{R}^2} Q_H(u)\cdot u_x \wedge u_y \ ,$$

where $Q_H \colon \mathbb{R}^3 \to \mathbb{R}^3$ is any vector field such that div $Q_H = H$. We can write $E_H(u) = E_0(u) + 2V_H(u)$, where $E_0(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2$ is the Dirichlet integral, and

$$V_H(u) = \int_{\mathbb{R}^2} Q_H(u)\cdot u_x \wedge u_y$$

is the so-called H-volume functional.

REMARK 9. This name for the functional V_H is motivated by the fact that if u is a regular parametrization of some $\mathbb{S}^2$-type surface M, then $V_H(u)$ equals the H-weighted algebraic volume of the bounded region enclosed by M. As a remarkable example, consider the mapping $\omega \colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\omega(z) = \begin{pmatrix} \mu x \\ \mu y \\ 1-\mu \end{pmatrix} \ , \quad \mu = \mu(z) = \frac{2}{1+|z|^2} \ , \tag{51}$$

where, as usual, $z = (x, y) \in \mathbb{R}^2$. Notice that ω is a (1-degree) conformal parametrization of the unit sphere $\mathbb{S}^2$ centered at the origin. Indeed ω solves $(B)_H$ with $H \equiv 1$. One has that $E_0(\omega) = 4\pi$ = area of the unit sphere $\mathbb{S}^2$, and, by the Gauss-Green theorem,

$$V_H(\omega) = -\int_{B_1} H(q)\, dq \ , \tag{52}$$

where B_1 denotes the unit ball in $\mathbb{R}^3$. Notice also that for every $n \in \mathbb{Z}\setminus\{0\}$ the mapping $\omega^n(z) = \omega(z^n)$ (in complex notation) is a n-degree parametrization of $\mathbb{S}^2$ and $V_H(\omega^n) = nV_H(\omega)$.

Keeping into account of the shape of the functional E_H, the natural functional space to be considered as a domain of E_H seems to be the Sobolev space

$$H^1 := \{v \circ \omega \mid v \in H^1(\mathbb{S}^2, \mathbb{R}^3)\}$$

where $\omega \colon \mathbb{R}^2 \to \mathbb{S}^2$, defined in (51), is the inverse of the stereographic projection. Clearly, H^1 is a Hilbert space, endowed with the norm

$$\|u\|^2_{H^1} = \int_{\mathbb{R}^2} (|\nabla u|^2 + \mu^2|u|^2) \ ,$$

it is isomorphic to $H^1(\mathbb{S}^2, \mathbb{R}^3)$, and it can also be defined as the completion of $C^\infty_c(\mathbb{R}^2, \mathbb{R}^3)$ with respect to the Dirichlet norm.

REMARK 10. 1. Since in general Q_H is not bounded (e.g., if $H \equiv 1$, then $Q_H(u) = \frac{1}{3}u$), the H-volume functional V_H as well as the energy E_H turn out to be well defined only for $u \in H^1 \cap L^\infty$. But we can take advantage from the generalized isoperimetric inequality, due to Steffen [39] and stated in Theorem 11 for functions in $H_0^1(D^2, \mathbb{R}^3)$. In fact, using the conformal invariance, the same inequality holds true also for functions in H^1 and, in this more general version, it guarantees that V_H and E_H can be extended on the whole space H^1 in a continuous way.
2. The functionals V_H and E_H are of class C^1 on H^1 only in some special cases, like, for instance, when H is constant far out. For an arbitrary function H (smooth and bounded), we can just consider the derivatives along directions in a (dense) subspace of H^1: for every $u \in H^1$ and for every $\varphi \in H^1 \cap L^\infty$ there exists

$$\partial_\varphi E_H(u) = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} H(u)\varphi \cdot u_x \wedge u_y \ . \tag{53}$$

In particular, from (53) one can recognize that if $u \in H^1$ is a critical point of E_H, namely $\partial_\varphi E_H(u) = 0$ for all $\varphi \in H^1 \cap L^\infty$, then u is a weak solution of $(B)_H$. In addition, by (53) one can see that the H-volume functional does not depend on the choice of the vector field Q_H.

REMARK 11. The functional E_H inherits all the invariances of problem $(B)_H$, and in particular $E_H(u \circ g) = E_H(u)$ for every conformal diffeomorphism of $\mathbb{S}^2 \approx \mathbb{R}^2 \cup \{\infty\}$. Since the conformal group of $\mathbb{S}^2$ is noncompact, this reflects into a lack of compactness in the variational problem associated to $(B)_H$, similarly to what we saw for the Plateau problem.

For several reasons, it is often meaningful to investigate the existence of H-bubbles having further properties concerning their energy or their location. Here is a list of some problems that will be discussed in the next subsections.

(i) Calling $\mathcal{B}_H$ the set of H-bubbles and assuming that $\mathcal{B}_H$ is nonempty (as it happens, for instance if H is constant, with a nonzero value, far away), is it true that $\inf_{u \in \mathcal{B}_H} E_H(u) > -\infty$?

(ii) Assuming $\mathcal{B}_H$ nonempty and $\mu_H := \inf_{u \in \mathcal{B}_H} E_H(u) > -\infty$, is μ_H attained in $\mathcal{B}_H$?

(iii) Find conditions on H ensuring the existence of an H-bubble u, possibly with minimal energy, that is, with $E_H(u) = \mu_H$.

(iv) Study the H-bubble problem in some perturbative setting, like for instance, $H(u) = H_0 + \varepsilon H_1(u)$, with $H_0 \in \mathbb{R} \setminus \{0\}$, H_1 smooth real function on $\mathbb{R}^3$, and $|\varepsilon|$ small.

7.1. On the minimal energy level for H-bubbles

Here we take $H \in C^1(\mathbb{R}^3) \cap L^\infty$ and, denoting by $\mathcal{B}_H$ the set of H-bubbles and assuming $\mathcal{B}_H \neq \emptyset$, we set

$$\mu_H = \inf_{u \in \mathcal{B}_H} E_H(u) \,. \tag{54}$$

In this subsection we will make some considerations about the minimal energy level μ_H and about the corresponding minimization problem (54). The results presented here are contained in [16].

To begin, we notice that if H is constant and nonzero, i.e., $H(u) \equiv H_0 \in \mathbb{R} \setminus \{0\}$, then by Theorem 15, $\omega^0 := \frac{1}{H_0}\omega$ belongs to $\mathcal{B}_{H_0}$ and $E_{H_0}(\omega^0) = \frac{4\pi}{3H_0^2} = \mu_{H_0}$.

REMARK 12. In case of a variable H, it is easy to see that in general it can be $\mathcal{B}_H \neq \emptyset$ and $\mu_H = -\infty$. Indeed, if there exists $u \in \mathcal{B}_H$ with $E_H(u) < 0$ then, setting $u^n(z) = u(z^n)$, for any $n \in \mathbb{N}$ the function u^n solves $(B)_H$, namely $u^n \in \mathcal{B}_H$, and $E_H(u^n) = nE_H(u)$. Consequently $\mu_H = -\infty$. One can easily construct examples of functions $H \in C^1(\mathbb{R}^3) \cap L^\infty$ for which there exist H-bubbles with negative energy. For instance, suppose that $H(u) = 1$ as $|u| = 1$, so that the mapping ω defined in (51) is an H-bubble. By (52), $E_H(\omega) = 4\pi - \int_{B_1} H(q)\, dq$. Hence, for a suitable definition of H in the unit ball B_1, one gets $E_H(\omega) < 0$.

The previous remark shows that in order that μ_H is finite, no H-bubbles with negative energy must exist. In particular, one needs some condition which prevents H to have too large variations. To this extent, in the definition of the vector field Q_H such that div $Q_H = H$, it seems convenient to choose

$$Q_H(u) = m_H(u)u \,, \quad m_H(u) = \int_0^1 H(su)s^2\, ds \,.$$

Taking any H-bubble u, since $\partial_u E_H(u) = 0$, and using the identity $3m_H(u) + \nabla m_H(u) \cdot u = H(u)$, one has

$$\begin{aligned} E_H(u) &= E_H(u) - \frac{1}{3}\partial_u E_H(u) \\ &= \frac{1}{6}\int_{\mathbb{R}^2} |\nabla u|^2 - \frac{2}{3}\int_{\mathbb{R}^2} \nabla m_H(u) \cdot u\, u \cdot u_x \wedge u_y \\ &\geq \left(\frac{1}{6} - \frac{\bar{M}_H}{3}\right)\int_{\mathbb{R}^2} |\nabla u|^2 \end{aligned} \tag{55}$$

where

$$\bar{M}_H := \sup_{u \in \mathbb{R}^3} |\nabla m_H(u) \cdot u\, u| \,.$$

Hence, if $\bar{M}_H \leq \frac{1}{2}$, then $\mu_H \geq 0$.

Now, let us focus on the simplest case in which H is assumed to be constant far out. This hypothesis immediately implies that $\mathcal{B}_H$ is nonempty and the minimization problem defined by (54) reduces to investigate the semicontinuity of the energy functional E_H along a sequence of H-bubbles. As shown by Wente in [47], in general E_H is not globally semicontinuous with respect to weak convergence, even if H is constant. However, as we will see in the next result, under the condition $\bar{M}_H < \frac{1}{2}$, semicontinuity holds true at least along a sequence of solutions of $(B)_H$.

THEOREM 17. *Let $H \in C^1(\mathbb{R}^3)$ satisfy*

(h₁) *$H(u) = H_\infty \in \mathbb{R} \setminus \{0\}$ as $|u| \geq R$, for some $R > 0$,*

(h₂) *$\bar{M}_H < \frac{1}{2}$.*

Then there exists $\omega \in \mathcal{B}_H$ such that $E_H(\omega) = \mu_H$. Moreover $\mu_H \leq \frac{4\pi}{3H_\infty^2}$.

Proof. First, we observe that by $(\mathbf{h_1})$, $\mathcal{B}_H \neq \emptyset$, since the spheres of radius $|H_\infty|^{-1}$ placed in the region $|u| \geq R$ are H-bubbles. In particular, this implies that $\mu_H \leq \frac{4\pi}{3H_\infty^2}$. Now, take a sequence $(u^n) \subset \mathcal{B}_H$ with $E_H(u^n) \to \mu_H$. Since the problem $(B)_H$ is invariant with respect to the conformal group, we may assume that $\|\nabla u^n\|_\infty = |\nabla u^n(0)| = 1$ (normalization conditions).

Step 1 (Uniform global estimates): we may assume

$$\sup \|\nabla u^n\|_2 < +\infty \text{ and } \sup \|u^n\|_\infty < +\infty \,.$$

The first bound follows by (55), by $(\mathbf{h_2})$, and by the fact that (u^n) is a minimizing sequence for the energy in $\mathcal{B}_H$. As regards the second estimate, first we observe that using Theorem 7 one can prove that

$$\sup_n \operatorname{diam} u^n =: \rho < +\infty \,,$$

where, in general, $\operatorname{diam} u = \sup_{z,z' \in \mathbb{R}^2} |u(z) - u(z')|$. If $\|u^n\|_\infty \leq R + \rho$, set $\tilde{u}^n = u^n$. If $\|u^n\|_\infty > R + \rho$, then by the assumption $(\mathbf{h_1})$, u^n solves $\Delta u = 2H_\infty u_x \wedge u_y$. Let $p_n \in \operatorname{range} u^n$ be such that $|p_n| = \|u^n\|_\infty$. Set $q_n = \left(1 - \frac{R+\rho}{|p_n|}\right) p_n$ and $\tilde{u}^n = u^n - q_n$. Then $\|\tilde{u}^n\|_\infty \leq R + \rho$, and $|\tilde{u}^n(z)| \geq R$ for every $z \in \mathbb{R}^2$. Hence, also $\tilde{u}^n \in \mathcal{B}_H$, and $E_H(\tilde{u}^n) = E_{H_\infty}(\tilde{u}^n) = E_H(u^n)$. Therefore $(\tilde{u}^n)$ is a minimizing sequence of H-bubbles satisfying the required uniform estimates.

Step 2 (Local "ε-regularity" estimates): there exist $\varepsilon > 0$ and, for every $s \in (1, +\infty)$ a constant $C_s > 0$ (depending only on $\|H\|_\infty$), such that if u is a weak solution of $(B)_H$, then

$$\|\nabla u\|_{L^2(D_R(z))} \leq \varepsilon \;\Rightarrow\; \|\nabla u\|_{H^{1,s}(D_{R/2}(z))} \leq C_s \|\nabla u\|_{L^2(D_R(z))}$$

for every $R \in (0, 1]$ and for every $z \in \mathbb{R}^2$.

These ε-regularity estimates are an adaptation of a similar result obtained by Sacks and Uhlenbeck in their celebrated paper [37]. We omit the quite technical proof of this step and we refer to [15] for the details.

Step 3 (Passing to the limit): there exists $u \in H^1 \cap C^1(\mathbb{R}^2, \mathbb{R}^3)$ such that, for a subsequence, $u^n \to u$ weakly in H^1 and strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$.

By the uniform estimates stated in the step 1, we may assume that the sequence (u^n) is bounded in H^1. Hence, there exists $u \in H^1$ such that, for a subsequence, still denoted (u^n), one has that $u^n \to u$ weakly in H^1. Now, fix a compact set K in $\mathbb{R}^2$. Since $\|\nabla \omega^n\|_\infty = 1$, there exists $R > 0$ and a finite covering $\{D_{R/2}(z_i)\}_{i \in I}$ of K such that $\|\nabla u^n\|_{L^2(D_R(z_i))} \le \varepsilon$ for every $n \in \mathbb{N}$ and $i \in I$. Using the ε-regularity estimates stated in the step 2, and since (u^n) is bounded in L^∞, we have that $\|u^n\|_{H^{2,s}(D_{R/2}(z_i))} \le \bar{C}_{s,R}$ for some constant $\bar{C}_{s,R} > 0$ independent of $i \in I$ and $n \in \mathbb{N}$. Then the sequence (u^n) is bounded in $H^{2,p}(K, \mathbb{R}^3)$. For $s > 2$ the space $H^{2,s}(K, \mathbb{R}^3)$ is compactly embedded into $C^1(K, \mathbb{R}^3)$. Hence $u^n \to u$ strongly in $C^1(K, \mathbb{R}^3)$. By a standard diagonal argument, one concludes that $u^n \to u$ strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$.

Step 4: u is an H-bubble.

For every $n \in \mathbb{N}$ one has that if $\varphi \in C^\infty_c(\mathbb{R}^2, \mathbb{R}^3)$ then

$$\int_{\mathbb{R}^2} \nabla u^n \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} H(u^n)\varphi \cdot u^n_x \wedge u^n_y = 0 \; .$$

By step 3, passing to the limit, one immediately infers that u is a weak solution of $(B)_H$. According to Remark 8, u is a classical, conformal solution of $(B)_H$. In addition, u is nonconstant, since $|\nabla u(0)| = \lim |\nabla u^n(0)| = 1$. Hence $u \in \mathcal{B}_H$.

Step 5 (Semicontinuity inequality): $E_H(u) \le \liminf E_H(u^n)$.

By the strong convergence in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$, for every $R > 0$, one has

(56) $$E_H(u^n, D_R) \to E_H(u, D_R)$$

where we denoted

$$E_H(u^n, \Omega) = \frac{1}{2} \int_\Omega |\nabla u^n|^2 + 2 \int_\Omega m_H(u^n) u^n \cdot u^n_x \wedge u^n_y$$

(and similarly for $E_H(u, \Omega)$). Now, fixing $\epsilon > 0$, let $R > 0$ be such that

(57) $$E_H(u, \mathbb{R}^2 \setminus D_R) \le \epsilon$$

(58) $$\int_{\mathbb{R}^2 \setminus D_R} |\nabla u|^2 \le \epsilon \; .$$

By (57) and (56) we have

$$\begin{aligned} E_H(u) &\le E_H(u, D_R) + \epsilon \\ &= E_H(u^n, D_R) + \epsilon + o(1) \\ &= E_H(u^n) - E_H(u^n, \mathbb{R}^2 \setminus D_R) + \epsilon + o(1) \end{aligned} \tag{59}$$

with $o(1) \to 0$ as $n \to +\infty$. Since every u^n is an H-bubble, using the divergence theorem, for any $R > 0$ one has

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2\setminus D_R}|\nabla u^n|^2 &= 3E_H(u^n,\mathbb{R}^2\setminus D_R)-\int_{\partial D_R}u^n\cdot\frac{\partial u^n}{\partial\nu}\\ &\quad +2\int_{\mathbb{R}^2\setminus D_R}(H(u^n)-3m_H(u^n))u^n\cdot u^n_x\wedge u^n_y\,.\end{aligned}$$

We can estimate the last term as in (55), obtaining that

$$\begin{aligned}-E_H(u^n,\mathbb{R}^2\setminus D_R) &\le -\frac{1}{3}\int_{\partial D_R}u^n\cdot\frac{\partial u^n}{\partial\nu}-\left(\frac{1}{6}-\frac{\bar{M}_H}{3}\right)\int_{\mathbb{R}^2\setminus D_R}|\nabla u^n|^2\\ &\le -\frac{1}{3}\int_{\partial D_R}u^n\cdot\frac{\partial u^n}{\partial\nu}\,,\end{aligned} \tag{60}$$

because of the assumption $(\mathbf{h_2})$. Using again the C^1_{loc} convergence of u^n to u, as well as the fact that u is an H-bubble, we obtain that

$$\begin{aligned}\lim_{n\to+\infty}\left|\int_{\partial D_R}u^n\cdot\frac{\partial u^n}{\partial\nu}\right| &= \left|\int_{\partial D_R}u\cdot\frac{\partial u}{\partial\nu}\right|\\ &= \left|\int_{\mathbb{R}^2\setminus D_R}\Big(u\cdot\Delta u+|\nabla u|^2\Big)\right|\\ &= \left|\int_{\mathbb{R}^2\setminus D_R}\Big(2H(u)u\cdot u_x\wedge u_y+|\nabla u|^2\Big)\right|\\ &\le (\|u\|_\infty\|H\|_\infty+1)\int_{\mathbb{R}^2\setminus D_R}|\nabla u|^2\\ &\le (\|u\|_\infty\|H\|_\infty+1)\,\epsilon\end{aligned} \tag{61}$$

thanks to (58). Finally, (59), (60) and (61) imply

$$E_H(u)\le E_H(u^n)+C\epsilon+o(1)$$

for some positive constant C independent of ϵ and n. Hence, the conclusion follows. □

7.2. Existence of minimal H-bubbles

Here we study the case of a prescribed mean curvature function $H \in C^1(\mathbb{R}^3)$ asymptotic to a constant at infinity and, in particular, we discuss a result obtained in [15] about the existence of H-bubbles with minimal energy, under global assumptions on the prescribed mean curvature H.

Before stating this result, we need some preliminaries. First, we observe that, by the generalized isoperimetric inequality stated in Theorem 11 and since E_H is invariant under dilation, for a nonzero, bounded function H, the volume functional V_H turns out to be essentially cubic and $u \equiv 0$ is a strict local minimum for E_H in the space of smooth functions $C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. Moreover, if H is nonzero on a sufficiently large set (as it happens if H is asymptotic to a nonzero constant at infinity), $E_H(v) < 0$ for some $v \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. Hence E_H has a mountain pass geometry on $C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. Let us introduce the value

$$c_H = \inf_{\substack{u \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3) \\ u \neq 0}} \sup_{s>0} E_H(su) ,$$

which represents the mountain pass level along radial paths. Now, the existence of minimal H-bubbles can be stated as follows.

THEOREM 18. *Let $H \in C^1(\mathbb{R}^3)$ satisfy*

($\mathbf{h_3}$) *$H(u) \to H_\infty$ as $|u| \to \infty$, for some $H_\infty \in \mathbb{R}$,*

($\mathbf{h_4}$) *$\sup_{u \in \mathbb{R}^3} |\nabla H(u) \cdot u\, u| =: M_H < 1$,*

($\mathbf{h_5}$) *$c_H < \frac{4\pi}{3H_\infty^2}$.*

Then there exists an H-bubble $\bar{u}$ with $E_H(\bar{u}) = c_H = \inf_{u \in \mathcal{B}_H} E_H(u)$.

The assumption ($\mathbf{h_4}$) is a stronger version of the condition ($\mathbf{h_2}$) (indeed $2\bar{M}_H \leq M_H$), and it essentially guarantees that the value c_H is an admissible minimax level.

The assumption ($\mathbf{h_5}$) is variational in nature, and it yields a comparison between the radial mountain pass level c_H for the energy functional E_H and the corresponding level for the problem at infinity, in the spirit of concentration-compactness principle by P.-L. Lions [35]. Indeed, the problem at infinity corresponds to the constant curvature H_∞ and, in this case, one can evaluate $c_{H_\infty} = \frac{4\pi}{3H_\infty^2}$.

The hypothesis ($\mathbf{h_5}$) can be checked in terms of H in some cases. For instance, ($\mathbf{h_5}$) holds true when $|H(u)| \geq |H_\infty| > 0$ for all $u \in \mathbb{R}$ but $H \not\equiv H_\infty$, or when $|H(u)| > |H_\infty| > 0$ for $|u|$ large, or when $H_\infty = 0$ and $E_H(v) < 0$ for some $v \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. On the other hand, one can show that if $H \in C^1(\mathbb{R}^3)$ satisfies ($\mathbf{h_3}$), ($\mathbf{h_4}$), and $|H(u)| \leq |H_\infty|$ for all $u \in \mathbb{R}^3$, then ($\mathbf{h_5}$) fails and, in this case, Theorem 18 gives no information about the existence of H-bubbles.

As a preliminary result, we state some properties about the value c_H, which make clearer the role of the assumption ($\mathbf{h_4}$).

LEMMA 12. *Let $H \in C^1(\mathbb{R}^3)$ be such that $M_H < 1$. The following properties hold:*

(i) *if $u \in \mathcal{B}_H$ then $E_H(u) \geq c_H$;*

(ii) *if $\lambda \in (0, 1]$ then $c_{\lambda H} \geq c_H$;*

(iii) *if $(H_n) \subset C^1(\mathbb{R}^3)$ is a sequence converging uniformly to H and $M_{H_n} < 1$ for all $n \in \mathbb{N}$, then $\limsup c_{H_n} \leq c_H$.*

Proof. (i) Let $u \in \mathcal{B}_H$ and consider the mapping $s \mapsto f(s) := E_H(su)$ for $s \geq 0$. We know that $s = 1$ is a stationary point for f since u is a critical point of E_H. Moreover, if $\bar{s} > 0$ is a stationary point for f, then

$$0 = f'(\bar{s}) = \bar{s} \int_{\mathbb{R}^2} |\nabla u|^2 + 2\bar{s}^2 \int_{\mathbb{R}^2} H(\bar{s}u) u \cdot u_x \wedge u_y$$

and consequently

$$\begin{aligned} f''(\bar{s}) &= \int_{\mathbb{R}^2} |\nabla u|^2 + 4\bar{s} \int_{\mathbb{R}^2} H(\bar{s}u) u \cdot u_x \wedge u_y + 2\bar{s}^2 \int_{\mathbb{R}^2} \nabla H(\bar{s}u) \cdot u \; u \cdot u_x \wedge u_y \\ &= -\int_{\mathbb{R}^2} |\nabla u|^2 + 2 \int_{\mathbb{R}^2} \nabla H(\bar{s}u) \cdot \bar{s}u \; \bar{s}u \cdot u_x \wedge u_y \\ &\leq -(1 - M_H) \int_{\mathbb{R}^2} |\nabla u|^2 \,. \end{aligned}$$

Hence, there exists only one stationary point $\bar{s} > 0$ for f and $\bar{s} = 1$. Moreover $\max_{s \geq 0} E_H(su) = E_H(u)$. Since $C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ is dense in H^1 with respect to the Dirichlet norm, for every $\epsilon > 0$ there exists $v \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ such that $|E_H(sv) - E_H(su)| < \epsilon$ for all $s \geq 0$ in a compact interval. This is enough to obtain the desired estimate.

The statements (ii) and (iii) follow by the definition of c_H, and by using arguments similar to the proof of (i).

□

Proof of Theorem 18. . We just give an outline of the proof and we refer to [15] for all the details.

First part: The case H constant far out.

Firstly one proves the result under the additional condition ($\mathbf{h_1}$). Since $\bar{M}_H \leq \frac{1}{2} M_H < \frac{1}{2}$ one can apply Theorem 17 to infer the existence of an H-bubble at the minimal level μ_H. Then one has to show that $c_H = \mu_H$, which is an essential information in order to give up the extra assumption ($\mathbf{h_1}$), performing an approximation procedure on the prescribed mean curvature function H. From Lemma 12, part (i), one gets $\mu_H \geq c_H$. The opposite inequality needs more work and its proof is obtained in few steps.

Step 1: Approximating compact problems.

Let us introduce the family of Dirichlet problems given by

$$(D)_{H,\alpha} \qquad \begin{cases} \operatorname{div}((1 + |\nabla u|^2)^{\alpha-1} \nabla u) = 2H(u) u_x \wedge u_y & \text{in } D^2 \\ u = 0 & \text{on } \partial D^2 \,, \end{cases}$$

where $\alpha > 1$, α close to 1. This kind of approximation is in essence the same as in a well known paper by Sacks and Uhlenbeck [37] and it turns out to be particularly helpful in order to get uniform estimates. Solutions to $(D)_{H,\alpha}$ can be obtained as critical points of the functional

$$E_H^\alpha(u) = \frac{1}{2\alpha}\int_{D^2}((1+|\nabla u|^2)^\alpha - 1) + 2V_H(u)$$

defined on $H_0^{1,2\alpha} := H_0^{1,2\alpha}(D^2,\mathbb{R}^3)$. Since $H_0^{1,2\alpha}$ is continuously embedded into $H_0^1 \cap L^\infty$, the functional E_H^α is of class C^1 on $H_0^{1,2\alpha}$. Moreover, for $\alpha > 1$, α close to 1, E_H^α admits a mountain pass geometry at a level $c_H^\alpha > 0$, and it satisfies the Palais-Smale condition, because the embedding of $H_0^{1,2\alpha}$ into L^∞ is compact. Then, an application of the mountain pass lemma (Theorem 13) gives the existence of a critical point $u^\alpha \in H_0^{1,2\alpha}$ for E_H^α at level c_H^α, namely a nontrivial weak solution to $(D)_{H,\alpha}$.

Step 2: Uniform estimates on u^α. The family of solutions (u^α) turn out to satisfy the following uniform estimates:

(62) $$\limsup_{\alpha\to 1} E_H^\alpha(u^\alpha) \le c_H\ ,$$

(63) $$C_0 \le \|\nabla u^\alpha\|_2 \le C_1 \text{ for some } 0 < C_0 < C_1 < +\infty\ ,$$

(64) $$\sup_\alpha \|u^\alpha\|_\infty < +\infty\ .$$

The inequality (62) is proved by showing that $\limsup_{\alpha\to 1} c_H^\alpha \le c_H$, which can be obtained using ($\mathbf{h_5}$), the definitions of c_H^α and c_H, and the fact that $E_H^\alpha(u) \to E_H(u)$ as $\alpha \to 1$ for every $u \in C_c^\infty(D^2,\mathbb{R}^3)$. As regards (63), the upper bound follows by an estimate similar to (55), whereas the lower bound is a consequence of the generalized isoperimetric inequality. In both the estimates one uses the bound $\bar{M}_H < \frac{1}{2}$. Finally, (64) is proved with the aid of a nice result by Bethuel and Ghidaglia [8] which needs the condition that H is constant far out (here we use the additional assumption ($\mathbf{h_1}$)).

Now, taking advantage from the previous uniform estimates, one can pass to the limit as $\alpha \to 1$ and one finds that the weak limit u of (u^α) is a solution of

$(D)_H$ $$\begin{cases} \Delta u = 2H(u)u_x \wedge u_y & \text{in } D^2 \\ u = 0 & \text{on } \partial D^2\ . \end{cases}$$

A nonexistence result by Wente [48] implies that $u \equiv 0$. Hence a lack of compactness occurs by a blow up phenomenon.

Step 3: Blow-up.

Let us define

$$v^\alpha(z) = u^\alpha(z_\alpha + \epsilon_\alpha z)$$

with $z_\alpha \in \mathbb{R}^2$ and $\epsilon_\alpha > 0$ chosen in order that $\|\nabla v^\alpha\|_\infty = |\nabla v^\alpha(0)| = 1$. Notice that $\epsilon_\alpha \to 0$ and the sets $\Omega_\alpha := \{z \in \mathbb{R}^2 \ :\ |z_\alpha + \epsilon_\alpha z| < 1\}$ are discs which become larger and larger as $\alpha \to 1$. Moreover $v^\alpha \in C_c(\mathbb{R}^2,\mathbb{R}^3) \cap H^1$ is a weak solution to

$$\begin{cases} \Delta v^\alpha = -\frac{2(\alpha-1)}{\epsilon_\alpha^2+|\nabla v^\alpha|^2}(\nabla^2 v^\alpha, \nabla v^\alpha)\nabla v^\alpha + \frac{2\epsilon_\alpha^{2(\alpha-1)}H(v^\alpha)}{(\epsilon_\alpha^2+|\nabla v^\alpha|^2)^{\alpha-1}} v_x^\alpha \wedge v_y^\alpha & \text{in } D_\alpha \\ v = 0 & \text{on } \partial D_\alpha\ , \end{cases}$$

satisfying the same uniform estimates as u^α for the Dirichlet and L^∞ norms, as well as the previous normalization conditions on its gradient. Using a refined version (adapted to the above system) of the ε-regularity estimates similar to the step 2 in the proof of Theorem 17, one can show that there exists $u \in H^1$ such that $v^\alpha \to u$ weakly in H^1 and strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$, and u is a λH-bubble for some $\lambda \in (0, 1]$. Here the value λ comes out as limit of $\epsilon_\alpha^{2(\alpha-1)}$ when $\alpha \to 1$. It remains to show that actually $\lambda = 1$. Indeed, one can show that $E_{\lambda H}(u) \le \lambda \liminf E^\alpha_H(u^\alpha)$. Using (62) and Lemma 12, parts (i) and (ii), one infers that $c_H \le c_{\lambda H} \le E_H(u) \le \lambda c_H$. Therefore $\lambda = 1$ and u is an H-bubble, with $E_H(u) = c_H$. In particular $\mu_H \le c_H$ and actually, by Lemma 12, part (i), $\mu_H = c_H$, which was our goal.

Second part: Removing the extra assumption ($\mathbf{h_1}$).
It is possible to construct a sequence $(H_n) \subset C^1(\mathbb{R}^3)$ converging uniformly to H and satisfying ($\mathbf{h_1}$) and $M_{H_n} \le M_H$. By the first part of the proof, for every $n \in \mathbb{N}$ there exists an H_n-bubble u^n with $E_{H_n}(u^n) = \mu_{H_n} = c_{H_n}$. Since $M_{H_n} \le M_H < 1$, by an estimate similar to (55), one deduces that the sequence (u^n) is uniformly bounded with respect to the Dirichlet norm. Moreover one has that that $\limsup E_{H_n}(u^n) = \limsup c_{H_n} \le c_H$, because of Lemma 12, part (iii). In order to get also a uniform L^∞ bound, one argues by contradiction. Suppose that (u^n) is unbounded in L^∞. Using Theorem 7, one can prove that the sequence of values diam u^n is bounded. Consequently, the sequence (u^n) moves at infinity and, roughly speaking, it accumulates on a solution u^∞ of the problem at infinity, that is on an H_∞-bubble. In addition, as in the proof of Theorem 17, the semicontinuity inequality $\liminf E_{H_n}(u^n) \ge E_{H_\infty}(u^\infty)$ holds true. Since the problem at infinity corresponds to a constant mean curvature H_∞, by Theorem 15, one has that $E_{H_\infty}(u^\infty) \ge \mu_{H_\infty} = \frac{4\pi}{3H_\infty^2}$. On the other hand, $E_{H_n}(u^n) = c_{H_n}$, and then $c_H \ge \limsup c_{H_n} \ge \frac{4\pi}{3H_\infty^2}$, in contradiction with the assumption ($\mathbf{h_5}$). Therefore (u^n) satisfies the uniform bounds

$$\sup \|\nabla u^n\|_2 < +\infty \;, \; \sup \|u^n\|_\infty < +\infty \;.$$

Now one can repeat essentially the same argument of the proof of Theorem 17 to conclude that, after normalization, u^n converges weakly in H^1 and strongly in $C^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$ to an H-bubble $\bar{u}$. Moreover

$$E_H(\bar{u}) \le \liminf E_H(u^n) = \liminf c_{H_n} \le c_H \;.$$

Since $E_H(\bar{u}) \ge c_H$ (see Lemma 12, (i)), the conclusion follows.

□

In [17] it is proved that the existence result about minimal H-bubbles stated in Theorem 18 is stable under small perturbations of the prescribed curvature function. More precisely, the following result holds.

THEOREM 19. *Let $H \in C^1(\mathbb{R}^3)$ satisfy ($\mathbf{h_3}$)–($\mathbf{h_5}$), and let $H_1 \in C^1(\mathbb{R}^3)$. Then there is $\bar\varepsilon > 0$ such that for every $\varepsilon \in (-\bar\varepsilon, \bar\varepsilon)$ there exists an $(H + \varepsilon H_1)$-bubble u^ε. Furthermore, as $\varepsilon \to 0$, u^ε converges to some minimal H-bubble u in $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$.*

We remark that the energy of u^ε is close to the (unperturbed) minimal energy of H-bubbles. However in general we cannot say that u^ε is a minimal $(H + \varepsilon H_1)$-bubble.

Finally, we notice that Theorem 19 cannot be applied when the unperturbed curvature H is a constant, since assumption (h_3) is not satisfied. That case is studied in the next subsection.

7.3. H-bubbles in a perturbative setting

Here we study the H-bubble problem when the prescribed mean curvature is a perturbation of a nonzero constant. More precisely we investigate the existence and the location of nonconstant solutions to the problem

$$(B)_{H_\varepsilon} \qquad \begin{cases} \Delta u = 2H_\varepsilon(u)u_x \wedge u_y \quad \text{on } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |\nabla u|^2 < +\infty. \end{cases}$$

where

$$H_\varepsilon(u) = H_0 + \varepsilon H_1(u)$$

being $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ and $\varepsilon \in \mathbb{R}$, with $|\varepsilon|$ small. All the results of this subsection are taken from [18].

To begin, we observe that the unperturbed problem $(B)_{H_0}$ is invariant under translations on the image, since the mean curvature is the constant H_0. It admits a fundamental solution

$$\omega^0 = \frac{1}{H_0}\omega$$

(with ω defined by (51)), and a corresponding family of solutions of the form $\omega^0 \circ g + p$ where g is any conformal diffeomorphism of $\mathbb{R}^2 \cup \{\infty\}$ and p runs in $\mathbb{R}^3$.

Notice that the translation invariance on the image is broken for $\varepsilon \neq 0$, when the perturbation H_1 is switched on, but problem $(B)_{H_\varepsilon}$ maintains the conformal invariance for every ε.

An important role for the existence of H_ε-bubbles is played by the following Poincaré-Melnikov function:

$$\Gamma(p) := -\int_{B_{1/|H_0|}(p)} H_1(q)\, dq$$

which measures the H_1-weighted volume of a ball centered at an arbitrary $p \in \mathbb{R}^3$ and with radius $1/|H_0|$. For future convenience, we point out that:

$$\Gamma(p) = V_{H_1}(\omega^0 + p)\ , \tag{65}$$

$$\nabla\Gamma(p) = \int_{\mathbb{R}^2} H_1(\omega^0 + p)\omega^0_x \wedge \omega^0_y\ . \tag{66}$$

The first equality is like (52), the second one can be obtained in a similar way, noting that div $(H_1(\cdot + p)e_i) = \partial_i H_i(\cdot + p)$ (e_1, e_2, e_3 denotes that canonical basis in $\mathbb{R}^3$, ∂_i means differentiation with respect to the i-th component).

The next result yields a necessary condition, expressed in terms of Γ, in order to have the existence of H_ε-bubbles approaching a sphere, as $\varepsilon \to 0$.

PROPOSITION 4. *Assume that there exists a sequence u^{ε_k} of H_{ε_k}-bubbles, with $\varepsilon_k \to 0$, and a point $p \in \mathbb{R}^3$ such that*

$$\|u^{\varepsilon_k} - (\omega^0 + p)\|_{C^1(\mathbb{S}^2,\mathbb{R}^3)} \to 0 \ \text{ as } k \to \infty.$$

Then p is a stationary point for Γ.

Proof. The maps u^{ε_k} solve $\Delta u^{\varepsilon_k} = 2H_0 u_x^{\varepsilon_k} \wedge u_y^{\varepsilon_k} + 2\varepsilon_k H_1(u^{\varepsilon_k}) u_x^{\varepsilon_k} \wedge u_y^{\varepsilon_k}$. Testing with the constant functions e_i $(i = 1, 2, 3)$ and passing to the limit, we get

$$0 = \int_{\mathbb{R}^2} H_1(u^{\varepsilon_k}) e_i \cdot u_x^{\varepsilon_k} \wedge u_y^{\varepsilon_k} \ = o(1) + \int_{\mathbb{R}^2} H_1(\omega^0 + p) e_i \cdot \omega_x^0 \wedge \omega_y^0 \ = o(1) + \partial_i \Gamma(p),$$

thanks to (66). Then the Proposition is readily proved. □

In the next result we consider the case in which Γ admits nondegenerate stationary points.

THEOREM 20. *If $\bar{p} \in \mathbb{R}^3$ is a nondegenerate stationary point for Γ, then there exists a curve $\varepsilon \mapsto u^\varepsilon$ of class C^1 from a neighborhood $I \subset \mathbb{R}$ of 0 into $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$ such that $u^0 = \omega^0 + \bar{p}$ and, for every $\varepsilon \in I$, u^ε is an H_ε-bubble, without branch points.*

In the case of extremal points for Γ, we can weaken the nondegeneracy condition. More precisely, we have the following result.

THEOREM 21. *If there exists a nonempty compact set $K \subset \mathbb{R}^3$ such that*

$$\max_{p\in\partial K} \Gamma(p) < \max_{p\in K} \Gamma(p) \ \text{ or } \ \min_{p\in\partial K} \Gamma(p) > \min_{p\in K} \Gamma(p)\ ,$$

then for $|\varepsilon|$ small enough there exists an H_ε-bubble u^ε, without branch points, and such that

$$\|u^\varepsilon - (\omega^0 + p_\varepsilon)\|_{C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)} \to 0 \ \text{ as } \varepsilon \to 0,$$

where $p_\varepsilon \in K$ is such that $\Gamma(p_\varepsilon) \to \max_K \Gamma$, or $\Gamma(p_\varepsilon) \to \min_K \Gamma$, respectively.

To prove Theorems 20 and 21 we adopt a variational-perturbative method introduced by Ambrosetti and Badiale in [1] and subsequently used with success to get existence and multiplicity results for a wide class of variational problems in some perturbative setting (see, e.g., [2] and [3]).

Firstly, we observe that solutions to problem $(B)_{H_\varepsilon}$ can be obtained as critical points of the energy functional

$$E_{H_\varepsilon}(u) = E_{H_0}(u) + 2\varepsilon V_{H_1}(u)\ .$$

Notice that E_{H_0} is the energy functional corresponding to the unperturbed problem $(B)_{H_0}$. Since in our argument we will need enough regularity for E_{H_ε}, a first (technical) difficulty concerns the functional setting (see Remark 10, 2). We can overcome this problem, either multiplying H_1 by a suitable cut-off function and proving some a priori estimates on the solutions we will find, or taking as a domain of E_{H_ε} a Sobolev space smaller than H^1, like for instance the space

$$W^{1,s} = \{v \circ \omega \ : \ v \in W^{1,s}(\mathbb{S}^2, \mathbb{R}^3)\}$$

with $s > 2$ fixed. Let us follow this second strategy, taking for simplicity $s = 3$. Hence E_{H_ε} is of class C^2 on $W^{1,3}$, since $H_1 \in C^2$ and $W^{1,3}$ is compactly embedded into L^∞.

Secondly, we point out that the unperturbed energy functional E_{H_0} admits a manifold Z of critical points that can be parametrized by $G \times \mathbb{R}^3$, where G is the conformal group of $\mathbb{S}^2 \approx \mathbb{R}^2 \cup \{\infty\}$, having dimension 6, and $\mathbb{R}^3$ keeps into account of the translation invariance on the image.

Thanks to some key results already known in the literature, see e.g. [32], Z is a nondegenerate manifold, that is

$$T_u Z = \ker E''_{H_0}(u) \quad \text{for every } u \in Z \ ,$$

where $T_u Z$ denotes the tangent space of Z at u, whereas $\ker E''_{H_0}(u)$ is the kernel of the second differential of E_{H_0} at u. This allows us to apply the implicit function theorem to get, taking account also of the G-invariance of E_{H_ε}, for $|\varepsilon|$ small, a 3-dimensional manifold $\mathcal{Z}_\varepsilon$ close to Z, constituting a natural constraint for the perturbed functional E_{H_ε}. More precisely, defining

$$(T_{\omega^0} Z)^\perp := \{v \in H^1 \mid \int_{\mathbb{R}^2} \nabla v \cdot \nabla u = 0 \ \ \forall u \in T_{\omega^0} Z\} \ ,$$

we can prove the following result.

LEMMA 13. *Let $R > 0$ be fixed. Then there exist $\bar\varepsilon > 0$, and a map $\eta^\varepsilon(p) \in W^{1,3}$ defined and of class C^1 on $(-\bar\varepsilon, \bar\varepsilon) \times B_R \subset \mathbb{R} \times \mathbb{R}^3$, such that $\eta^0(p) = 0$ and*

$$\begin{aligned} E'_{H_\varepsilon}(\omega + p + \eta^\varepsilon(p)) &\in T_{\omega^0} Z \\ \eta^\varepsilon(p) &\in (T_{\omega^0} Z)^\perp \\ \int_{\mathbb{S}^2} \eta^\varepsilon(p) &= 0. \end{aligned}$$

Moreover, for every fixed $\varepsilon \in (-\bar\varepsilon, \bar\varepsilon)$ the set $\mathcal{Z}_\varepsilon^R := \{\omega^0 + p + \eta^\varepsilon(p) \mid |p| < R\}$ is a natural constraint for E_{H_ε}, that is, if $u \in \mathcal{Z}_\varepsilon^R$ is such that $dE_{H_\varepsilon}\big|_{\mathcal{Z}_\varepsilon^R}(u) = 0$, then $E'_{H_\varepsilon}(u) = 0$.

We refer to [18] for the proof of Lemma 13. Now, the problem is reduced to look for critical points of the function $f_\varepsilon \colon B_R \to \mathbb{R}$ defined by

$$f_\varepsilon(p) = E_{H_\varepsilon}(\omega^0 + p + \eta^\varepsilon(p)) \quad (p \in B_R) \ . \tag{67}$$

This step gives the finite dimensional reduction of the problem. The proofs of Theorems 20 and 21 can be completed as follows.

Proof of Theorem 20. Let $\bar{p} \in \mathbb{R}^3$ be a nondegenerate critical point of Γ and let $R > |\bar{p}|$. One can show that the function f_ε defined in (67) satisfies:

$$\nabla f_\varepsilon(p) = 2\varepsilon G(\varepsilon, p) \tag{68}$$

where

$$G(\varepsilon, p) = \int_{\mathbb{R}^2} H_1(\omega^0 + p + \eta^\varepsilon(p))(\omega^0 + \eta^\varepsilon(p))_x \wedge (\omega^0 + \eta^\varepsilon(p))_y \,.$$

By (66), one has that $G(0, p) = \nabla\Gamma(p)$ and, in addition, $\partial_i G_k(0, p) = \partial^2_{ik}\Gamma(p)$. Hence $G(0, \bar{p}) = 0$, because $\bar{p}$ is a stationary point of Γ. Moreover, since $\bar{p}$ is nondegenerate, $\nabla_p G(0, \bar{p})$ is invertible. Therefore by the implicit function theorem, there exists a neighborhood I of 0 (in $\mathbb{R}$) and a C^1 mapping $\varepsilon \mapsto p^\varepsilon \in \mathbb{R}^3$ defined on I, such that $p^0 = \bar{p}$ and $G(\varepsilon, p^\varepsilon) = 0$ for all $\varepsilon \in I$. Hence, by (67), (68) and by Lemma 13, we obtain that the function

$$\varepsilon \mapsto u^\varepsilon := \omega^0 + p^\varepsilon + \eta^\varepsilon(p^\varepsilon) \qquad (\varepsilon \in I)$$

defines a C^1 curve from I into $W^{1,3}$ of H_ε-bubbles, passing through $\omega^0+\bar{p}$ when $\varepsilon = 0$. It remains to prove that the curve $\varepsilon \mapsto u^\varepsilon$ is of class C^1 from I into $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$. This can be obtained by a boot-strap argument. Indeed u^ε solves $\Delta u^\varepsilon = F^\varepsilon$ on $\mathbb{R}^2$, where $F^\varepsilon = 2H_\varepsilon(u^\varepsilon)u^\varepsilon_x \wedge u^\varepsilon_y$. Since $\varepsilon \mapsto u^\varepsilon$ is of class C^1 from I into $W^{1,3}$ we have that $\varepsilon \mapsto F^\varepsilon$ is of class C^1 from I into $L^{3/2}$. Now, regularity theory yields that the mapping $\varepsilon \mapsto u^\varepsilon$ turns out of class C^1 from I into $W^{2,3/2}$. This implies that $\varepsilon \mapsto du^\varepsilon$ is C^1 from I into L^6, by Sobolev embedding. Hence $\varepsilon \mapsto F^\varepsilon$ belongs to $C^1(I, L^3)$. Consequently, again by regularity theory, $\varepsilon \mapsto u^\varepsilon$ is of class C^1 from I into $W^{2,3}$. By the embedding of $W^{2,3}$ into $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$, the conclusion follows. Lastly, we point out that u^ε has no branch points because $u^\varepsilon \to \omega^0 + \bar{p}$ in $C^{1,\alpha}(\mathbb{S}^2, \mathbb{R}^3)$ as $\varepsilon \to 0$, and ω^0 is conformal on $\mathbb{R}^2$.

□

Proof of Theorem 21. Since $\eta^\varepsilon(p)$ is of class C^1 with respect to the pair (ε, p), and $\eta^0(p) = 0$, we have that

$$\|\eta^\varepsilon(p)\|_{W^{1,3}} = O(\varepsilon) \ \text{ uniformly for } p \in B_R, \ \text{ as } \varepsilon \to 0 \,. \tag{69}$$

Now we show that

$$f_\varepsilon(p) = E_{H_0}(\omega^0) + 2\varepsilon\Gamma(p) + O(\varepsilon^2) \text{ as } \varepsilon \to 0, \ \text{ uniformly for } p \in B_R \,. \tag{70}$$

Indeed, set

$$\begin{aligned} R^\varepsilon(p) \quad &:= \quad f_\varepsilon(p) - E_{H_0}(\omega^0) - 2\varepsilon\Gamma(p) \\ &= \quad E_{H_0}(\omega^0 + \eta^\varepsilon(p)) - E_{H_0}(\omega^0) \\ &\qquad +2\varepsilon\Big(V_{H_1}(\omega^0 + p + \eta^\varepsilon(p)) - V_{H_1}(\omega^0 + p)\Big)\,. \end{aligned}$$

Using $E'_{H_0}(\omega^0) = 0$ and the decomposition $V_{H_0}(u+v) = V_{H_0}(u)+V_{H_0}(v)+H_0\int_{\mathbb{R}^2} u\cdot v_x \wedge v_y + H_0\int_{\mathbb{R}^2} v\cdot u_x\wedge u_y$ we compute

$$\begin{aligned} E_{H_0}(\omega^0+\eta^\varepsilon(p)) - E_{H_0}(\omega^0) &= E_{H_0}(\eta^\varepsilon(p)) + 2V_{H_0}(\eta^\varepsilon(p)) \\ &\quad +2H_0\int_{\mathbb{R}^2}\omega^0\cdot\eta^\varepsilon(p)_x\wedge\eta^\varepsilon(p)_y \\ &= O(\|d\eta^\varepsilon(p)\|_3^2) \end{aligned}$$

Therefore, using also (69), we infer that

$$\begin{aligned} R^\varepsilon(p)\varepsilon^{-2} &= O(\|d\eta^\varepsilon(p)\|_3^2)\varepsilon^{-2} + 2\left(V_{H_1}(\omega^0+p+\eta^\varepsilon(p)) - V_{H_1}(\omega^0+p)\right)\varepsilon^{-1} \\ &= O(1) + 2(dV_{H_1}(\omega^0+p)\eta^\varepsilon(p) + \|\eta^\varepsilon(p)\|_{W^{1,3}}o(1))\varepsilon^{-1} = O(1), \end{aligned}$$

and (70) follows. Now, let K be given according to the assumption and take $R > 0$ so large that $K \subset B_R$. The hypothesis on K and (70) imply that for $|\varepsilon|$ small, there exists $p_\varepsilon \in K$ such that $u^\varepsilon := \omega^0 + p_\varepsilon + \eta^\varepsilon(p_\varepsilon)$ is a stationary point for E_{H_ε} constrained to $\mathcal{Z}_\varepsilon^R$. According to Lemma 13, $E'_{H_\varepsilon}(u^\varepsilon) = 0$, namely u^ε is an H_ε-bubble. Moreover, $\Gamma(p_\varepsilon) \to \max_K \Gamma$ (or $\Gamma(p_\varepsilon) \to \min_K \Gamma$) as $\varepsilon \to 0$. To prove that $\|u^\varepsilon - (p_\varepsilon + \omega^0)\|_{C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)} \to 0$ as $\varepsilon \to 0$ one can follow a boot-strap argument, as in the last part of the proof of Theorem 20.

□

The assumptions on Γ in Theorems 20 and 21 can be made explicit in terms of H_1 when $|H_0|$ is large. In particular, as a first consequence of the above existence theorems we obtain the following result, which says that nondegenerate critical points as well as topologically stable extremal points of the perturbation term H_1 are concentration points of H_ε-bubbles, in the double limit $\varepsilon \to 0$ and $|H_0| \to \infty$.

THEOREM 22. *Assume that one of the following conditions is satisfied:*

(i) *there exists a nondegenerate stationary point $\bar p \in \mathbb{R}^3$ for H_1;*

(ii) *there exists a nonempty compact set $K \subset \mathbb{R}^3$ such that* $\max_{p\in\partial K} H_1(p) < \max_{p\in K} H_1(p)$ *or* $\min_{p\in\partial K} H_1(p) > \min_{p\in K} H_1(p)$.

Then, for every $H_0 \in \mathbb{R}$ with $|H_0|$ large, there exists $\varepsilon_{H_0} > 0$ such that for every $\varepsilon \in [-\varepsilon_{H_0}, \varepsilon_{H_0}]$ there is a smooth H_ε-bubble $u^{H_0,\varepsilon}$ without branch points. Moreover

$$\lim_{|H_0|\to\infty}\lim_{\varepsilon\to 0}\|u^{H_0,\varepsilon} - p_\varepsilon\|_{C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)} = 0$$

where $p_\varepsilon \equiv \bar p$ if (i) *holds, or $p_\varepsilon \in \mathbb{R}^3$ is such that $p_\varepsilon \in K$ and $H_1(p_\varepsilon) \to \max_K H_1$, or $H_1(p_\varepsilon) \to \min_K H_1$ if* (ii) *holds. In addition, under the condition* (i)*, the map $\varepsilon \mapsto u^{H_0,\varepsilon}$ defines a C^1 curve in $C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)$.*

As a further application of Theorem 21, we consider a perturbation H_1 having some decay at infinity.

THEOREM 23. *If $H_1 \in L^1(\mathbb{R}^3) + L^2(\mathbb{R}^3)$, then for $|\varepsilon|$ small enough there exist $p_\varepsilon \in \mathbb{R}^3$ and a smooth H_ε-bubble u^ε, without branch points, such that $\|u^\varepsilon - (\omega^0 + p_\varepsilon)\|_{C^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)} \to 0$ as $\varepsilon \to 0$, and (p_ε) is uniformly bounded with respect to ε.*

We refer to [18] for the proofs of Theorems 22 and 23.

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POINCARÉ-BIRKHOFF FIXED POINT THEOREM AND PERIODIC SOLUTIONS OF ASYMPTOTICALLY LINEAR PLANAR HAMILTONIAN SYSTEMS

Abstract. This work, which has a self contained expository character, is devoted to the Poincaré-Birkhoff (PB) theorem and to its applications to the search of periodic solutions of nonautonomous periodic planar Hamiltonian systems. After some historical remarks, we recall the classical proof of the PB theorem as exposed by Brown and Neumann. Then, a variant of the PB theorem is considered, which enables, together with the classical version, to obtain multiplicity results for asymptotically linear planar hamiltonian systems in terms of the gap between the Maslov indices of the linearizations at zero and at infinity.

1. The Poincaré-Birkhoff theorem in the literature

In his paper [28], Poincaré conjectured, and proved in some special cases, that an area-preserving homeomorphism from an annulus onto itself admits (at least) two fixed points when some "twist" condition is satisfied. Roughly speaking, the twist condition consists in rotating the two boundary circles in opposite angular directions. This concept will be made precise in what follows.
Subsequently, in 1913, Birkhoff [4] published a complete proof of the existence of at least one fixed point but he made a mistake in deducing the existence of a second one from a remark of Poincaré in [28]. Such a remark guarantees that the sum of the indices of fixed points is zero. In particular, it implies the existence of a second fixed point in the case that the first one has a nonzero index.
In 1925 Birkhoff not only corrected his error, but he also weakened the hypothesis about the invariance of the annulus under the homeomorphism T. In fact Birkhoff himself already searched a version of the theorem more convenient for the applications. He also generalized the area-preserving condition.

Before going on with the history of the theorem we give a precise statement of the classical version of Poincaré-Birkhoff fixed point theorem and make some remarks.
In the following we denote by $\mathcal{A}$ the annulus $\mathcal{A} := \{(x, y) \in \mathbb{R}^2 \,:\, r_1^2 \le x^2 + y^2 \le r_2^2\,,\ 0 < r_1 < r_2\}$ and by C_1 and C_2 its inner and outer boundaries, respectively.

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Moreover we consider the covering space $H := \mathbb{R} \times \mathbb{R}_0^+$ of $\mathbb{R}^2 \setminus \{(0,0)\}$ and the projection associated to the polar coordinate system $\Pi : H \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ defined by $\Pi(\vartheta, r) = (r \cos\vartheta, r \operatorname{sen}\vartheta)$. Given a continuous map $\varphi : D \subset \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$, a map $\widetilde{\varphi} : \Pi^{-1}(D) \longrightarrow H$ is called a lifting of φ to H if

$$\Pi \circ \widetilde{\varphi} = \varphi \circ \Pi.$$

Furthermore for each set $D \subset \mathbb{R}^2 \setminus \{(0,0)\}$ we set $\tilde{D} := \Pi^{-1}(D)$.

THEOREM 1 (POINCARÉ-BIRKHOFF THEOREM). *Let $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ be an area-preserving homeomorphism such that both boundary circles of $\mathcal{A}$ are invariant under ψ (i.e. $\psi(C_1) = C_1$ and $\psi(C_2) = C_2$). Suppose that ψ admits a lifting $\widetilde{\psi}$ to the polar coordinate covering space given by*

$$\widetilde{\psi}(\vartheta, r) = (\vartheta + g(\vartheta, r), f(\vartheta, r)), \tag{1}$$

where g and f are $2\pi-$periodic in the first variable. If the twist condition

$$g(\vartheta, r_1)\, g(\vartheta, r_2) < 0 \qquad \forall\, \vartheta \in \mathbb{R} \quad \textit{[twist condition]} \tag{2}$$

holds, then ψ admits at least two fixed points in the interior of $\mathcal{A}$.

The proof of Theorem 1 guarantees the existence of two fixed points (called F_1 and F_2) of $\widetilde{\psi}$ such that $F_1 - F_2 \neq k(2\pi, 0)$, for any $k \in \mathbb{Z}$. This fact will be very useful in the applications of the theorem to prove the multiplicity of periodic solutions of differential equations. Of course the images of F_1, F_2 under the projection Π are two different fixed points of ψ.

We make now some remarks on the assumptions of the theorem.

REMARK 1. We point out that it is essential to assume that the homeomorphism is area-preserving. Indeed, let us consider an homeomorphism $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ which admits the lifting $\widetilde{\psi}(\vartheta, r) = (\vartheta + \alpha(r), \beta(r))$, where α and β are continuous functions verifying $2\pi > \alpha(r_1) > 0 > \alpha(r_2) > -2\pi$, $\beta(r_i) = r_i$ for $i \in \{1, 2\}$, β is strictly increasing and $\beta(r) > r$ for every $r \in (r_1, r_2)$. This homeomorphism, which does not preserve the area, satisfies the twist condition, but it has no fixed points. Also its projection has no fixed points.

REMARK 2. The homeomorphism ψ preserves the standard area measure $\mathrm{d}x\mathrm{d}y$ in $\mathbb{R}^2$ and hence its lift $\widetilde{\psi}$ preserves the measure $r\mathrm{d}r\mathrm{d}\vartheta$. We remark that it is possible to consider a lift in the Poincaré-Birkhoff theorem which preserves $\mathrm{d}r\mathrm{d}\vartheta$ instead of $r\mathrm{d}r\mathrm{d}\vartheta$ and still satisfies the twist condition. In fact, let us consider the homeomorphism T of $\mathbb{R} \times [r_1, r_2]$ onto itself defined by $T(\vartheta, r) = (\vartheta, a\, r^2 + b)$, where $a = \dfrac{1}{r_1 + r_2}$ and $b = \dfrac{r_1 r_2}{r_1 + r_2}$. The homeomorphism T preserves the twist and transforms the measure $r\mathrm{d}r\mathrm{d}\vartheta$ in a multiple of $\mathrm{d}r\mathrm{d}\vartheta$. Thus, if we define $\widetilde{\psi}^* := T \circ \widetilde{\psi} \circ T^{-1}$, we note that it preserves the measure $\mathrm{d}r\mathrm{d}\vartheta$. Furthermore, there is a bijection between fixed points F

of $\widetilde{\psi}^*$ and fixed points $T^{-1}(F)$ of $\widetilde{\psi}$. Finally, it is easy to verify that $\widetilde{\psi}^*$ is the lifting of an homeomorphism ψ^* which satisfies all the assumptions of Theorem 1. This remark implies that Theorem 1 is equivalent to Theorem 7 in Section 2.

It is interesting to observe that if slightly stronger assumptions are required in Theorem 1, then its proof is quite simple (cf. [25]). Indeed, we have the following proposition.

PROPOSITION 1. *Suppose that all the assumptions of Theorem* 1 *are satisfied and that*

(3) $\qquad g(\vartheta, \cdot)$ *is strictly decreasing* (*or strictly increasing*) *for each* ϑ.

Then, ψ *admits at least two fixed points in the interior of* $\mathcal{A}$.

Proof. According to (2) and (3), it follows that for every $\vartheta \in \mathbb{R}$ there exists a unique $r(\vartheta) \in (r_1, r_2)$ such that $g(\vartheta, r(\vartheta)) = 0$. By the periodicity of g in the first variable, we have that $g(\vartheta + 2k\pi, r(\vartheta)) = g(\vartheta, r(\vartheta)) = 0$ for every $k \in \mathbb{Z}$ and $\vartheta \in \mathbb{R}$. Hence as $g(\vartheta + 2k\pi, r(\vartheta + 2k\pi)) = 0$, we deduce from the uniqueness of $r(\vartheta)$ that $r : \vartheta \longmapsto r(\vartheta)$ is a $2\pi-$periodic function. Moreover, we claim that it is continuous too. Indeed, by contradiction, let us assume that there exist $\vartheta \in \mathbb{R}$ and a sequence ϑ_n converging to ϑ which admits a subsequence ϑ_{n_k} satisfying $\lim\limits_{k\to+\infty} r(\vartheta_{n_k}) = b \neq r(\vartheta)$. Passing to the limit, from the equality $g(\vartheta_{n_k}, r(\vartheta_{n_k})) = 0$, we immediately obtain $g(\vartheta, b) = 0 = g(\vartheta, r(\vartheta))$, which contradicts $b \neq r(\vartheta)$.
By construction, $\widetilde{\psi}(\vartheta, r(\vartheta)) = (\vartheta + g(\vartheta, r(\vartheta))\, ,\ f(\vartheta, r(\vartheta))\,) = (\vartheta\ ,\ f(\vartheta, r(\vartheta))\,)$. Hence, each point of the continuous closed curve $\Gamma \subset \mathcal{A}$ defined by

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \,:\, x = r(\vartheta)\, \cos\vartheta\ ,\ \ y = r(\vartheta)\, \mathrm{sen}\,\vartheta\ ,\ \ \vartheta \in \mathbb{R}\}$$

is "radially" mapped into another one under the operator ψ. Being ψ area-preserving and recalling the invariance of the boundary circles C_1, C_2 of $\mathcal{A}$ under ψ, we can deduce that the region bounded by the curves C_1 and Γ encloses the same area as the region bounded by the curves C_1 and $\psi(\Gamma)$. Therefore, there exist at least two points of intersection between Γ and $\psi(\Gamma)$. In fact as the two regions mentioned above have the same measure, we can write

$$\int_0^{2\pi}\int_{r_1}^{r(\vartheta)} r\,\mathrm{d}r\,\mathrm{d}\vartheta \;=\; \int_0^{2\pi}\int_{r_1}^{f(\vartheta, r(\vartheta))} r\,\mathrm{d}r\,\mathrm{d}\vartheta\ ,$$

which implies $\displaystyle\int_0^{2\pi}\Big(r^2(\vartheta) - f^2(\vartheta, r(\vartheta))\Big)\,\mathrm{d}\vartheta = 0$. Being the integrand continuous and $2\pi-$periodic, it vanishes at least at two points which give rise to two distinct fixed points of $\widetilde{\psi}(\cdot, r(\cdot))$ in $[0, 2\pi)$. Hence, we have found two fixed points of ψ and the proposition follows.

□

Morris [26] applied this version of the theorem to prove the existence of infinitely many $2\pi-$periodic solutions for

$$x'' + 2x^3 = e(t),$$

where e is continuous, $2\pi-$periodic and it satisfies

$$\int_0^{2\pi} e(t)\, \mathrm{d}t = 0.$$

If we assume monotonicity of $\vartheta + g(\vartheta, r)$ in ϑ, for each r, then also in this case the existence of at least one fixed point easily follows (cf. [25]).

PROPOSITION 2. *Assume that all the hypotheses of Theorem* 1 *hold. Moreover, suppose that*

(4) $\quad \vartheta + g(\vartheta, r)$ *is strictly increasing* (*or strictly decreasing*) *in* ϑ *for each* r.

Then, the existence of at least one fixed point follows, when ψ *is differentiable.*

Proof. Let us suppose that $\vartheta \longmapsto \vartheta + g(\vartheta, r)$ is strictly increasing for every $r \in [r_1, r_2]$. Thus, since $\dfrac{\partial\,(\vartheta + g(\vartheta, r))}{\partial\,\vartheta} > 0$ for every r, it follows that the equation $\vartheta^* = \vartheta + g(\vartheta, r)$ defines implicitly ϑ as a function of ϑ^* and r. Moreover, taking into account the $2\pi-$periodicity of g in the first variable, it turns out that $\vartheta = \vartheta(\vartheta^*, r)$ satisfies $\vartheta(\vartheta^* + 2\pi, r) = \vartheta(\vartheta^*, r) + 2\pi$ for every ϑ^* and r. We set $r^* = f(\vartheta, r)$. Combining the area-preserving condition and the invariance of the boundary circles under ψ, then the existence of a generating function $W(\vartheta^*, r)$ such that

$$\left\{\begin{array}{lcl} \vartheta & = & \dfrac{\partial\, W}{\partial r}(\vartheta^*, r) \\[2ex] r^* & = & \dfrac{\partial\, W}{\partial \vartheta^*}(\vartheta^*, r) \end{array}\right. \tag{5}$$

is guaranteed by the Poincaré Lemma.
Now we consider the function $w(\vartheta^*, r) = W(\vartheta^*, r) - \vartheta^* r$. Since, according to (5), the following equalities hold

$$\left\{\begin{array}{lcl} \dfrac{\partial\, w}{\partial \vartheta^*} & = & r^* - r \\[2ex] \dfrac{\partial\, w}{\partial r} & = & \vartheta - \vartheta^*, \end{array}\right.$$

the critical points of w give rise to fixed points of ψ.

It is easy to verify that w has period 2π in ϑ^*. Indeed, according to the hypothesis of boundary invariance and to (5), we get

$$W(\vartheta^* + 2\pi, r_1) - W(\vartheta^*, r_1) = \int_{\vartheta^*}^{\vartheta^*+2\pi} r^*(s, r_1)\, \mathrm{d}s = \int_{\vartheta^*}^{\vartheta^*+2\pi} r_1\, \mathrm{d}s = 2\pi\, r_1.$$

Furthermore, combining (5) with the equality $\vartheta(\vartheta^* + 2\pi, r) = \vartheta(\vartheta^*, r) + 2\pi$, we deduce

$$\begin{aligned} W(\vartheta^* + 2\pi, r) - W(\vartheta^* + 2\pi, r_1) &= \int_{r_1}^{r} \vartheta(\vartheta^* + 2\pi, s)\, \mathrm{d}s \\ &= \int_{r_1}^{r} \vartheta(\vartheta^*, s)\, \mathrm{d}s + 2\pi(r - r_1) \\ &= W(\vartheta^*, r) - W(\vartheta^*, r_1) + 2\pi(r - r_1). \end{aligned}$$

Finally, we infer that

$$\begin{aligned} w(\vartheta^* + 2\pi, r) - w(\vartheta^*, r) &= W(\vartheta^* + 2\pi, r) - W(\vartheta^*, r) - 2\pi\, r \\ &= W(\vartheta^* + 2\pi, r_1) - W(\vartheta^*, r_1) - 2\pi\, r_1 = 0, \end{aligned}$$

and the periodicity of w in the first variable follows.
Consider now the external normal derivatives of w

$$(6) \qquad \left.\frac{\partial w}{\partial n}\right|_{\widetilde{C_1}} = (r^* - r, \vartheta - \vartheta^*) \cdot (0, -1) = \vartheta^* - \vartheta,$$

$$(7) \qquad \left.\frac{\partial w}{\partial n}\right|_{\widetilde{C_2}} = (r^* - r, \vartheta - \vartheta^*) \cdot (0, 1) = \vartheta - \vartheta^*.$$

The twist condition (2) implies that $\vartheta^* - \vartheta$ has opposite signs on the two boundary circles. Hence, by (6) and (7), the two external normal derivatives in $\widetilde{C_1}$ and $\widetilde{C_2}$ have the same sign. Being w a $2\pi-$periodic function in ϑ^*, critical point theory guarantees the existence of a maximum or a minimum of w in the interior of the covering space $\widetilde{\mathcal{A}}$. Such a point is the required critical point of w.

□

It is interesting to notice that as a consequence of the periodicity of g and f in ϑ, the existence of a second fixed point (a saddle) follows from critical point theory.

As we previously said, in order to apply the twist fixed point theorem to prove the existence of periodic solutions to planar Hamiltonian systems, Birkhoff tried to replace the invariance of the annulus by a weaker assumption. Indeed, he was able to require that only the inner boundary is invariant under T. He also generalized the area-preserving condition. More precisely, in his article [5] the homeomorphism T is defined on a region R bounded by a circle C and a closed curve Γ surrounding C. Such an homeomorphism takes values on a region R_1 bounded by C and by a closed curve Γ_1 surrounding C. Under these hypotheses, Birkhoff proved the following theorem

THEOREM 2. *Let $T : R \longrightarrow R_1$ be an homeomorphism such that $T(C) = C$ and $T(\Gamma) = \Gamma_1$, with Γ and Γ_1 star-shaped around the origin. If T satisfies the twist condition, then either*

- *there are two distinct invariant points P of R and R_1 under T*

or
• *there is a ring in* R *(or* R_1*) around* C *which is carried into part of itself by* T *(or* T^{-1}*).*

Since Birkhoff's proof was not accepted by many mathematicians, Brown and Neumann [6] decided to publish a detailed and convincing proof (based on the Birkhoff's one) of Theorem 1. In the same year, Neumann in [27] studied generalizations of such a theorem. For completeness, we will recall the proof given in [6] and also the details of a remark stated in [27] in the next section.

After Birkhoff's contribution, many authors tried to generalize the hypothesis of invariance of the annulus, in view of studying the existence of periodic solutions for problems of the form

$$x'' + f(t,x) = 0\,,$$

with $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuous and T-periodic in t.
In this sense we must emphasize the importance of the works by Jacobowitz and W-Y Ding. In his article [22] Jacobowitz (see also [23]), gave a version of the twist fixed point theorem in which the area-preserving twist homeomorphism is defined on an annulus whose internal boundary (roughly speaking) degenerates into a point, while the external one is a simple curve around it. More precisely, he first considered two simple curves $\Gamma_i = (\vartheta_i(\cdot), r_i(\cdot))$, $i = 1, 2$, defined in [0, 1], with values in the (ϑ, r) half-plane $r > 0$, such that $\vartheta_i(0) = -\pi$, $\vartheta_i(1) = \pi$, $\vartheta_i(s) \in (-\pi, \pi)$ for each $s \in (0, 1)$ and $r_i(0) = r_i(1)$. Then, he considered the corresponding 2π-periodic extensions, which he called again Γ_i. Denoting by A_i the regions bounded by the curve Γ_i (included) and the axis $r = 0$ (excluded), Jacobowitz proved the following theorem

THEOREM 3. *Let* $\psi : A_1 \longrightarrow A_2$ *be an area-preserving homeomorphism, defined by*

$$\psi(\vartheta, r) = (\vartheta + g(\vartheta, r),\ f(\vartheta, r)),$$

where
- g *and* f *are* $2\pi-$*periodic in the first variable;*
- $g(\vartheta, r) < 0$ *on* Γ_1*;*
- $\liminf_{r\to 0} g(\vartheta, r) > 0$.

Then, ψ *admits at least two fixed points, which do not differ from a multiple of* $(2\pi, 0)$.

Unfortunately the proof given by Jacobowitz is not very easy to follow. Subsequently, using the result by Jacobowitz, W-Y Ding in [15] and [16] treated the case in which also the inner boundary can vary under the area-preserving homeomorphism. He considered an annular region $\mathcal{A}$ whose inner boundary C_1 and the outer one C_2 are two closed simple curves. By D_i he denoted the open region bounded by C_i, $i = 1, 2$. Using the result by Jacobowitz, he proved the following theorem

THEOREM 4. *Let* $T : \mathcal{A} \longrightarrow T(\mathcal{A}) \subset \mathbb{R}^2 \setminus \{(0, 0)\}$ *be an area-preserving home-*

omorphism. Suppose that

(a) C_1 *is star-shaped about the origin;*

(b) T *admits a lifting* $\widetilde{T}$ *onto the polar coordinate covering space, defined by*

$$\widetilde{T}(\vartheta, r) = (\vartheta + g(\vartheta, r),\ f(\vartheta, r)),$$

where f *and* g *are* $2\pi-$*periodic in the first variable,* $g(\vartheta, r) > 0$ *on the lifting of* C_1 *and* $g(\vartheta, r) < 0$ *on the lifting of* C_2*;*

(c) *there exists an area-preserving homeomorphism* $T_0 : \overline{D_2} \longrightarrow \mathbb{R}^2$*, which satisfies* $T_{0|\mathcal{A}} = T$ *and* $(0, 0) \in T_0(D_1)$.

Then, $\widetilde{T}$ *has at least two fixed points such that their images under the usual covering projection* Π *are two different fixed points of* T.

We point out that condition (c) cannot be removed.
Indeed, we can define $\mathcal{A} := \{(x, y) : 2^{-2} < x^2 + y^2 < 2^2\}$ and consider an homeomorphism $T : \mathcal{A} \longrightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$ whose lifting is given by $\widetilde{T}(\vartheta, r) = \left(\vartheta + 1 - r, \sqrt{r^2 + 1}\right)$. It easily follows that $\widetilde{T}$ preserves the measure $r\mathrm{d}r\mathrm{d}\vartheta$ and, consequently, T preserves the measure $\mathrm{d}x\mathrm{d}y$. Moreover, the twist condition is satisfied, being $g(\vartheta, r) = 1 - r$ positive on $r = \dfrac{1}{2}$ and negative on $r = 2$. We also note that it is not possible to extend the homeomorphism into the interior of the circle of radius $1/2$ as an area-preserving homeomorphism, and hence (c) is not satisfied. Since $f(\vartheta, r) = \sqrt{r^2 + 1} > r$ for every $r \in \left(\dfrac{1}{2}, 2\right)$, we can conclude that $\widetilde{T}$ has no fixed points.

In [29], Rebelo obtained a proof for Jacobowitz and Ding versions of the Poincaré-Birkhoff theorem directly from Theorem 7.
The W-Y Ding version of the theorem seems the most useful in terms of the applications. In 1998, Franks [18] proved a quite similar result using another approach. In fact he considered an homeomorphism f from the open annulus $\mathcal{A} = \mathrm{S}^1 \times (0, 1)$ into itself. He replaced the area-preserving condition with the weaker condition that every point of $\mathcal{A}$ is non-wandering under f. We recall that a point x is non-wandering under f if for every neighbourhood U of x there is an $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.
Being $\widetilde{f}$, from the covering space $\widetilde{\mathcal{A}} = \mathbb{R} \times (0, 1)$ onto itself, a lift of f, it is said that there is a positively returning disk for $\widetilde{f}$ if there is an open disk $U \subset \widetilde{A}$ such that $\widetilde{f}(U) \cap U = \emptyset$ and $\widetilde{f}^n(U) \cap (U + k) \neq \emptyset$ for some $n,\ k > 0$. A negatively returning disk is defined similarly, but with $k < 0$. We recall that by $U + k$ it is denoted the set $\{(x + k, t) : (x, t) \in U\}$. Franks generalized the twist condition on a closed annulus assuming the existence of both a positive and a negative returning disk on the open annulus, since this hypothesis holds if the twist condition is verified. Under these generalized assumptions, Franks obtained the existence of a fixed point (for the open annulus). However, he observed that reducing to the case of the closed annulus, one can conclude the existence of two fixed points.

On the lines of Birkhoff [5], some mathematicians generalized the Poincaré-Birkhoff theorem, replacing the area-preserving requirement by a more general topological condition. Among others, we quote Carter [8], who, as Birkhoff, considered an homeomorphism g defined on an annulus $\mathcal{A}$ bounded by the unit circle T and a simple, closed, star-shaped around the origin curve γ that lies in the exterior of T. She also supposed that $g(T) = T$, $g(\gamma)$ is star-shaped around the origin and lies in the exterior of T. Before stating her version of the Poincaré-Birkhoff theorem, we only remark that a simple, closed curve in $\mathcal{A}$ is called essential if it separates T from γ.

THEOREM 5. *If g is a twist homeomorphism of the annulus $\mathcal{A}$ and if g has at most one fixed point in the interior of $\mathcal{A}$, then there is an essential, simple, closed curve C in the interior of $\mathcal{A}$ which meets its image in at most one point.* (*If the curve C intersects its image, the point of intersection must be the fixed point of g in the interior of $\mathcal{A}$*).

We point out that Theorem 2 can be seen as a consequence of Theorem 5 above.

Recently, in [24], Margheri, Rebelo and Zanolin proved a modified version of the Poincaré-Birkhoff theorem generalizing the twist condition. They assumed that the points of the external boundary circle rotate in one angular direction, while only some points of the inner boundary circle move in the opposite direction. The existence of one fixed point is guaranteed. More precisely, they proved the following

THEOREM 6. *Let $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ be an area-preserving homeomorphism in $\mathcal{A} = \mathbb{R} \times [0, R]$, $R > 0$ such that*

$$\psi(\vartheta, r) = (\vartheta_1, r_1),$$

with

$$\begin{cases} \vartheta_1 = \vartheta + g(\vartheta, r) \\ r_1 = f(\vartheta, r), \end{cases}$$

where f and g are $2\pi-$periodic in the first variable and satisfy the conditions

- $f(\vartheta, 0) = 0$, $f(\vartheta, R) = R$ *for every* $\vartheta \in \mathbb{R}$ (*boundary invariance*),
- $g(\vartheta, R) > 0$ *for every* $\vartheta \in \mathbb{R}$ *and there is* $\overline{\vartheta}$ *such that* $g(\overline{\vartheta}, 0) < 0$ (*modified twist condition*).

Then, ψ admits at least a fixed point in the interior of $\mathcal{A}$.

2. Proof of the classical version of the Poincaré-Birkhoff theorem

In this section we recall the proof of the classical version of the Poincaré-Birkhoff theorem given by Brown and Neumann [6] and give the details of the proof of an important remark (see Remark 3 below) made by Neumann in [27].

THEOREM 7. *Let us define* $\widetilde{\mathcal{A}} = \mathbb{R} \times [r_1, r_2]$, $0 < r_1 < r_2$. *Moreover, let* $h : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}$ *be an area-preserving homeomorphism satisfying*

$$h(x, r_2) = (x - s_1(x), r_2)\,,$$

$$h(x, r_1) = (x + s_2(x), r_1)\,,$$

$$h(x + 2\pi, y) = h(x, y) + (2\pi, 0),$$

for some $2\pi-$*periodic positive continuous functions* s_1, s_2. *Then,* h *has two distinct fixed points* F_1 *and* F_2 *which are not in the same periodic family, that is* $F_1 - F_2$ *is not an integer multiple of* $(2\pi, 0)$.

Note that Theorem 7 and Theorem 1 are the same. In fact, taking into account Remark 2, Theorem 7 corresponds to Theorem 1 choosing $h = \widetilde{\psi}$.

Before giving the proof of the theorem we give some useful preliminary definitions and results.

We define the direction from P to Q, setting $\mathrm{D}(P, Q) = \dfrac{Q - P}{\|Q - P\|}$, whenever P and Q are distinct points of $\mathbb{R}^2$. If we consider $X \subset \mathbb{R}^2$, $\mathcal{C}$ a curve in X and $h : X \longrightarrow \mathbb{R}^2$ an homeomorphism with no fixed points, then we will denote by $i_h(\mathcal{C})$ the index of $\mathcal{C}$ with respect to h. This index represents the total rotation that the direction $\mathrm{D}(P, h(P))$ performs as P moves along $\mathcal{C}$. In order to give a precise definition, we set $\mathcal{C} : [a, b] \longrightarrow \mathbb{R}^2$ and define the map $\overline{\mathcal{C}} : [a, b] \longrightarrow \mathrm{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ by $\overline{\mathcal{C}}(t) := \mathrm{D}(\mathcal{C}(t)\,,\ h(\mathcal{C}(t))\,)$. If we denote by $\pi : \mathbb{R} \longrightarrow \mathrm{S}^1$ the covering map $\pi(r) = (\cos r, \operatorname{sen} r)$, then we can lift the function $\overline{\mathcal{C}}$ into $\widetilde{\mathcal{C}} : [a, b] \longrightarrow \mathbb{R}$ assuming $\overline{\mathcal{C}} = \pi \circ \widetilde{\mathcal{C}}$. Finally, we set

$$i_h(\mathcal{C}) = \frac{\widetilde{\mathcal{C}}(b) - \widetilde{\mathcal{C}}(a)}{2\pi}\,,$$

which is well defined, since it is independent of the lifting.

This index satisfies the following properties:

1. For a one parameter continuous family of curves $\mathcal{C}$ or homeomorphisms h, $i_h(\mathcal{C})$ varies continuously with the parameter. (*Homotopy lifting property*).

2. If $\mathcal{C}$ runs from a point A to a point B, then $i_h(\mathcal{C})$ is congruent modulo 1 to $\frac{1}{2\pi}$ times the angle between the directions $\mathrm{D}(A, h(A))$ and $\mathrm{D}(B, h(B))$.

3. If $\mathcal{C} = \mathcal{C}_1\mathcal{C}_2$ consists of $\mathcal{C}_1$ and $\mathcal{C}_2$ laid end to end (i.e. $\mathcal{C}_1 = \mathcal{C}|_{[a,c]}$ and $\mathcal{C}_2 = \mathcal{C}|_{[c,b]}$ with $a < c < b$), then $i_h(\mathcal{C}) = i_h(\mathcal{C}_1) + i_h(\mathcal{C}_2)$. In particular, $i_h(-\mathcal{C}) = -i_h(\mathcal{C})$.

4. $i_h(\mathcal{C}) = i_{h^{-1}}(h(\mathcal{C}))$.

As a consequence of properties 1 and 2 we have that in order to calculate the index we can make first an homotopy on $\overline{\mathcal{C}}$ so long as we hold the endpoints fixed, this will be very important in what follows.

In the following it will be useful to consider an extension of the homeomorphism $h : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}$ to all $\mathbb{R}^2$.

To this aim, we introduce the following notations:

$$H_+ = \{(x, y) \in \mathbb{R}^2 : y \geq r_2\},$$

$$H_- = \{(x, y) \in \mathbb{R}^2 : y \leq r_1\}$$

and consider the extension of h (which we still denote by h)

$$h(x, y) := \begin{cases} (x - s_1(x), y) & y \geq r_2 \\ (x + s_2(x), y) & y \leq r_1 \\ h(x, y) & r_1 < y < r_2 . \end{cases}$$

The following lemma will be essential in order to prove the theorem.

LEMMA 1. *Suppose that all the assumptions of Theorem* 7 *are satisfied and that* h *has at most one family of fixed points of the form* $(2k\pi, r^*)$ *with* $r^* \in (r_1, r_2)$. *Then, for any curve* $\mathcal{C}$ *running from* H_- *to* H_+ *and not passing through any fixed point of* h*,*

(a) $i_h(\mathcal{C}) \equiv \dfrac{1}{2}$ (mod 1),

(b) $i_h(\mathcal{C})$ *is independent of* $\mathcal{C}$.

Proof of the lemma. From Property 2 of the index, it is easy to deduce that part (a) is verified.

Let us now consider two curves $\mathcal{C}_i$ $(i = 1, 2)$ running from $A_i \in H_-$ to $B_i \in H_+$ and not passing through any fixed point of h. Our aim consists in proving that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same index. Let us take a curve $\mathcal{C}_3$ from B_1 to B_2 in H_+ and a curve $\mathcal{C}_4$ from A_2 to A_1 in H_-. Being $\mathrm{D}(P, h(P))$ constant in H_+ and H_-, we immediately deduce that $i_h(\mathcal{C}_3) = i_h(\mathcal{C}_4) = 0$. Now, we can calculate the index of the closed curve $\mathcal{C}' := \mathcal{C}_1\, \mathcal{C}_3\, (-\mathcal{C}_2)\, \mathcal{C}_4$. In particular, from Property 3 we get

$$i_h(\mathcal{C}') = i_h(\mathcal{C}_1) + i_h(\mathcal{C}_3) + i_h(-\mathcal{C}_2) + i_h(\mathcal{C}_4) = i_h(\mathcal{C}_1) - i_h(\mathcal{C}_2).$$

Hence, in order to prove (b), it remains to show that such an index is zero. To this purpose, we give some further definitions. We denote by $\mathrm{Fix}(h)$ the fixed point set of h and by $\pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), A_1)$ the fundamental group of $\mathbb{R}^2 \setminus \mathrm{Fix}(h)$ in the base-point A_1. We recall that such a fundamental group is the set of all the loops (closed curves defined on closed intervals and taking values in $\mathbb{R}^2 \setminus \mathrm{Fix}(h)$) based on A_1, i.e. whose initial and final points coincide with A_1. The fundamental group is generated by paths which start from A_1, run along a curve $\mathcal{C}_0$ to near a fixed point (if there are any), loop around this fixed point and return by $-\mathcal{C}_0$ to A_1. Hence, since $\mathcal{C}'$ belongs to $\pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), A_1)$, it is deformable into a composition of such paths. Thus, it is sufficient to show that i_h is zero for any path belonging to the set of generators of the fundamental group. Since h has at most one family of fixed points of the form

$(2k\pi, r^*)$ with $r^* \in (r_1, r_2)$, then a loop surrounding a single fixed point can be deformed into the loop $\mathcal{D}' := \mathcal{D}_1\,\mathcal{D}_2\,\mathcal{D}_3\,\mathcal{D}_4$, where
$\mathcal{D}_1$ covers $[-\pi, \pi] \times \{r_0\}$ with $r_0 < r_1$, moving horizontally from $\widetilde{A}_1 = (-\pi, r_0)$ to $\widetilde{A}_2 = (\pi, r_0)$;
$\mathcal{D}_2$ covers $\{\pi\} \times [r_0, r_3]$ with $r_3 > r_2$, moving vertically from $\widetilde{A}_2$ to $\widetilde{A}_3 = (\pi, r_3)$;
$\mathcal{D}_3$ covers $[-\pi, \pi] \times \{r_3\}$, moving horizontally from $\widetilde{A}_3 = (\pi, r_3)$ to $\widetilde{A}_4 = (-\pi, r_3)$;
$\mathcal{D}_4$ covers $\{-\pi\} \times [r_0, r_3]$, moving vertically from $\widetilde{A}_4$ to $\widetilde{A}_1$.
Roughly speaking, $\mathcal{D}'$ is the boundary curve of a rectangle with vertices $(\pm\pi, r_0), (\pm\pi, r_3)$.
As $\mathcal{D}_1$ and $\mathcal{D}_3$ lie in H_- and H_+ respectively, their index is zero. Moreover, being $h(x, y) - (x, 0)$ a $2\pi-$periodic function in its first variable, it follows that $i_h(\mathcal{D}_4) = -i_h(\mathcal{D}_2)$. Thus, Property 3 of the index ensures that $i_h(\mathcal{D}') = 0$. This completes the proof.

□

Proof of Theorem 7. To prove the theorem, we will argue by contradiction. Assume that h has at most one family of fixed points $F = (\vartheta^*, r^*) + k\,(2\pi, 0)$, with $k \in \mathbb{Z}$. It is not restrictive to suppose $\vartheta^* = 0$. Indeed, we can always reduce to this case with a simple change of coordinates. In order to get the contradiction, we will construct two curves, with different indices, satisfying the hypotheses of Lemma 1.

Now we define the set

$$W = \{(x, y) \in \mathbb{R}^2 \,:\, 2k\pi + \frac{\pi}{2} \le x \le 2k\pi + \frac{3}{2}\pi \ , \quad k \in \mathbb{Z}\}\,.$$

Since the fixed points of h (if there are any) are of the form $(2k\pi, r^*)$, we can conclude that h has no fixed points in this region. Moreover, there exists $\varepsilon > 0$ such that

$$\varepsilon < \|P - h(P)\| \qquad \forall\, P \in W. \tag{8}$$

Indeed, by the periodicity of $(x, 0) - h(x, y)$ in its first variable, it is sufficient to find $\varepsilon > 0$ which satisfies the above inequality only for every $P \in W_1 := \{(x, y) \,:\, \frac{\pi}{2} \le x \le \frac{3}{2}\pi\}$. If we choose $\varepsilon < \min s_i$, for $i \in \{1, 2\}$ the inequality is satisfied on the sets $W_1 \cap H_\pm$. On the region $V := \{(x, y) \,:\, \frac{\pi}{2} \le x \le \frac{3}{2}\pi\ , \, r_1 \le y \le r_2\}$, the function $\|\mathrm{Id} - h\|$ is continuous and positive, hence it has a minimum on V, which is positive too.

Define the area-preserving homeomorphism $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T(x, y) = (x, y + \frac{\varepsilon}{2}(|\cos x| - \cos x))\,.$$

We point out that it moves only points of W and $\|T(P) - P\| \le \varepsilon$ for every $P \in \mathbb{R}^2$. Combining this fact with (8), we deduce that $T \circ h$ (just like h) has no fixed points in W. Furthermore, fixed points of $T \circ h$ coincide with the ones of h in $\mathbb{R}^2 \setminus W$ and, consequently, in $\mathbb{R}^2$.

Let us introduce the following sets

$$D_0 := H_- \setminus (T \circ h)^{-1} H_-\,,$$

$$D_1 := (T \circ h)\, D_0 = (T \circ h)\, H_- \setminus H_- ,$$
$$D_i := (T \circ h)^i\, D_0 \qquad \forall\, i \in \mathbb{Z} .$$

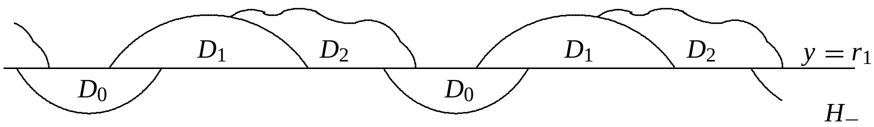

Figure 1: Some of the sets D_i

We immediately observe that $D_0 \subset H_-$, while $D_1 \subset \mathbb{R}^2 \setminus H_- = \{(x, y) : y > r_1\}$. Since $(T \circ h)(\mathbb{R}^2 \setminus H_-) \subset \mathbb{R}^2 \setminus H_-$, we can easily conclude that $D_i \subset \mathbb{R}^2 \setminus H_-$ for every $i \geq 1$. Hence, $D_i \cap D_0 = \emptyset$ for every $i \geq 1$. This implies that $D_k \cap D_j = \emptyset$ whenever $j \neq k$. Since $(T \circ h)^{-1} H_- \subset H_-$, we also get $D_i \subset H_-$ for every $i < 0$.
Furthermore, as T, h and, consequently, $(T \circ h)$ are area-preserving homeomorphisms, every D_i has the same area in the rolled-up plane $\mathbb{R}^2 / ((x, y) \equiv (x + 2\pi, y))$ and its value is 2ε. Thus, as the sets D_j are disjoint and contained in $\mathbb{R}^2 \setminus H_-$ for every $j \geq 1$, they must exhaust $\widetilde{\mathcal{A}}$ and hence intersect H_+. In particular, there exists $n > 0$ such that $D_n \cap H_+ \neq \emptyset$. Since $D_n \subset (T \circ h)^n H_-$, we also obtain that $(T \circ h)^n H_- \cap H_+ \neq \emptyset$. For such an $n > 0$, we can consider a point $P_n \in (T \circ h)^n H_- \cap H_+$ with maximal $y-$coordinate. The point P_n is not unique, but it exists since, by periodicity, it is sufficient to look at the compact region $(T \circ h)^n H_- \cap \{(x, y) : 0 \leq x \leq 2\pi , y \geq r_1\}$. Let us define

$$P_i = (x_i, y_i) := (T \circ h)^{i-n}\, P_n , \qquad i \in \mathbb{Z}.$$

Clearly, $P_n \in H_+$ and $P_0 = (T \circ h)^{-n}\, P_n \in H_-$. Moreover, $P_{i+1} = (T \circ h)\, P_i$ for every $i \in \mathbb{Z}$. Hence, recalling that $(T \circ h)\, H_+ \subset H_+$ and $(T \circ h)^{-1} H_- \subset H_-$, we obtain $P_{n+1} \in H_+$ and $P_{-1} \in H_-$.
Let us denote by $\mathcal{C}_0$ the straight line segment from P_{-1} to P_0 and let

$$\mathcal{C}_i = (T \circ h)^i\, \mathcal{C}_0 , \qquad i \in \mathbb{Z}.$$

In particular, the curve $\mathcal{C}_i$ runs from P_{i-1} to P_i. Furthermore, let us define the curve $\mathcal{C} := \mathcal{C}_0\, \mathcal{C}_1 \ldots \mathcal{C}_{n-1}\, \mathcal{C}_n$. Thus, $(T \circ h)(\mathcal{C}) = \mathcal{C}_1\, \mathcal{C}_2 \ldots \mathcal{C}_n\, \mathcal{C}_{n+1}$.
We have constructed a curve $\mathcal{C}$ running from H_- to H_+. Now, we will show that it does not pass through any fixed point of h and we will calculate its index. First, we need to list and prove some properties that this curve satisfies.

1. The curve $\mathcal{C}\, \mathcal{C}_{n+1} = \mathcal{C}_0 \ldots \mathcal{C}_{n+1}$ has no double points;
2. No point of $\mathcal{C}$ has larger $y-$coordinate than P_{n+1};
3. No point of $(T \circ h)(\mathcal{C})$ has smaller $y-$coordinate than P_{-1}.

In order to prove Property 1, we first observe that as $\mathcal{C}_0$ has no double points and $T \circ h$ is a homeomorphism, each curve $\mathcal{C}_i$ has no double points. Hence, we only need to show that $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for every $i \neq j$, exception made for the common endpoint when $|i - j| = 1$. We initially prove that this is true when i and j are both negative. We recall that the functions $f := (\mathrm{Id} + s_2) : \mathbb{R} \longrightarrow \mathbb{R}$ and f^{-1} are strictly monotone, being both continuous and bijective. From the positiveness of s_2, it immediately follows that both functions are strictly increasing. Moreover, $f^{-1}(x_0) \leq x \leq x_0$, whenever $(x, y) \in \mathcal{C}_0$. Thus, since $\mathcal{C}_0 \subset H_-$ and since f^{-1} is an increasing function, it turns out that $f^{-2}(x_0) \leq x \leq f^{-1}(x_0)$, whenever $(x, y) \in \mathcal{C}_{-1} = h^{-1}(T^{-1}(\mathcal{C}_0))$. In general, we have

$$\mathcal{C}_i \subset \{(x, y) : f^{\,i-1}(x_0) \leq x \leq f^{\,i}(x_0)\} \qquad \forall\, i < 0$$

and $\mathcal{C}_i$ intersects the boundaries of this strip only in its endpoints (because this is true for $\mathcal{C}_0$ and f^{-1} is strictly increasing). Thus, $\mathcal{C}_l$ and $\mathcal{C}_s$ intersect at most in a endpoint, if we choose l and s negative. In general, if we take $\mathcal{C}_i$ and $\mathcal{C}_j$ with $i \neq j$, then there exists $k < 0$ such that $(T \circ h)^k$ transforms such curves in two curves $\mathcal{C}_l$ and $\mathcal{C}_s$ with l and s both negative. Finally, the previous step guarantees that $\mathcal{C}_s \cap \mathcal{C}_l$ and hence $\mathcal{C}_i \cap \mathcal{C}_j$ are empty, if we exclude the intersection in the common endpoint.

Property 2 is easily proved. In fact, it is immediate to show that $\mathcal{C} \subset (T \circ h)^n\, H_-$. Thus, from the maximal choice involving the $y-$coordinate of P_n, we can conclude that for every $(x, y) \in \mathcal{C}$, we obtain $y \leq y_n$. Moreover, since $P_n \in H_+$ and $P_{n+1} = (T \circ h) P_n$, we can conclude that $y_n \leq y_{n+1}$. This completes the proof of Property 2.

With respect to Property 3, we remark that if we take $y \geq y_{-1}$ and if we define $(x', y') := (T \circ h)(x, y)$, then $y' \geq y_{-1}$. This is a consequence of the fact that $P_{-1} \in H_-$. Moreover, $y_0 \geq y_{-1}$ and hence $\mathcal{C}_0 \subset \{(x, y) : y \geq y_{-1}\}$. Thus, for every $(x, y) \in \mathcal{C}_1 = (T \circ h)\, \mathcal{C}_0$, we get $y \geq y_{-1}$. By induction, Property 3 follows.

Property 1 guarantees that $\mathcal{C}$ does not pass through any fixed point of $T \circ h$ and, consequently, of h.

We are interested in calculating the index of $\mathcal{C}$. More precisely, we will show that its value is exactly $\frac{1}{2}$. First, we will calculate $i_{(T \circ h)}(\mathcal{C})$.

The curve $\mathcal{C}$ runs from P_{-1} to P_n. Thus, recalling that $(T \circ h)\,(P_{-1}) = P_0$ and $(T \circ h)\,(P_n) = P_{n+1}$, let us consider the angle ϑ between $\mathrm{D}(P_{-1}, P_0)$ and $\mathrm{D}(P_n, P_{n+1})$. Since, by construction,

$$P_0 = (x_0, y_0) = (x_{-1} + s_2(x_{-1}), y_{-1} + \delta_2)\,, \qquad 0 \leq \delta_2 \leq \varepsilon,$$

$$P_{n+1} = (x_{n+1}, y_{n+1}) = (x_n - s_1(x_n), y_n + \delta_1)\,, \qquad 0 \leq \delta_1 \leq \varepsilon,$$

then we can write the explicit expression of ϑ

$$\vartheta = \pi - \left(\operatorname{arctg}\left(\frac{\delta_1}{s_1(x_n)} \right) + \operatorname{arctg}\left(\frac{\delta_2}{s_2(x_{-1})} \right) \right).$$

By Property 2 of the index, we can conclude that

$$\begin{aligned} i_{(T \circ h)}(\mathcal{C}) &= \frac{\vartheta}{2\pi} \,(\mathrm{mod}\, 1) \\ &= \frac{1}{2} - \frac{1}{2\pi} \left(\operatorname{arctg}\left(\frac{\delta_1}{s_1(x_n)} \right) + \operatorname{arctg}\left(\frac{\delta_2}{s_2(x_{-1})} \right) \right) (\mathrm{mod}\, 1)\,. \end{aligned}$$

From the choice of ε, we get $0 \le \delta_i \le \varepsilon < \min s_i$ for $i \in \{1, 2\}$. This implies that both $\operatorname{arctg}\left(\frac{\delta_1}{s_1(x_n)}\right)$ and $\operatorname{arctg}\left(\frac{\delta_2}{s_2(x_{-1})}\right)$ belong to the interval $[0, \frac{\pi}{4}[$. Consequently, $\dfrac{1}{4} < \dfrac{\vartheta}{2\pi} \le \dfrac{1}{2}$.

Our aim now consists in proving that we can cut mod 1 in the previous formula for $i_{(T\circ h)}(\mathcal{C})$. For this purpose, we will construct a suitable homotopy.

Let $P : [-1, 0] \longrightarrow \mathbb{R}^2$ be a parametrization of $\mathcal{C}_0$. Setting $P(t+1) := (T \circ h)(P(t))$ for $t \in [-1, n]$, we extend the given parametrization of $\mathcal{C}_0$ into a parametrization $P : [-1, n+1] \longrightarrow \mathbb{R}^2$ of $\mathcal{C}\,\mathcal{C}_{n+1}$. Clearly, if we restrict P to the interval $[-1, n]$, we obtain a parametrization of $\mathcal{C}$. Moreover, it is immediate to see that $P(i) = P_i$ for every integer $i \in \{-1, 0, \dots, n+1\}$. In order to calculate $i_{(T\circ h)}(\mathcal{C})$, by definition, we will consider the map $\overline{P} : [-1, n] \longrightarrow S^1$ given by

$$\overline{P}(t) := \mathrm{D}(\, P(t)\,,\ (T \circ h)(P(t))\,) = \mathrm{D}(P(t), P(t+1))\,.$$

Let us define now $\overline{P}_0 : [-1, 2n+1] \longrightarrow S^1$, setting

$$\overline{P}_0 = \begin{cases} \overline{P}(t) & -1 \le t \le n \\ \overline{P}(n) & n \le t \le 2n+1. \end{cases} \tag{9}$$

Of course, in order to evaluate the index we can use $\overline{P}_0$ instead of $\overline{P}$.

Now, we are in position to write the required homotopy. We will introduce a family of maps $\overline{P}_\lambda : [-1, 2n+1] \longrightarrow S^1$, with $0 \le \lambda \le n+2$. We will define this family treating separately the cases $0 \le \lambda \le n+1$ and $n+1 \le \lambda \le n+2$.

We develop the first case. The homotopy that we will exhibit will carry the initial map $\overline{P}_0$, which deals with the rotation of $\mathrm{D}\,(P, (T \circ h)(P))$ as P moves along $\mathcal{C}$, into the map $\overline{P}_{n+1}$ defined by

$$\overline{P}_{n+1}(t) = \begin{cases} \mathrm{D}\,(P(-1), P(t+1)) & -1 \le t \le n \\ \mathrm{D}\,(P(t-n-1), P(n+1)) & n \le t \le 2n+1. \end{cases} \tag{10}$$

This map corresponds to a rotation obtained if we initially move $(T \circ h)(P)$ along $(T \circ h)(\mathcal{C})$ from $(T \circ h)(P_{-1}) = P_0$ to P_{n+1}, holding P_{-1} fixed, and then we move P along $\mathcal{C}$ from P_{-1} to P_n, holding P_{n+1} fixed.

More precisely, when $0 \le \lambda \le n+1$, we set

$$\overline{P}_\lambda(t) = \begin{cases} \mathrm{D}\,(P(-1), P(t+1)) & -1 \le t \le \lambda - 1 \\ \mathrm{D}\,(P(t-\lambda), P(t+1)) & \lambda - 1 \le t \le n \\ \mathrm{D}\,(P(t-\lambda), P(n+1)) & n \le t \le n+\lambda \\ \mathrm{D}\,(P(n), P(n+1)) & n+\lambda \le t \le 2n+1. \end{cases}$$

Clearly, the above definition of $\overline{P}_\lambda$ in the case $\lambda = 0$ and $\lambda = n+1$ is compatible with (9) and (10), respectively. Furthermore, we note that $\overline{P}_\lambda(t)$ is always of the form

$D(P(t_0), P(t_1))$ with $-1 \le t_0 < t_1 \le n+1$. By Property 1 of $\mathcal{C}$, we deduce that $P(t_0) \ne P(t_1)$, hence $\overline{P}_\lambda$ is well defined for every $0 \le \lambda \le n+1$.

We consider now the second case: $n+1 \le \lambda \le n+2$. The homotopy we will exhibit will carry the map $\overline{P}_{n+1}$ into the map $\overline{P}_{n+2}$ defined by

$$\text{(11)} \quad \overline{P}_{n+2}(t) = \begin{cases} D(P(-1), P'(t+1)) & -1 \le t \le n \\ D(P''(t-n-1), P(n+1)) & n \le t \le 2n+1, \end{cases}$$

where by $P' : [0, n+1] \longrightarrow \mathbb{R}^2$ and $P'' : [-1, n] \longrightarrow \mathbb{R}^2$ we denote the straight line segments from $P(0)$ to $P(n+1)$ and from $P(-1)$ to $P(n)$, respectively.

The map $\overline{P}_{n+2}$ corresponds to a rotation obtained if we initially move $(T \circ h)(P)$ along the straight line segment P' from P_0 to P_{n+1}, holding P_{-1} fixed, and then if we move P along the straight line segment P'' from P_{-1} to P_n, holding P_{n+1} fixed.

More precisely, for every $0 \le \mu \le 1$, we define

$$\overline{P}_{n+1+\mu}(t) = \begin{cases} D(P(-1), (1-\mu) P(t+1) + \mu P'(t+1)) \\ \qquad\qquad\qquad -1 \le t \le n \\ D((1-\mu)P(t-n-1) + \mu P''(t-n-1), P(n+1)) \\ \qquad\qquad\qquad n \le t \le 2n+1. \end{cases}$$

Clearly, the above definition of $\overline{P}_{n+1+\mu}$ in the case $\mu = 0$ and $\mu = 1$ is compatible with (10) and (11), respectively.

Moreover, the homotopy is well defined. To prove this, we will show first that $P(-1)$ is never equal to $Q := (1-\mu)P(t+1) + \mu P'(t+1)$ for any $t \in [-1, n]$. Indeed, by Property 3 of $\mathcal{C}$, we deduce that Q has larger $y-$coordinate than $P(-1)$, except possibly when $t = -1$ or $\mu = 0$. However, in both these cases $Q = P(t+1)$ for some $t \in [-1, n]$. Since in this interval $t+1 \ge 0 > -1$, then Property 1 of $\mathcal{C}$ guarantees that $P(t+1) \ne P(-1)$. Hence, $P(-1) \ne Q$.

Analogously, by applying Property 2 and Property 1 of $\mathcal{C}$, we can conclude that $(1-\mu)P(t-n-1) + \mu P''(t-n-1) \ne P(n+1)$. Thus, the homotopy is well defined.

In particular, $\overline{P}_{n+2}$ defined in the interval $[-1, 2n+1]$ describes an increase in the angle which corresponds exactly to ϑ, calculated above. Thus as a consequence of the homotopy property, we conclude that

$$i_{(T\circ h)}(\mathcal{C}) = \frac{\vartheta}{2\pi} = \frac{1}{2} - \frac{1}{2\pi}\left(\operatorname{arctg}\left(\frac{\delta_1}{s_1(x_n)}\right) + \operatorname{arctg}\left(\frac{\delta_2}{s_2(x_{-1})}\right)\right).$$

From the previous calculations, we get

$$\frac{1}{4} < i_{(T\circ h)}(\mathcal{C}) \le \frac{1}{2}.$$

Our aim consists now in proving that $i_h(\mathcal{C}) = \frac{1}{2}$.

To this end, we define for every $s \in [0,1]$ the map $T_s : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, setting

$$T_s(x, y) = \left(x\,,\ y + \left(\frac{s\,\varepsilon}{2}\right)(|\cos x| - \cos x)\right).$$

In particular, $T_0 = \mathrm{Id}$ and $T_1 = T$. Arguing as before, we can easily see that

$$(12) \qquad i_{(T_s \circ h)}(\mathcal{C}) = \frac{1}{2} - \frac{1}{2\pi}\left(\operatorname{arctg}\left(\frac{s\,\delta_1}{s_1(x_n)}\right) + \operatorname{arctg}\left(\frac{s\,\delta_2}{s_2(x_{-1})}\right)\right) \pmod 1 .$$

Since this congruence becomes an equality in the case $s = 1$, by the continuity of the index we infer that also the congruence in (12) is an equality for every $s \in [0, 1]$. Hence, when $s = 0$ we can conclude that

$$(13) \qquad i_h(\mathcal{C}) = \frac{1}{2}.$$

In order to get the contradiction with Lemma 1, we need to construct another curve $\mathcal{C}'$ running from H_- to H_+, having index different from $\frac{1}{2}$. To this aim, we can repeat the whole argument replacing h with h^{-1}. Now everything works as before except the fact that the directions along which the two boundaries of $\widetilde{\mathcal{A}}$ move under h and under h^{-1} are opposite. In such a way, we find a curve $\widehat{\mathcal{C}}$ from H_- to H_+ with $i_{h^{-1}}(\widehat{\mathcal{C}}) = -\dfrac{1}{2}$. Let us define $\mathcal{C}' := h^{-1} \circ \widehat{\mathcal{C}} : H_- \longrightarrow H_+$. By Property 4 of the index, we finally infer

$$i_h\left(\mathcal{C}'\right) = i_{h^{-1}}\left(h\left(\mathcal{C}'\right)\right) = i_{h^{-1}}(\widehat{\mathcal{C}}) = -\frac{1}{2}.$$

If we compare the above equality with the equality in (13), we get the desired contradiction with Lemma 1.

□

As a consequence of the above proof, Neumann in [27] provides the following useful remark

REMARK 3. If h satisfies all the assumptions of Theorem 7 and it has a finite number of families of fixed points (finite number of fixed points in $[0, 2\pi] \times [r_1, r_2]$), then there exist fixed points with positive and negative indices.

We recall that the definition of index of a fixed point coincides with $i_h(\alpha)$ for a small circle α surrounding the fixed point when it has a positive (counter-clockwise) direction. Given a fixed point F, we will denote by $ind(F)$ its index.

Proof. Let us denote by F_i ($i = 1, 2, \ldots, k$) the distinct fixed points in $[0, 2\pi] \times (r_1, r_2)$, belonging to different periodic families. Theorem 7 guarantees that $k \geq 2$.
It is not restrictive to assume that $F_i \in (0, 2\pi) \times (r_1, r_2)$ since we suppose that the number of families of fixed points is finite. As in the proof of Theorem 7, we extend the homeomorphism h to an homeomorphism in the whole $\mathbb{R}^2$, and we still denote it by h.
If we fix $r_0 < r_1$, arguing as in the proof of Lemma 1, we can construct a loop $\mathcal{D}' := \mathcal{D}_1\,\mathcal{D}_2\,\mathcal{D}_3\,\mathcal{D}_4 \in \pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), (0, r_0))$, where
$\mathcal{D}_1$ covers $[0, 2\pi] \times \{r_0\}$, moving horizontally from $(0, r_0)$ to $(2\pi, r_0)$;
$\mathcal{D}_2$ covers $\{2\pi\} \times [r_0, r_3]$ with $r_3 > r_2$, moving vertically from $(2\pi, r_0)$ to $(2\pi, r_3)$;

$\mathcal{D}_3$ covers $[0, 2\pi] \times \{r_3\}$, moving horizontally from $(2\pi, r_3)$ to $(0, r_3)$;
$\mathcal{D}_4$ covers $\{0\} \times [r_0, r_3]$, moving vertically from $(0, r_3)$ to $(0, r_0)$.
In particular, $\mathcal{D}'$ moves with a positive orientation and, by construction, the only fixed points of h it surrounds are exactly the fixed points F_i, $i \in \{1, \dots, k\}$. We note that $i_h(\mathcal{D}_1) = i_h(\mathcal{D}_3) = 0$, since the curves $\mathcal{D}_1$ and $\mathcal{D}_3$ respectively lie in H_- and H_+ and $\mathrm{D}(P, h(P))$ is constant in these regions. Furthermore, being $h(x, y) - (x, 0)$ a $2\pi-$periodic function in its first variable, it follows that $i_h(\mathcal{D}_4) = -i_h(\mathcal{D}_2)$. Thus, Property 3 of the index guarantees that $\mathcal{D}'$ has index zero.

We recall that the fundamental group $\pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), (0, r_0))$ is generated by paths which start from $(0, r_0)$, run along a curve $\mathcal{C}_0$ to near a fixed point, loop around this fixed point and return by $-\mathcal{C}_0$ to $(0, r_0)$. It is possible to show that the generating paths, whose composition is deformable into the closed curve $\mathcal{D}'$, surround only the inner fixed points F_i. Consequently, the following equality holds

$$0 = i_h(\mathcal{D}') = \sum_{j=1}^{k} ind(F_j)\,. \tag{14}$$

This means that the sum of the fixed point indices is zero. We remark that such a result could have been directly obtained from the Lefschetz fixed point theorem.

Next step consists in constructing two curves with opposite indices, running from H_- to H_+ and not passing through any fixed point of h.

Since the number of fixed points in $[0, 2\pi] \times [r_1, r_2]$ is finite, it is possible to consider a non-empty vertical strip $\widehat{W} = [\alpha, \beta] \times \mathbb{R}$, for some $\alpha, \beta \in (0, 2\pi)$, which does not contain any F_i. Let us extend $2\pi-$periodically $\widehat{W}$ into the set

$$\bigcup_{m\in\mathbb{Z}} \big(\widehat{W} + (2m\pi, 0)\big) := \{(x, y) \in \mathbb{R}^2 : 2m\pi + \alpha \le x \le 2m\pi + \beta,\ \ m \in \mathbb{Z}\},$$

that we still denote by $\widehat{W}$.

Arguing as in the proof of Theorem 7, we can find a positive constant $\varepsilon < \min s_i$ for every $i \in \{1, 2\}$, satisfying $\varepsilon < \|P - h(P)\|$ for every $P \in \widehat{W}$. Let us now introduce the area-preserving homeomorphism $\widehat{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by setting

$$\widehat{T}_{|[0,2\pi]\times\mathbb{R}}(x, y) := \begin{cases} (x, y + \varepsilon \cos\Big(\frac{\pi}{2(\beta-\alpha)}\,(2x - \beta - \alpha)\Big) & \text{for } x \in [\alpha, \beta] \\ (x, y) & \text{otherwise}\,. \end{cases}$$

Fixed points of $\widehat{T} \circ h$ coincide with the ones of h in $\mathbb{R}^2$. If we proceed exactly as in the proof of Theorem 7, considering the homeomorphism $\widehat{T}$ instead of T and the set $\widehat{W}$ instead of W, we are able to construct a curve $\mathcal{C}$ of index $\frac{1}{2}$, which runs from $P_{-1} \in H_-$ to $P_n \in H_+$ and does not pass through any fixed point of h. Analogously, we can find another curve $\mathcal{C}'$ of index $-\frac{1}{2}$ running from $P'_{-1} \in H_-$ to $P'_n \in H_+$.

Let us consider now the closed curve $\mathcal{F} := \mathcal{C}\,\mathcal{B}\,(-\mathcal{C}')\,\mathcal{B}'$, where $\mathcal{B}$ is the straight line segment from P_n to P'_n; while $\mathcal{B}'$ is the straight line segment from P'_{-1} to P_{-1}. In particular, $\mathcal{B}$ lies in H_+ and connects the curve $\mathcal{C}$ to the curve $-\mathcal{C}'$; while $\mathcal{B}'$ lies in H_- and connects the curve $-\mathcal{C}'$ to the curve $\mathcal{C}$.

Since, by construction, $\mathcal{B}$ and $\mathcal{B}'$ have index equal to zero, we infer from Property 2 of the index that

$$i_h(\mathcal{F}) = i_h(\mathcal{C}) - i_h(\mathcal{C}') = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1\,.$$

Moreover, the loop $\mathcal{F}$ belongs to the fundamental group $\pi_1(\mathbb{R}^2 \setminus \mathrm{Fix}(h), P_{-1})$ and surrounds a finite number of fixed points. Each of them is of the form $F_i + m(2\pi, 0)$ for some $i \in \{1, 2 \ldots, k\}$ and some integer $m \in \mathbb{Z}$. Since $ind(F_i + m(2\pi, 0)) = ind(F_i)$ for every $m \in \mathbb{Z}$, we can deduce that

$$1 = i_h(\mathcal{F}) = \sum_{j=1}^{k} \nu(\mathcal{F}, F_j)\; ind(F_j)\,, \tag{15}$$

where the integer $\nu(\mathcal{F}, F_j)$ coincides with the sum of all the signs corresponding to the directions of every loop in which $\mathcal{F}$ can be deformed in a neighbourhood of every point of the form $F_j + m(2\pi, 0)$ surrounded by $\mathcal{F}$. From (15), we infer that there exists $j^* \in \{1, 2 \ldots, k\}$ such that $\nu(\mathcal{F}, F_{j^*})\, ind(F_{j^*}) > 0$ and, consequently, $ind(F_{j^*}) \neq 0$.

Hence, recalling that the sum of the fixed point indices is zero (cf. (14)), we can conclude the existence of at least a fixed point with positive index and a fixed point with negative one. This completes the proof.

□

3. Applications of the Poincaré-Birkhoff theorem

In this section we are interested in the applications of the Poincaré-Birkhoff fixed point theorem to the study of the existence and multiplicity of T-periodic solutions of Hamiltonian systems, that is systems of the form

$$\begin{cases} x' &= \frac{\partial H}{\partial y}(t, x, y) \\ \\ y' &= -\frac{\partial H}{\partial x}(t, x, y) \end{cases} \tag{16}$$

where $H : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous scalar function that we assume $T-$periodic in t and C^2 in $z = (x, y)$.

Under these conditions uniqueness of Cauchy problems associated to system (16) is guaranteed. Hence for each $z_0 = (x_0, y_0) \in \mathbb{R}^2$ and $t_0 \in \mathbb{R}$ there is a unique solution $(x(t), y(t))$ of system (16) such that

$$(x(t_0), y(t_0)) = (x_0, y_0) := z_0. \tag{17}$$

In the following we will denote such a solution by

$$z(t; t_0, z_0) := (x(t; t_0, z_0), y(t; t_0, z_0)) := (x(t; t_0, (x_0, y_0)), y(t; t_0, (x_0, y_0))).$$

For simplicity we set $z(t; z_0) := (x(t; z_0), y(t; z_0)) := (x(t; 0, z_0), y(t; 0, z_0))$. If we suppose that H satisfies further conditions which imply global existence of the solutions of Cauchy problems, then the Poincaré operator

$$\tau : z_0 = (x_0, y_0) \to (x(T; (x_0, y_0)), y(T; (x_0, y_0)))$$

is well defined in $\mathbb{R}^2$ and it is continuous. Also fixed points of the Poincaré operator are initial conditions of periodic solutions of system (16) and as a consequence of the Liouville theorem, the Poincaré operator is an area-preserving map. Hence it is natural to try to apply the Poincaré-Birkhoff fixed point theorem in order to prove the existence of periodic solutions of the Hamiltonian systems.

Before giving a version of the Poincaré-Birkhoff fixed point theorem useful for the applications, we previously introduce some notation.
Let $z : [0, T] \to \mathbb{R}^2$ be a continuous function satisfying $z(t) \neq (0, 0)$ for every $t \in [0, T]$ and $(\vartheta(\cdot), r(\cdot))$ a lifting of $z(\cdot)$ to the polar coordinate system. We define the rotation number of z, and denote it by $\mathrm{Rot}(z)$ as

$$\mathrm{Rot}(z) := \frac{\vartheta(T) - \vartheta(0)}{2\pi}.$$

Note that $\mathrm{Rot}(z)$ counts the counter-clockwise turns described by the vector $\overrightarrow{0\,z(s)}$ as s moves in the interval $[0, T]$. In what follows, we will use the notation $\mathrm{Rot}(z_0)$ to indicate $\mathrm{Rot}(z(\cdot; z_0))$.

From the Poincaré-Birkhoff Theorem 4, we can obtain the following multiplicity result.

THEOREM 8 ([29]). *Let $\mathcal{A} \subset \mathbb{R}^2 \setminus \{(0, 0)\}$ be an annular region surrounding $(0, 0)$ and let C_1 and C_2 be its inner and outer boundaries, respectively. Assume that C_1 is strictly star-shaped with respect to $(0, 0)$ and that $z(\cdot; t_0, z_0)$ is defined in $[t_0, T]$ for every $z_0 \in C_2$ and $t_0 \in [0, T]$. Suppose that*

i) $z(t; t_0, z_0) \neq (0, 0) \qquad \forall\, t_0 \in [0, T[\ ,\ \forall\, z_0 \in C_1\ ,\ \forall\, t \in [t_0, T]$;

ii) there exist $m_1, m_2 \in \mathbb{Z}$ with $m_1 \geq m_2$ such that

$$\mathrm{Rot}(z_0) > m_1 \qquad \forall\, z_0 \in C_1,$$

$$\mathrm{Rot}(z_0) < m_2 \qquad \forall\, z_0 \in C_2.$$

Then, for each integer l with $l \in [m_2, m_1]$, there are two fixed points of the Poincaré map which correspond to two periodic solutions of the Hamiltonian system having l as T-rotation number.

Sketch of the proof. The idea of the proof consists in applying Theorem 4 to the area-preserving Poincaré map $\tau : z_0 \longrightarrow z(T; z_0)$, considering different liftings of it. For each integer l with $m_2 \leq l \leq m_1$, it is possible to consider the liftings

$$\widetilde{\tau}_l(\vartheta, r) := (\vartheta + 2\pi(\mathrm{Rot}\,\Pi(\vartheta, r) - l)\,,\ \|\,\tau(\Pi(\vartheta, r))\,\|\,)\,.$$

Since $z(t; z_0) \neq (0,0)$ for every $t \in [0,T]$ and for every $z_0 \in \mathcal{A}$, the liftings are well defined. We note that as a consequence of i), $(0,0)$ belongs to the image of the interior of C_1 and also that

$$\begin{aligned} \mathrm{Rot}(z_0) - l \geq \mathrm{Rot}(z_0) - m_1 > 0 \qquad & \forall\, z_0 \in C_1, \\ \mathrm{Rot}(z_0) - l \leq \mathrm{Rot}(z_0) - m_2 < 0 \qquad & \forall\, z_0 \in C_2. \end{aligned}$$

Hence we can easily conclude that assumptions (a) and (b) in Theorem 4 are satisfied. Moreover it is easy to show that also assumption (c) is verified. Hence, from Theorem 4 we infer the existence of at least two fixed points (ϑ_l^i, r_l^i), $i = 1, 2$, of $\widetilde{\tau}_l$ whose images z_l^i under the projection Π are two different fixed points of τ. Since (ϑ_l^i, r_l^i) are fixed points of $\widetilde{\tau}_l$, we get that $\mathrm{Rot}(z_l^i) = l$ for every $i \in \{1, 2\}$. We can finally conclude that $z(\cdot; z_l^1)$ and $z(\cdot; z_l^2)$ are the searched $T-$periodic solutions.

□

There are many examples in the literature of the application of the Poincaré-Birkhoff theorem in order to study the existence and multiplicity of $T-$periodic solutions of the equation

$$x'' + f(t,x) = 0\,, \tag{18}$$

with $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuous and T-periodic in t. Note that if we consider the system

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -f(t, x(t)), \end{cases} \tag{19}$$

this system is a particular case of system (16) and its solutions give rise to solutions of equation (18). Hence we can consider equation (18) as a particular case of an Hamiltonian system and everything we mentioned above holds for the case of this equation.

Among the mathematicians who studied existence and multiplicity of periodic solutions for equation (18) via the Poincaré-Birkhoff theorem, we quote Jacobowitz [22], Hartman [20], Butler [7]. We remark that in order to reach the results, in all of these papers the authors assumed the validity of the condition $f(t,0) \equiv 0$.

With respect to the particular case of the nonlinear Duffing's equation

$$x'' + g(x) = p(t),$$

we mention the papers [15], [11], [13], [12], [17], [32], in which the Poincaré-Birkhoff theorem was applied in order to prove the existence of periodic solutions with prescribed nodal properties. Among the applications of the Poincaré-Birkhoff theorem to the analysis of periodic solutions to nonautonomous second order scalar differential equations depending on a real parameter s, we refer to the paper [10] by Del Pino, Manásevich and Murua, which studies the following equation

$$x'' + g(x) = s(1 + h(t))$$

and also the paper [30] by Rebelo and Zanolin, which deals with the equation

$$x'' + g(x) = s + w(t,x).$$

Finally, we quote Hausrath, Manásevich [21] and Ding, Zanolin [14] for the treatment of periodically perturbed Lotka-Volterra systems of type

$$\begin{cases} x' = x\,(a(t) - b(t)\,y) \\ y' = y\,(-d(t) + c(t)\,x). \end{cases}$$

We describe now recent results obtained in [24] in which a modified version of the Poincaré-Birkhoff fixed point theorem is obtained and applied, together with the classical one, in order to obtain existence and multiplicity of periodic solutions for Hamiltonian systems. In their paper the authors study system (16) assuming that $z = 0$ is an equilibrium point, i.e. $H'_z(t,0) \equiv 0$, and that it is an asymptotically linear Hamiltonian system. This implies that it admits linearizations at zero and infinity. More precisely, as $H'_z(t,0) \equiv 0$, if we consider the continuous and $T-$periodic function with range in the space of symmetric matrices given by $t \to B_0(t) := H''_z(t,0),\ t \in \mathbb{R}$, we have

$$J\,H'_z(t,z) = J\,B_0(t)\,z + o(\|z\|), \qquad \text{when } z \to 0.$$

Moreover, by definiton of asymptotically linear system, there exists a continuous, $T-$periodic function $B_\infty(\cdot)$ such that $B_\infty(t)$ is a symmetric matrix for each $t \in \mathbb{R}$, satisfying

$$J\,H'_z(t,z) = J\,B_\infty(t)\,z + o(\|z\|), \qquad \text{when } \|z\| \to \infty.$$

We remark that system (16) can be equivalently written in the following way

$$z' = J\,H'_z(t,z)\,, \qquad z = (x,y)\,, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{20}$$

Before going on with the description of the results obtained in [24], we recall some results present in the literature dealing with the study of asymptotically linear Hamiltonian systems.
In [2] and [3], Amann and Zehnder considered asymptotically linear systems in $\mathbb{R}^{2N}$ of the form of system (20) with

$$\sup_{t\in[0,T],\, z\in\mathbb{R}^{2N}} \|H''_z(t,z)\| < +\infty$$

and which admit autonomous linearizations at zero and at infinity

$$z' = J\,B_0\,z\,, \qquad z' = J\,B_\infty\,z\,,$$

respectively. In these papers an index i depending on B_0 and B_∞ was introduced and the existence of at least one nontrivial $T-$periodic solution combining nonresonance conditions at infinity with the sign assumption $i > 0$ was achieved. The authors also remarked that in the planar case $N = 1$ the condition $i > 0$ corresponds to the twist condition in the Poincaré-Birkhoff theorem.

Some years later, Conley and Zehnder studied in [9] Hamiltonian systems with bounded Hessian, considering the general case in which the linearized systems at zero and at infinity

$$z' = J\,B_0(t)z \qquad \text{and} \qquad z' = J\,B_\infty(t)z$$

can be nonautonomous. The authors assumed nonresonance conditions for the linearized systems at zero and at infinity. Hence, after defining the Maslov indices associated to the above linearizations at zero and infinity, denoted respectively by i_T^0 and i_T^∞, they proved the following result.

THEOREM 9. *If $i_T^0 \neq i_T^\infty$, then there exists a nontrivial $T-$periodic solution of $z' = J\,H_z'(t,z)$. If this solution is nondegenerate, then there exists another $T-$periodic solution.*

Note that in this last theorem the existence of more than two solutions is not guaranteed, even if $|i_T^0 - i_T^\infty|$ is large. This is in contrast with the fact that in the paper [9] and for the case $N = 1$ the authors mention that the Maslov index is a measure of the twist of the flow. In fact, if this is the case, a large $|i_T^0 - i_T^\infty|$ should imply large gaps between the twists of the flow at the origin and at infinity. Hence, the Poincaré-Birkhoff theorem would provide the existence of a large number of periodic solutions.

The main goal in [24] consists in clarifying the relation between i_T^0, i_T^∞ and the twist condition in the Poincaré-Birkhoff theorem, when $N = 1$, obtaining multiplicity results in the case when $|i_T^0 - i_T^\infty|$ is large.

Now we give a glint of the notion of Maslov index in the plane. We will follow [1] (see also [19]).
Let us consider the following planar Cauchy problem

$$\begin{cases} z' = J\,B(t)z \\ z(0) = w, \end{cases} \tag{21}$$

where $B(t)$ is a T-periodic continuous path of symmetric matrices. The matrix $\Psi(t)$ is called the fundamental matrix of the system (21) if it satisfies $\Psi(t)\,w = z(t; w)$. Clearly, $\Psi(0) = \mathrm{Id}$. Moreover, it is well known that as $B(t)$ is symmetric, the fundamental matrix $\Psi(t)$ is a *symplectic matrix* for each $t \in [0, T]$. We recall that a matrix A of order two is symplectic if it verifies

$$A^T\,J\,A = J, \tag{22}$$

where J is as in (20). Since we are working in a planar setting, condition (22) is equivalent to

$$\det A = 1\,,$$

from which it follows immediately that the symplectic 2×2 matrices form a group, usually denoted by $Sp(1)$.

We will show that, under a nonresonance condition on (21), it is possible to associate to the path $t \to \Psi(t)$ of symplectic matrices with $\Psi(0) = \mathrm{Id}$ an integer, the $T-$Maslov index $i_T(\Psi)$.
The system $z' = J\,B(t)z$ is said to be $T-$*nonresonant* if the only T-periodic solution it admits is the trivial one or, equivalently, if

$$\det(\mathrm{Id} - \Psi(T)) \neq 0\,,$$

where Ψ is the fundamental matrix of (21).

Before introducing the Maslov indices, we need to recall some properties of $Sp(1)$. If we take $A \in Sp(1)$, then A can be uniquely decomposed as

$$A = P \cdot O\,,$$

where $P \in \{\widetilde{P} \in Sp(1) : \widetilde{P}$ is symmetric and positive definite$\} \approx \mathbb{R}^2$ and O is symplectic and orthogonal. In particular, O belongs to the group of the rotations $SO(2) \approx S^1$. Thus we can conclude that

$$Sp(1) \approx \mathbb{R}^2 \times S^1 \approx \{z \in \mathbb{R}^2 : |z| < 1\} \times S^1 = \text{ the interior of a torus.}$$

Hence, as $[0,1) \times \mathbb{R} \times \mathbb{R}$ is a covering space of the interior of the torus, we can parametrize $Sp(1)$ with $(r,\sigma,\vartheta) \in [0,1) \times \mathbb{R} \times \mathbb{R}$. In [1] a parametrization

$$\begin{array}{rccl} \Phi : & [0,1)\times\mathbb{R}\times\mathbb{R} & \longrightarrow & Sp(1) \\ & (r,\sigma,\vartheta) & \to & \Phi(r,\sigma,\vartheta) = P(r,\sigma)\,R(\vartheta) \end{array}$$

is given, where ϑ is the angular coordinate on S^1 and (r,σ) are polar coordinates in $\{z \in \mathbb{R}^2 : |z| < 1\}$. In such a parametrization, for each $k \in \mathbb{Z}$ and $\sigma \in \mathbb{R}$, $\Phi(0,\sigma,2k\pi) = \mathrm{Id}$ and $\Phi(0,\sigma,2(k+1)\pi) = -\mathrm{Id}$ (for the details see [1]). The following sets are essential in order to define the $T-$Maslov index:

$$\begin{aligned} \Gamma^+ : &= \{A \in Sp(1) : \det(\mathrm{Id} - A) > 0\} \\ &= \Phi\{(r,\sigma,\vartheta) : r < \sin^2\vartheta \text{ and } |\vartheta| < \frac{\pi}{2} \quad \text{or} \quad |\vartheta| \geq \frac{\pi}{2}\}\,, \end{aligned}$$

$$\Gamma^- := \{A \in Sp(1) : \det(\mathrm{Id}-A) < 0\} = \Phi\{(r,\sigma,\vartheta) : r > \sin^2\vartheta \text{ and } |\vartheta| < \frac{\pi}{2}\}\,,$$

$$\Gamma^0 := \{A \in Sp(1) : \det(\mathrm{Id}-A) = 0\} = \Phi\{(r,\sigma,\vartheta) : r = \sin^2\vartheta \text{ and } |\vartheta| < \frac{\pi}{2}\}\,.$$

The set Γ^0 is called the resonant surface and it looks like a two-horned surface with a singularity at the identity.

Now we are in position to associate to each path $t \to \Psi(t)$ defined from $[0,T]$ to $Sp(1)$, satisfying $\Psi(0) = \mathrm{Id}$ and $\Psi(T) \notin \Gamma^0$ an integer which will be called the Maslov index of Ψ. To this aim we extend such a path $t \to \Psi(t) \in Sp(1)$ in $[T, T+1]$, without intersecting Γ^0 and in such a way that

- $\Psi(T+1) = -\mathrm{Id}$, if $\Psi(T) \in \Gamma^+$,

- $\Psi(T+1)$ is a standard matrix with $\vartheta = 0$, if $\Psi(T) \in \Gamma^-$.

We define the *T-Maslov index* $i_T(\Psi)$ as the (integer) number of half turns of $\Psi(t)$ in $Sp(1)$, as t moves in $[0, T+1]$, counting each half turn ± 1 according to its orientation.

In order to compare Theorem 8 with Theorem 9 it is necessary to find a characterization of the Maslov indices in terms of the rotation numbers. To this aim, in [24] a lemma which provides a relation between the $T-$Maslov index of system

$$z' = J\, B(t) z \tag{23}$$

and the rotation numbers associated to the solutions of (23) was given.

LEMMA 2. *Let Ψ be the fundamental matrix of system* (23) *and let i_T and ψ be, respectively, its T-Maslov index and the Poincaré map defined by*

$$\psi \,:\, w \;\to\; \Psi(T)w\,.$$

Consider the $T-$rotation number $\mathrm{Rot}_w(T)$ *associated to the solution of* (23) *satisfying $z(0) = w \in S^1$. Then,*
a) $i_T = 2\ell + 1$ with $\ell \in \mathbb{Z}$ if and only if

$$\deg(\mathrm{Id} - \psi, B(1), 0) = 1 \textit{ and } \ell < \min_{w \in S^1} \mathrm{Rot}_w(T) \le \max_{w \in S^1} \mathrm{Rot}_w(T) < \ell + 1;$$

b) $i_T = 2\ell$ with $\ell \in \mathbb{Z}$ if and only if

$$\deg(\mathrm{Id} - \psi, B(1), 0) = -1 \textit{ and } \ell - \frac{1}{2} < \min_{w \in S^1} \mathrm{Rot}_w(T) \le \max_{w \in S^1} \mathrm{Rot}_w(T) < \ell + \frac{1}{2};$$

moreover, in this case there are $w_1, w_2 \in S^1$ such that

$$\mathrm{Rot}_{w_1}(T) < \ell < \mathrm{Rot}_{w_2}(T)\,.$$

In the statement of Lemma 2 the $T-$rotation number $\mathrm{Rot}_w(T)$ associated to the solution of (23) with $z(0) = w \in S^1$ was considered. We observe that from the linearity of system (23) it follows that $\mathrm{Rot}_w(T) = \mathrm{Rot}_{\lambda w}(T)$ for every $\lambda > 0$.
Now, we are in position to make a first comparison between Theorem 8 and Theorem 9.
Let us consider the second order scalar equation

$$x'' + q(t,x)x = 0\,, \tag{24}$$

where the continuous function $q : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is $T-$periodic in its first variable t and it satisfies

$$q(t,0) \equiv q_0 \in \mathbb{R} \quad \text{and} \quad \lim_{|x| \to +\infty} q(t,x) = q_\infty \in \mathbb{R}, \tag{25}$$

uniformly with respect to $t \in [0, T]$. Hence, the linearizations of (24) at zero and infinity are respectively $x'' + q_0 x = 0$ and $x'' + q_\infty x = 0$. We observe that equation (24) can be equivalently written in the following form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -q(t,x)x \end{pmatrix} = J \begin{pmatrix} q(t,x)x \\ y \end{pmatrix}.$$

Analogously, the corresponding linearizations at zero and infinity are given respectively by

$$z' = J \begin{pmatrix} q_0 & 0 \\ 0 & 1 \end{pmatrix} z = \begin{pmatrix} 0 & 1 \\ -q_0 & 0 \end{pmatrix} z$$

and

$$z' = J \begin{pmatrix} q_\infty & 0 \\ 0 & 1 \end{pmatrix} z = \begin{pmatrix} 0 & 1 \\ -q_\infty & 0 \end{pmatrix} z.$$

If we choose $q_0 = -\frac{1}{2}$ and $q_\infty = 5$, we can easily deduce that there exists $r_0 > 0$ such that $\mathrm{Rot}_w(T) \in (-\frac{1}{2}, \frac{1}{2})$ for every $\|w\| = r_0$ and there exists $R_0 > r_0$ such that $\mathrm{Rot}_w(T) \in (-3, -2)$ for every $\|w\| = R_0$. Hence applying Theorem 8 we can guarantee the existence of four nontrivial T-periodic solutions to equation (24). On the other hand, since $i_T^0 = 0$ and $i_T^\infty = -5$, Theorem 9 ensures the existence of at least one periodic solution.

We recall that, even if the gap between q_0 and q_∞ is large, Theorem 9 guarantees only the existence of at least one solution (or at least two solutions if the first one is nondegenerate) while it is quite clear that the number of nontrivial periodic solutions we can find by applying Theorem 8 depends on the gap between q_0 and q_∞.

On the other hand, there are particular situations in which Theorem 9 can be applied, while Theorem 8 cannot, because the twist condition is not satisfied.

For instance, let us set $q_0 = -\frac{1}{2}$ and $q_\infty = \frac{1}{2}$. The corresponding indices are different, since $i_T^0 = 0$, as before, and $i_T^\infty = -1$. Hence, from Theorem 9, we know that there exists a periodic solution of (24). As far as the rotation numbers are concerned, one can prove the existence of $R_0 > r_0$ such that $\mathrm{Rot}_w(T) \in (-1, 0)$ for every $\|w\| = R_0$; while, from Lemma 2, there exist $w_1,\ w_2 \in \mathbb{R}^2$ with $\|w_i\| = r_0$ $(i = 1, 2)$ such that $-\frac{1}{2} < \mathrm{Rot}_{w_1}(T) < 0 < \mathrm{Rot}_{w_2}(T) < \frac{1}{2}$. Consequently, the twist condition is not verified and Theorem 8 is not applicable. In [24] the authors tried to sharpen the results obtained via the Poincaré-Birkhoff theorem in order to obtain periodic solutions in cases like this one. For this purpose, they developed a suitable version of the Poincaré-Birkhoff theorem. Before describing this result we can obtain a first result of multiplicity of T-periodic solutions for system (16) which is a consequence of Lemma 2 and of Theorem 8.

We will use the notation: for each $s \in \mathbb{R}$, we denote by $\lfloor s \rfloor$ the integer part of s, while we denote by $\lceil s \rceil$ the smallest integer larger than or equal to s.

COROLLARY 1. *Assume that $z' = J\,H_z'(t,z)$ is asymptotic at infinity and at zero to the $T-$periodic and $T-$nonresonant linear systems $z' = J\,B_\infty(t)z$ and $z' = J\,B_0(t)z$,* *respectively. Let* i_T^∞ *and* i_T^0 *be the corresponding* $T-$*Maslov indices. If* $i_T^0 \neq i_T^\infty$ *then the Hamiltonian system admits at least*

- $|i_T^\infty - i_T^0|$ *nontrivial* $T-$*periodic solutions if* i_T^0 *and* i_T^∞ *are odd;*
- $|i_T^\infty - i_T^0| - 2$ *nontrivial* $T-$*periodic solutions if* i_T^0 *and* i_T^∞ *are even;*
- $2\left\lfloor \dfrac{|i_T^\infty - i_T^0|}{2} \right\rfloor$ *nontrivial* $T-$*periodic solutions otherwise.*

REMARK 4. If i_T^0 and i_T^∞ are either consecutive integers or consecutive even integers, the previous corollary does not guarantee the existence of $T-$periodic solutions. Indeed in these cases the twist condition in Theorem 8 is not satisfied.
However, if i_T^0 and i_T^∞ are consecutive integers the excision property of the degree implies the existence of a $T-$periodic solution.

THEOREM 10 (MODIFIED POINCARÉ-BIRKHOFF THEOREM). *Let* $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ *be an area-preserving homeomorphism in* $\mathcal{A} = \mathbb{R} \times [0, R]$, $R > 0$ *such that*

$$\psi(\vartheta, r) = (\vartheta_1, r_1),$$

with

$$\begin{cases} \vartheta_1 = \vartheta + g(\vartheta, r) \\ r_1 = f(\vartheta, r)\,, \end{cases}$$

where f *and* g *are* $2\pi-$*periodic in the first variable and satisfy the conditions*

- $f(\vartheta, 0) = 0,\ f(\vartheta, R) = R$ *for every* $\vartheta \in \mathbb{R}$ (*boundary invariance*),
- $g(\vartheta, R) > 0$ *for every* $\vartheta \in \mathbb{R}$ *and there is* $\overline{\vartheta}$ *such that* $g(\overline{\vartheta}, 0) < 0$ (*modified twist condition*).

Then, ψ *admits at least a fixed point in the interior of* $\mathcal{A}$*. If* ψ *admits only one fixed point in the interior of* $\mathcal{A}$*, then its fixed point index is nonzero.*

Idea of the proof. By contradiction, it is assumed that there are no fixed points in the interior of $\mathcal{A}$. As in the proof of Theorem 7, the homeomorphism ψ is extended to an homeomorphism $\widehat{\psi} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. If the fixed point set of $\widehat{\psi}$ is not empty, it is union of vertical closed halflines in the halfplane $r \leq 0$ with origin on the line $r = 0$.
Without loss of generality, one can assume $\overline{\vartheta} \in (0, 2\pi)$. Hence, denoting by $\mathcal{N}$ the maximal strip contained in $\mathbb{R}\times]-\infty, 0]$ such that $(\overline{\vartheta}, 0) \in \mathcal{N}$ and $g(\vartheta, 0) < 0$ in $\mathcal{N}$, the following important property of $\mathcal{N}$ holds:
if $(\vartheta, r) \in \cup_{k\in\mathbb{Z}}(\mathcal{N} + (2k\pi, 0))$, then for each $n > 0$ we have that $\widehat{\psi}^{-n}(\vartheta, r)$ belongs to the connected component of $\cup_{k\in\mathbb{Z}}(\mathcal{N} + (2k\pi, 0))$ which contains (ϑ, r).
Then the proof follows steps analogous to those in Theorem 7, taking into account this property. The contradiction follows from the existence of a curve Γ with $i_{\widehat{\psi}}(\Gamma) =$

$-\frac{1}{2}$, which runs from the point $(\overline{\vartheta}, 0)$ to a point into $\{(\vartheta, r) : r \geq R\}$ and such that $i_{\widehat{\psi}^{-1}}(\widehat{\psi}(\Gamma)) = \frac{1}{2}$.

The fact that if ψ admits only one fixed point in the interior of $\mathcal{A}$, then its fixed point index is nonzero can be proved following similar steps to those in the proof of Remark 3. Now it will be important to take into account the property of $\mathcal{N}$ mentioned above.

□

At this point, the authors in [24] obtain a variant of Theorem 10 in which the invariance of the outer boundary is not assumed. The proof of this Corollary follows the same steps as the proof of Theorem 1 in [29].

Let Γ_1 be a circle with center in the origin and radius $R > 0$ and Γ_2 be a simple closed curve surrounding the origin. For each $i \in \{1, 2\}$ we denote by $\mathcal{B}_i$ the finite closed domain bounded by Γ_i. Let $\widetilde{\Gamma}_i$ be the lifting of Γ_i and $\widetilde{A}_i$ be the lifting of $\mathcal{A}_i$, where $\mathcal{A}_i := \mathcal{B}_i \setminus \{0\}$. Then, the following result holds.

COROLLARY 2. *Let $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ be an area-preserving homeomorphism. Assume that ψ admits a lifting which can be extended to an homeomorphism $\widetilde{\psi} : \widetilde{\mathcal{A}}_1 \cup \{(\vartheta, r) : r = 0\} \to \widetilde{\mathcal{A}}_2 \cup \{(\vartheta, r) : r = 0\}$ given by $\widetilde{\psi}(\vartheta, r) = (\vartheta + g(\vartheta, r), f(\vartheta, r))$, where g and f are $2\pi-$periodic in the first variable. Moreover, suppose that $g(\vartheta, r) > 0$ for every $(\vartheta, r) \in \widetilde{\Gamma}_1$ and there is $\overline{\vartheta}$ such that $g(\overline{\vartheta}, 0) < 0$ (modified twist condition). Then, $\widetilde{\psi}$ admits at least a fixed point in the interior of $\widetilde{\mathcal{A}}_1$ whose image under the usual covering projection Π is a fixed point of ψ in $\mathcal{A}_1$. If ψ admits only one fixed point in the interior of $\mathcal{A}_1$, then its fixed point index is nonzero.*

We point out that the proof of Corollary 2 cannot be repeated if we modify the twist condition by supposing that there is $(\overline{\vartheta}, \overline{r}) \in \widetilde{\Gamma}_1$ such that $g(\overline{\vartheta}, \overline{r}) > 0$, while $g(\vartheta, 0) < 0$ for every $\vartheta \in \mathbb{R}$.

Now, we will show how the application of Theorem 10 to the scalar equation (24) can improve the multiplicity results achieved by applying Theorem 8 and Theorem 9.

First let us set once again $q_0 = -\frac{1}{2}$ and $q_\infty = \frac{1}{2}$ in (25). By the modified Poincaré-Birkhoff theorem, there is a fixed point P_0 of ϕ (that corresponds to a nontrivial $T-$periodic solution) and, if it is the unique fixed point, then

$$ind(P_0) \neq 0\,.$$

We recall that Simon in [31] has shown that an isolated fixed point of an area-preserving homeomorphism in $\mathbb{R}^2$ has index less than or equal to 1. In particular, the fixed point index of P_0 satisfies

$$ind(P_0) \leq 1\,.$$

As the fixed point index of ψ changes from -1 (near the origin) to $+1$ (near infinity), there is at least another fixed point P_1 of ϕ. Hence, in this case we can guarantee the existence of at least two nontrivial $T-$periodic solutions. We recall that applying Theorem 9 only the existence of one nontrivial periodic solution could be guaranteed.

Now we choose $q_0 = -\frac{1}{2}$ and $q_\infty = 3$ in (25). The Poincaré-Birkhoff theorem, according to Theorem 8, guarantees that there are at least two fixed points P_1 and P_2 of ϕ (which correspond to two $T-$periodic solutions with rotation number -1). If they are unique, then from Remark 3 we obtain that

$$ind(P_1) = +1\,, \qquad ind(P_2) = -1\,.$$

Moreover, by the modified Poincaré-Birkhoff theorem there is a fixed point P_3 of ϕ (which corresponds to a $T-$periodic solution with rotation number 0) and, if it is unique,

$$0 \neq ind(P_3) \leq 1\,.$$

As the fixed point index of ϕ changes from -1 (near the origin) to $+1$ (near infinity), there exists at least a fourth fixed point P_4 of ϕ. Summarizing, Theorem 8 combined with the modified Poincaré-Birkhoff theorem guarantees the existence of at least four nontrivial $T-$periodic solutions to (24). We recall that also in this case Theorem 9 is applicable and it ensures that there exists at least one nontrivial periodic solution.

Finally, we state the main multiplicity theorem. We point out that the multiplicity results achieved in the above examples can be also obtained by applying the following theorem.

THEOREM 11. *Assume that the conditions of Corollary* 1 *hold.*
Then if $i_T^0 \neq i_T^\infty$ *the Hamiltonian system* (16) *admits at least* $\max\left\{1, 2\left\lfloor \frac{|i_T^\infty - i_T^0|}{2} \right\rfloor\right\}$ *nontrivial* $T-$*periodic solutions.*
If i_T^0 *is even then the Hamiltonian system admits at least* $2\left\lceil \frac{|i_T^\infty - i_T^0|}{2} \right\rceil$ *nontrivial* $T-$*periodic solutions.*

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DIFFERENTIAL EQUATIONS WITH INDEFINITE WEIGHT: BOUNDARY VALUE PROBLEMS AND QUALITATIVE PROPERTIES OF THE SOLUTIONS

Abstract. We describe the qualitative properties of the solutions of the second order scalar equation $\ddot{x} + q(t)g(x) = 0$, where q is a changing sign function, and consider the problem of existence and multiplicity of solutions which satisfy various different boundary conditions. In particular we outline some difficulties which arise in the use of the shooting approach.

1. Introduction

We discuss the second-order scalar nonlinear ordinary differential equation:

$$\ddot{x} + q(t)g(x) = 0, \tag{1}$$

where:

- $g : \mathbb{R} \to \mathbb{R}$ is continuous (maybe locally Lipschitz continuous on $\mathbb{R}$ or on $\mathbb{R} \setminus \{0\}$) and such that $g(s) \cdot s > 0$ for every $s \neq 0$
- the "weight" $q : \mathbb{R} \to \mathbb{R}$ is continuous (sometimes more stronger regularity assumptions will be needed and, in some applications, like the two point boundary value problem, it will be enough that q is defined in an interval I).

EXAMPLE 1. A simple case of (1) is the nonlinear Hill's equation:

$$\ddot{x} + q(t)|x|^{\gamma-1}x = 0, \qquad \gamma > 0 \tag{2}$$

(recall that the classical Hill's equation is the one with $\gamma = 1$).

The expression "indefinite weight" means that the function q changes sign.

Waltman [86] in a paper of 1965 studied the oscillating solutions of

$$\ddot{x} + q(t)x^{2n+1} = 0 \qquad n \in \mathbb{N},$$

*Lecture notes of the course held by F. Zanolin during the "Third Turin Forthnight on Nonlinear Analysis", Sept. 23 – Oct. 6 2001, Turin, Italy when the weight q is allowed to change sign. Many authors studied the oscillatory properties of equations like (1): Bhatia [16], Bobisud [17], Butler [19, 20], Kiguradze [52], Kwong and Wong [54], Onose [68], Wong [94, 95, 96].

The existence of periodic solutions for a large class of equations (including (2)) was considered by Butler: in the superlinear case ($\gamma > 1$) he found infinitely many "large solutions" [22] while in the sublinear one ($\gamma < 1$) he found infinitely many "small solutions" [23]. In both cases there are periodic solutions with an arbitrarily large number of zeros and Butler's results, which are valid with respect to a quite wide class of nonlinearities, have been improved in the superlinear case by Papini in [69, 70] and in the sublinear case by Bandle Pozio and Tesei [12], for the existence of small solutions, and by Liu and Zanolin [59] for what concerns large solutions.

Recently, many authors considered generalizations of (1) both in the direction of Hamiltonian systems (with respect to the problem of finding periodic or homoclinic solutions) and elliptic partial differential equations with Dirichlet boundary conditions. In particular Hamiltonian systems with changing sign weights were studied by Lassoued [55, 56], Avila and Felmer [10], Antonacci and Magrone [9], Ben Naoum, Troestler and Willem [13], Caldiroli and Montecchiari [25], Fei [37], Ding and Girardi [33], Girardi and Matzeu [41], Le and Schmitt [57], Liu [58], Schmitt and Wang [76], Felmer and Silva [39], Felmer [38], Ambrosetti and Badiale [8], Jiang [50]. On the other hand, the partial differential case was developed by Alama and Del Pino [1], Alama and Tarantello [2, 3], Amann and Lopez-Gómez [7], Badiale and Nabana [11], Berestycki, Capuzzo-Dolcetta and Nirenberg [14, 15], Khanfir and Lassoued [51], Le and Schmitt [57], Ramos, Terracini and Troestler [74]. Equations of the form:

$$\ddot{x} + q(t)x^{2n+1} = m(t)x + h(t),$$

with a changing sign q, were considered by Terracini and Verzini in [85] paired with either Dirichlet or periodic boundary conditions. They applied a suitable version of the Nehari method [67] in order to find solutions of the boundary value problem with prescribed nodal behavior. More precisely, if the domain $[0, T]$ of q is decomposed into the union of consecutive and adjacent closed intervals $I_1^+, I_1^-, I_2^+, I_2^-, \dots, I_k^+$ such that:

$$q \geq 0,\ q \not\equiv 0 \quad \text{in } I_i^+ \qquad \text{and} \qquad q \leq 0,\ q \not\equiv 0 \quad \text{in } I_i^-,$$

then they found k natural numbers $m_1^*, \dots, m_k^*$, one for each interval of positivity I_i^+, in such a way that, for every choice of k natural numbers $m_1, \dots, m_k$, with $m_i \geq m_i^*$ for all $i = 1, \dots, k$, there are two solutions of the boundary value problem which have exactly m_i zeros in I_i^+ and one zero in I_i^-.

An analogous situation was considered in [70, 71, 72] where, via a shooting approach, boundary value problems associated to (1) were studied, with a general nonlinearity g which has to be superlinear at infinity in some sense. In this case, after having arbitrarily chosen the natural numbers $m_i \geq m_i^*$ and a $(k-1)$-tuple $(\delta_1, \dots, \delta_{k-1})$, with $\delta_i \in \{0, 1\}$, we found two solutions with m_i zeros in I_i^+ and δ_i zeros in I_i^-.

On the other hand Capietto, Dambrosio and Papini [26] focused their attention on the existence of globally defined solutions of (1) with prescribed nodal behavior again in the case of g superlinear at infinity and q changing sign. They showed that the Poincaré map associated to (1) exhibits chaotic features.

It is the aim of these lectures to discuss some qualitative properties of the solutions and some difficulties which arise in the use of the shooting approach.

2. The shooting method

Equation (1) can be written as a first order system in the phase plane:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(t)g(x) \end{cases} \tag{3}$$

If we assume that the uniqueness for the Cauchy problems for (3) holds, then we denote by $z(t; t_0, p) = (x(t; t_0, p), y(t; t_0, p))$ the solution of (3) with $z(t_0; t_0, p) = p = (x_0, y_0) \in \mathbb{R}^2$. The shooting method is based on the theorem on the continuous dependence of the solutions with respect to the initial data: if $z(t; t_0, p)$ is defined on an interval $[\alpha, \beta] \ni t_0$ for some $t_0 \in \mathbb{R}$ and some $p \in \mathbb{R}^2$, then $z(t; t_0, p_1)$ is defined on $[\alpha, \beta]$ for each p_1 "near" p and we have that $z(\cdot; t_0, p_1) \to z(\cdot; t_0, p)$ uniformly on $[\alpha, \beta]$ as $p_1 \to p$.

Therefore there is a couple of problems if we wish to apply this method for the study of boundary value problems associated to (1) and (3). The first one is about the uniqueness, which is granted whenever q is locally integrable and g is locally Lipschitz continuous: in particular, if g behaves like $|x|^{\gamma-1}x$ near zero and $0 < \gamma < 1$, we might loose the uniqueness at zero.

The second problem is the global existence of the solutions, since the sole continuity of q does not imply that all the maximal solutions of (1) are globally defined, even if q is assumed to be greater than a positive constant, as shown by Coffman and Ullrich in [28]. Indeed they produce a weight $q(t) = 1 + \delta(t)$, with a function $\delta : [0, +\infty[\to \mathbb{R}$ which is positive and continuous, but has unbounded variation in every left neighborhood of some $\hat{t} > 0$, and they show that the equation:

$$\ddot{x} + (1 + \delta(t))x^3 = 0$$

has a solution which starts from $t_0 = 0$ and blows up as t tends to $\hat{t}$ from the left. On the other hand they prove that, if q is positive, continuous and has bounded variation in an interval $[a, b]$, then every solution of:

$$\ddot{x} + q(t)x^{2n+1} = 0$$

has $[a, b]$ as maximal interval of definition. If we consider a positive weight q which is continuously differentiable on $[a, b]$ and a function g such that $g(x) \cdot x > 0$ for $x \neq 0$, it is not difficult to show that the same conclusion holds for (1). Indeed, let us consider a solution x of (1) starting from $t = a$ and define the auxiliary function:

$$v(t) = \frac{1}{2}\dot{x}^2(t) + q(t)G(x(t)),$$

where $G(x) = \int_0^x g(s)ds$ is nonnegative by the sign assumption on g. The function v is surely defined in an interval $J \subseteq [a, b]$ with a as left end-point. For every t in J we have:

$$\begin{aligned} \dot{v}(t) &= \ddot{x}(t)\dot{x}(t) + q(t)g(x(t))\dot{x}(t) + \dot{q}(t)G(x(t)) \\ &= \dot{q}(t)G(x(t)). \end{aligned}$$

Since q is strictly positive and $\dot{q}$ is continuous on $[a, b]$ there is a constant $M \geq 1$ such that:

$$\dot{q}(t) \leq Mq(t) \qquad \forall\, t \in [a, b],$$

so that we obtain:

$$\dot{v}(t) \leq Mq(t)G(x(t)) \leq Mv(t) \qquad \forall\, t \in J.$$

Hence v satisfies the inequality:

$$v(t) \leq Me^{M(t-a)} \qquad \forall\, t \in J$$

and turns out to be bounded in J. This implies that $|\dot{x}(t)|$ and, therefore, $|x(t)|$ are bounded, too, and, thus, x must be defined up to b.

The argument just employed can be modified in order to cover also some cases in which q is nonnegative and vanishes somewhere. Indeed Butler observed that if one starts from $t = a$ then the solution is defined up to (and including) the first zero $t_0 > a$ of q provided that $\dot{q} \leq 0$ (or, more generally, q is decreasing) in a left neighborhood of t_0. Then the solution surely proceeds further t_0 simply by Peano's theorem about local existence. Similarly, if one looks for backward continuability, every solution starting from $t = b$ reaches the first zero $t_1 < b$ of q provided that q is monotone increasing in a right neighborhood of t_1. Therefore, if every interval $[a, b]$ in which q is nonnegative can be expressed as the union of a finite number of closed intervals (possibly degenerating to a single point) where q vanishes and of a finite number of open sub-intervals $]t_0, t_1[$, such that q is strictly positive in such intervals and is monotone increasing in a right neighborhood of t_0 and decreasing in a left neighborhood of t_1, then the argument above can be repeated a finite number of times in order to obtain the continuability of the solutions across $[a, b]$.

EXAMPLE 2. Let us see how the shooting method can be used to solve a Dirichlet boundary value problem associated to a superlinear Hill's equation like (2) with a nonnegative weight. To be precise we look for solutions of:

$$\begin{cases} \begin{array}{l} \dot{x} = y \\ \dot{y} = -q(t)|x|^{\gamma-1}x \end{array} \qquad t \in [0, T] \\ x(0) = x(T) = 0 \end{cases}$$

assuming that $\gamma > 1$ and that q is a nonnegative continuous function in $[0, T]$ which also satisfies the regularity assumptions discussed above in such a way that all the solutions of the differential equation are continuable along the interval $[0, T]$.

The idea is to consider all the solutions which have value 0 and slope $k \in \mathbb{R}$ for $t = 0$, that is $z(\cdot; 0, (0, k))$ with the notation previously introduced, and to determine for which values of k we have $x(T; 0, (0, k)) = 0$. In other words, we are considering the set of the solutions which start at $t = 0$ from the y-axis of the phase plane and we wish to select those which come back to the y-axis at $t = T$. One way to do this is to measure the angle spanned in the phase plane by the solution vector $z(t; 0, p)$ as t runs in $[0, T]$; indeed, if p lies on the y-axis, then $z(T; 0, p)$ is again on the x-axis if and only if the angle spanned by $z(t; 0, p)$ on $[0, T]$ is an integer multiple of π. Now, if $z(t) = (x(t), y(t))$ is a nontrivial solution of the differential equation, then $z(t) \neq (0, 0)$ for every $t \in [0, 1]$ by the uniqueness of the constant solution $(0, 0)$; hence we can define an angular function $\theta(t)$ such that:

$$x(t) = |z(t)| \cos\theta(t) \qquad \text{and} \qquad y(t) = |z(t)| \sin\theta(t)$$

and it is easy to see that it satisfies:

$$-\dot{\theta}(t) = \frac{y^2(t) + q(t)|x(t)|^{\gamma+1}}{y^2(t) + x^2(t)}.$$

Therefore the measure of the angle spanned by $z(t)$ can be obtained by integrating the last expression and it is given by:

$$\operatorname{rot}(p) = \frac{1}{\pi} \int_0^T \frac{y^2(t; 0, p) + q(t)|x(t; 0, p)|^{\gamma+1}}{y^2(t; 0, p) + x^2(t; 0, p)} dt.$$

Thus $z(\cdot; 0, (0, k))$ is a solution of the Dirichlet boundary value problem if and only if $\operatorname{rot}((0, k)) \in \mathbb{Z}$. Now, $\operatorname{rot}(p)$ is clearly a continuous function of p and in this case it can be proved that:

$$\operatorname{rot}(p) \to +\infty \qquad \text{as} \qquad |p| \to +\infty,$$

therefore, by the intermediate values theorem, our boundary value problem has infinitely many solutions. Moreover, the value $\operatorname{rot}(p)$ clearly gives information about how many times the curve $z(t; 0, p)$ crosses the y-axis in the phase plane as t runs from 0 to T and, more precisely, we have that if $\operatorname{rot}((0, k)) = j \in \mathbb{N}$ then $x(\cdot; 0, (0, k))$ has exactly j zeros in $[0, T[$.

The same technique can be used to solve Sturm–Liouville boundary value problems like the following one:

$$\begin{cases} \dot{x} = y & \qquad t \in [0, T] \\ \dot{y} = -q(t)|x|^{\gamma-1}x & \\ a_1 x(0) + b_1 y(0) = 0 & \\ a_2 x(T) + b_2 y(T) = 0 & \end{cases}$$

where $a_i^2 + b_i^2 \neq 0$, $i = 1, 2$, since the boundary conditions just mean that one looks for solutions which start at $t = 0$ on the straight line $a_1 x + b_1 y = 0$ in the phase plane and end at $t = T$ on the straight line $a_2 x + b_2 y = 0$.

On the other hand other boundary value problems, like the periodic one, are more difficult to be solved by the shooting method, as one has to use less trivial fixed points theorems.

What about if we do not have the continuability of the solutions? Hartman [44] avoids the use of the global continuability for an equation of the form:

$$\ddot{x} + f(t,x) = 0 \qquad \text{with} \qquad \lim_{x\to\pm\infty} \frac{f(t,x)}{x} = +\infty \text{ uniformly w.r.t. } t,$$

by assuming that:

$$f(t,0) \equiv 0 \qquad \text{and that} \qquad \frac{f(t,x)}{x} \text{ is bounded in a neighborhood of } x = 0.$$

The idea is that, if, on one hand, small solutions (that are those starting at a point suitably near to the origin of the phase plane) are continuable up to T by the theorem on continuous dependence on initial data, on the other, if a solution blows up before $t = T$, then it oscillates infinitely many times. Therefore $\mathrm{rot}(p)$ can be defined at least in a neighborhood of $p = (0,0)$, and it becomes unbounded either as $|p| \to +\infty$ for the superlinearity assumption on f or for those p's nearby some blowing up solution and, thus, the shooting argument can be still used.

Now we come to the general situation of (1). We denote by $G(x) = \int_0^x g(s)ds$ the primitive of the nonlinearity g and we assume that $G(x) \to +\infty$ as $s \to \pm\infty$. Let $G_l^{-1} : [0,+\infty[\to]-\infty, 0]$ and $G_r^{-1} : [0,+\infty[\to [0,+\infty[$ be, respectively, the left and the right inverse functions of G. We describe the phase plane portrait of two autonomous equations which model the situation of $q \geq 0$ and $q \leq 0$, respectively.

Consider a constant weight $q \equiv 1$; then equation (1) becomes:

$$\ddot{x} + g(x) = 0$$

or, equivalently:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) \end{cases} \tag{4}$$

Each non trivial solution (x, y) of (4) satisfies:

$$\frac{1}{2}y^2(t) + G(x(t)) = c \qquad \forall t$$

for some constant $c > 0$. Since the level sets of the function $(x, y) \mapsto \frac{1}{2}y^2 + G(x)$ are closed curves around the origin, every solution of (4) is periodic with a period $\tau^+(c)$ which depends only on the "energy" c of the solution and can be explicitly evaluated:

$$\tau^+(c) = \sqrt{2}\int_{G_l^{-1}(c)}^{G_r^{-1}(c)} \frac{ds}{\sqrt{c - G(s)}}, \qquad c > 0.$$

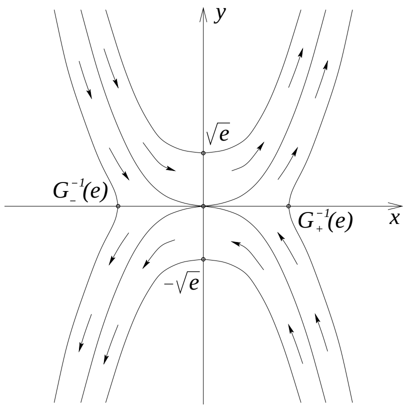

Figure 1: The phase portrait for (5) with $e = |c|$.

It is well known that the following facts hold:

- $\lim_{s\to\pm\infty} \dfrac{g(s)}{s} = +\infty \Longrightarrow \lim_{c\to+\infty} \tau^+(c) = 0$;
- if the ratio $g(s)/s$ *monotonically increases* to $+\infty$ as $s \to \pm\infty$ then $\tau^+(c)$ *monotonically decreases* to 0 as $c \to +\infty$.

On the other hand, if we take a constant weight $q \equiv -1$, then (1) becomes:

$$\ddot{x} - g(x) = 0$$

or, equivalently:

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x) \end{cases} \tag{5}$$

and each solution (x, y) of (5) satisfies:

$$\frac{1}{2}y^2(t) - G(x(t)) = c \qquad \forall t$$

for some real constant c.

The phase portrait is that of a saddle (see Figure 1) in which the four nontrivial and unbounded trajectories with "energy" $c = 0$ correspond to the stable (II and IV quadrants) and to the unstable (I and III quadrants) manifolds with respect to the only critical point $(0, 0)$. For each negative value c there are two unbounded trajectories with energy c : one of them lies in the half plane $x > 0$, crosses the positive x-axis at $(G_r^{-1}(-c), 0)$ and corresponds to convex and positive solutions x, and the other lies in $x < 0$, crosses the negative x-axis at $(G_l^{-1}(-c), 0)$ and corresponds to concave and negative solutions x. On the other hand, for each positive c there are two unbounded trajectories with energy c : one of them lies in $y > 0$, crosses the positive y-axis at $(0, \sqrt{2c})$ and corresponds to solutions x which are monotone increasing and have exactly one zero, while the other lies in $y < 0$, crosses the negative y-axis at $(0, -\sqrt{2c})$ and corresponds to solutions x which are monotone decreasing and have exactly one zero.

In this case we do not have any nontrivial periodic solution and, therefore, any period to evaluate; however, when g grows in a superlinear way towards infinity, *all* solutions with nonzero energy have a blow-up in finite time, both in the future and in the past (see [18]). Then we can compute the length of the maximal interval of existence of each trajectory and it turns out to be a function of the energy of the trajectory itself. Indeed, in the case of each of the two trajectories with positive energy c, that length is:

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \frac{ds}{\sqrt{c + G(s)}},$$

while for the trajectories with negative energy we have to distinguish between that on $x > 0$, whose maximal interval length is:

$$\sqrt{2} \int_{G_r^{-1}(-c)}^{+\infty} \frac{ds}{\sqrt{c + G(s)}},$$

and the other on $x < 0$, for which the length is:

$$\sqrt{2} \int_{-\infty}^{G_l^{-1}(-c)} \frac{ds}{\sqrt{c + G(s)}}.$$

If for every nonzero c we sum the length of the maximal intervals of the two corresponding trajectories, we obtain the following function:

$$\tau^-(c) = \begin{cases} \sqrt{2} \displaystyle\int_{-\infty}^{+\infty} \frac{ds}{\sqrt{c + G(s)}} & \text{if } c > 0 \\ \sqrt{2} \displaystyle\int_{-\infty}^{G_l^{-1}(-c)} \frac{ds}{\sqrt{c + G(s)}} + \sqrt{2} \int_{G_r^{-1}(-c)}^{+\infty} \frac{ds}{\sqrt{c + G(s)}} & \text{if } c < 0 \end{cases}$$

which, like τ^+, is infinitesimal for $c \to \pm\infty$ in the superlinear case:

$$\lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty, \ \left| \int^{\pm\infty} \frac{ds}{G(s)} \right| < +\infty, \ \liminf_{s \to +\infty} \frac{G(ks)}{G(s)} > 1 \Longrightarrow \lim_{c \to \pm\infty} \tau^-(c) = 0$$

(k is some constant larger than 1).

EXAMPLE 3. Consider again Hill's equation (2) with exponent $\gamma > 1$ and a piecewise constant weight function q which changes sign:

$$q(t) = \begin{cases} +1 & \text{if } 0 \le t \le t_0 \\ -1 & \text{if } t_0 < t \le T \end{cases}$$

for some $t_0 \in]0, T[$. Let us consider the behavior on $[0, T]$ of the solution x_k such that $x_k(0) = 0$ and $\dot{x}_k(0) = k > 0$ as the initial slope k increases. The problem is: is x_k defined on $[0, T]$ and which is its shape? Clearly a blow-up can appear only in $]t_0, T]$ and it depends on which trajectory of (5) the point $(x_k(t_0), \dot{x}_k(t_0))$ belongs to. Indeed we have that $\tau^-(c)$ tends to zero, as c tends to $\pm\infty$, for our $g(s) = |s|^{\gamma-1}s$ and, thus, all the orbits of (5) with an energy c such that $|c| \gg 0$ have a very small maximal interval of existence and are not defined on the whole $[t_0, T]$. On the other hand, all the solutions of (5) passing sufficiently near the two stable manifolds (that are the trajectories of (5) with zero energy which lie in the second and in the fourth quadrant) have a maximal interval of existence which is larger than $[t_0, T]$.

Now, we observe that all the trajectories of (4) intersect the stable manifolds of (5), but for some values of k the point $(x_k(t_0), \dot{x}_k(t_0))$ will be near to the stable manifolds, while for others it will lie far: it depends essentially on the value $\operatorname{rot}((0, k))$, that is on the measure of the angle spanned by the vector $(x(t), \dot{x}(t))$ as t goes from 0 to t_0. Since $\operatorname{rot}((0, k))$ tends to $+\infty$ together with k, it is possible to select a sequence of successive and disjoint intervals:

$$I_0 = [0, k_0[\,,\ I_1 =]h_1, k_1[\,, \ldots, I_j = \big]h_j, k_j\big[\,, \ldots$$

such that:

- if $k \in I_j$ then $(x_k(t_0), \dot{x}_k(t_0))$ lies near the stable manifolds of (5) and, hence, x_k is defined on $[0, T]$;

- initial slopes belonging to the same I_j determine solutions with the same number of zeros in $[0, t_0]$, but such a number increases together with j.

Moreover, since the stable manifolds separates the trajectories of (5) with positive and negative energy, it is possible to distinguish inside each I_j those initial slopes k such that x_k is monotone in $[t_0, T]$ and with exactly one zero therein, from those such that x_k has constant sign and is convex/concave in $[t_0, T]$. A generalization of this example is given by Lemma 4.

We remark that, when g is superlinear at infinity and q is an arbitrary function, the blow-up always occurs in the intervals where $q < 0$ at least for some "large" initial conditions, no matter how much q and g are regular. This was shown by Burton and Grimmer in [18]: they actually proved that, if $q < 0$, the convergence of one of the following two integrals:

$$\int_{-\infty} \frac{ds}{\sqrt{G(s)}} \qquad \text{and} \qquad \int^{+\infty} \frac{ds}{\sqrt{G(s)}},$$

is a necessary and sufficient condition for the existence of at least one exploding solution of (1).

3. Butler's theorems

In [21] Butler considers the problem of finding periodic solutions of equation (1), or of its equivalent first order system (3), assuming that:

- $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function such that $g(s) \cdot s > 0$ for $s \neq 0$;
- $\displaystyle\lim_{s\to\pm\infty} \frac{g(s)}{s} = +\infty$;
- $\displaystyle\left| \int^{\pm\infty} \frac{1}{\sqrt{G(s)}} ds \right| < +\infty$.

For example, a function g satisfying the first condition and:

$$|g(s)| \geq k|s| \log^{\alpha} |s| \qquad \text{if } s \gg 1,$$

for some $k > 0$ and $\alpha > 2$, satisfies also the other two assumptions. With respect to the weight function q, he supposes that it is a T-periodic and continuous function changing sign a finite number of times and that it is enough regular in the intervals in which it is nonnegative (e.g. q is piecewise monotone), in such a way that in these intervals the solutions cannot blow up; therefore, up to a time-shift, there are j zeros of q, $0 < t_1 < t_2 < \cdots < t_j < T$, such that:

- $q \leq 0$ and $q \not\equiv 0$ in $[0, t_1]$ and in $[t_{j-1}, t_j]$;
- $q \geq 0$ and $q \not\equiv 0$ in $[t_j, T]$;
- $q \not\equiv 0$ and either $q \geq 0$ or $q \leq 0$ in each other interval $[t_i, t_{i+1}]$.

Using the notation introduced at the beginning of Section 2 and recalling what has been said in [18], the value $z(t; t_0, p)$ is surely not defined for some $p \in \mathbb{R}^2$ if the interval between t_0 and t contains points in which q is negative. Therefore Butler introduces the following set of "good" initial conditions with respect to a fixed time interval:

$$\Omega_a^b = \{p \in \mathbb{R}^2 : z(t; a, p) \text{ is defined in the closed interval between } a \text{ and } b\}.$$

In general very little can be said about the shape of Ω_a^b : the theorem about the continuous dependence on initial data implies that it is open and our assumptions guarantee that it always contains the origin, since (1) admits the constant solution $x \equiv 0$. If $q \geq 0$ in $[a, b]$, then $\Omega_a^b = \mathbb{R}^2$, of course, and in particular one has that $\Omega_0^T = \Omega_0^{t_j}$. Clearly, if b lies between a and c then $\Omega_a^c \subset \Omega_a^b$.

One way to find T-periodic solutions of (1) or (3) is to write the T-periodic boundary condition $z(T) = z(0)$ in a way which puts in evidence the dependence on the initial value $z(0) = p$; indeed, we are essentially looking for initial conditions $p \in \mathbb{R}^2$

such that $z(T; 0, p) = p$, that is we search points in the plane where the vector field $p \mapsto z(T; 0, p) - p$ vanishes. If we introduce the following two auxililary functions:

$$\begin{aligned} \phi(p) &= \|z(T; 0, p)\| - \|p\| \\ \psi(p) &= \int_0^T \frac{y^2(t; 0, p) + q(t)g(x(t; 0, p))x(t; 0, p)}{y^2(t; 0, p) + x^2(t; 0, p)} dt, \end{aligned}$$

then the solution departing from p at $t = 0$ is T-periodic if and only if:

$$\begin{cases} \phi(p) = 0 \\ \psi(p) = 2k\pi \qquad \text{for some } k \in \mathbb{Z}. \end{cases}$$

Before entering more in the details of Butler's technique, let us fix some notation. If r is a positive number, then C_r will denote the circumference $\{p \in \mathbb{R}^2 : \|p\| = r\}$ with radius r; if, moreover, $R > r$, then $A[r, R]$ will be the closed annulus with boundary $C_r \cup C_R$. By the word "continuum" we mean, as usual, a compact and connected set.

Here is a first lemma about what happens in any interval of positivity for q.

LEMMA 1. *Assume that $q \geq 0$ and $q \not\equiv 0$ in $[a, b]$. Then for every $M > 0$ and $n \in \mathbb{N}$ there exist $r = r(M, n)$ and $R = R(M, n)$, with $0 < r < R$, such that:*

1. *$\|z(t; a, p)\| \geq M$ for all $t \in [a, b]$ and $\|p\| \geq r$;*
2. *$\Gamma \ni p \mapsto \arg z(b; a, p)$ is a n-fold covering of S^1, for any continuum $\Gamma \subset A[r, R]$ which does not intersect both axes and satisfies $\Gamma \cap C_r \neq \emptyset \neq \Gamma \cap C_R$.*

Roughly speaking, the second statement just means that the map $p \mapsto z(b; a, p)$ transforms any continuum crossing the annulus $A[r, R]$ into a continuum which turns around the origin at least n times. Observe that it is required that Γ "does not intersect both axes", that is it must be contained in one of the four half-planes generated by the coordinate axes: this prevents Γ itself from turning around the origin and escaping the twisting effect of the map $p \mapsto z(b; a, p)$.

REMARK 1. If q is nonnegative in $[a, b]$, then the mappings $p \mapsto z(b; a, p)$ and $p \mapsto z(a; b, p)$ are defined on $\mathbb{R}^2$, continuous and each one is the inverse of the other. Thus they are homeomorphisms of $\mathbb{R}^2$ onto itself and, in particular, map bounded sets into bounded sets.

Even if little can be said in general about the structure of a set Ω_a^b (it might be disconnected and its boundary might not be a continuous arc), Butler actually proved the following, when $[a, b]$ is an interval of negativity for q.

LEMMA 2. *Assume that $q \leq 0$ and $q \not\equiv 0$ on $[a, b]$. If $J \subset \mathbb{R}$ is any compact interval, then $\Omega_a^b \cap J \times \mathbb{R}$ is non-empty and bounded.*
The same holds for Ω_b^a.

This result is simple if $q \equiv -1$, since in this case it turns out that, on one hand, the set Ω_a^b must contain the stable manifolds $y = -\sqrt{2G(x)}$, for $x > 0$, and $y = \sqrt{2G(x)}$, for $x < 0$, while, on the other, it cannot contain any point from the trajectories $y = \pm\sqrt{2(c + G(x))}$, if $c > 0$ is such that $\tau^-(c) < b - a$ and this happens for every sufficiently large c.

LEMMA 3. *There are* $\alpha < 0 < \beta$ *and a continuous arc* $\gamma = (\gamma_1, \gamma_2) :]\alpha, \beta[\to \Omega_0^T$ *such that:*

1. $\gamma(0) = (0, 0)$;
2. $\lim_{s\to\alpha^+} \gamma_1(s) = \lim_{s\to\alpha^+} \gamma_2(s) = \lim_{s\to\beta^-} \gamma_1(s) = \lim_{s\to\beta^-} \gamma_2(s) = \infty$;
3. $\|z(t_j; 0, \gamma(s))\|$ *and* $\|z(T; 0, \gamma(s))\|$ *are uniformly bounded for* $s \in]\alpha, \beta[$.

Proof. Let us consider just the case $j = 1$, in which $t_{j-1} = 0$ and $t_1 = t_j$. The intersection between $\Omega_{t_j}^0$ and the y-axis $\{0\} \times \mathbb{R}$ determines, by Lemma 2, a bounded and open (relatively to the topology of the straight line) set which contains the origin. Therefore there are $\alpha < 0 < \beta$ such that the segment $\{0\} \times]\alpha, \beta[$ is contained in $\Omega_{t_j}^0$ while its end-points $(0, \alpha)$ and $(0, \beta)$ belong to $\partial\Omega_{t_j}^0$. By construction each solution departing from $\{0\} \times]\alpha, \beta[$ at time t_j is defined at least up to 0, hence we can set:

$$\gamma(s) = z(0; t_j, (0, s)) \qquad \text{for } s \in]\alpha, \beta[\, .$$

Since $\gamma(s)$ is the value at time 0 of a solution defined on $[0, t_j]$, we have that the support of γ lies in $\Omega_0^{t_j}$, which in turn coincides with Ω_0^T because in the last interval $[t_j, T]$ q is nonnegative and, therefore, solutions cannot blow up therein by our assumptions on q.

Clearly statement 1 is satisfied and Statement 2 follows from the fact that the points $(0, \alpha)$ and $(0, \beta)$ do not belong to $\Omega_{t_j}^0$: hence $z(t; t_j, (0, \alpha))$ and $z(t; t_j, (0, \beta))$ blow up somewhere in $[0, t_j]$ and an argument based on the continuous dependence on initial data shows that $\gamma_i(s)$ is unbounded when s ranges near α and β.

The definition of γ implies that:

$$z(t_j; 0, \gamma(s)) = (0, s) \qquad \text{for } s \in]\alpha, \beta[\, ,$$

thus $\|z(t_j; 0, \gamma(s))\|$ is bounded by $\max\{-\alpha, \beta\}$. Finally, observe that $z(T; 0, \gamma(s)) = z(T; t_j, (0, s))$ and that q is nonnegative on $[t_j, T]$; then also Statement 3 holds by Remark 1.

□

THEOREM 1. *Equation* (1) *has infinitely many* T*-periodic solutions.*

Proof. We start fixing some constants. By Lemma 2 the intersection of the y-axis with the set $\Omega_{t_j}^{t_{j-1}}$ is bounded by a constant A_1; therefore, if $z(t_j; 0, p)$ lies on the y-axis then $\|z(t_j; 0, p)\| = |y(t_j; 0, p)| \leq A_1$. Moreover, by Remark 1 the following constant:

$$A_2 = \max\{\|z(T; t_j, p)\| : \|p\| \leq A_1\}$$

exists and is finite. In particular, if γ is the curve given in Lemma 3, we have that:

$$\|z(t_j; 0, \gamma(s))\| \leq A_1 \qquad \text{and} \qquad \|z(T; 0, \gamma(s))\| \leq A_2 \qquad \text{for } s \in]\alpha, \beta[\,.$$

Now let A_3 be any real number such that $A_3 > A_2$ and let L_1 be the vertical straight line $\{A_3\} \times \mathbb{R}$. Let Ω be the connected component of Ω_0^T which contains the support of γ. By Lemma 2 and the fact that $\Omega \subset \Omega_0^{t_1}$, we have that the set $L_1 \cap \Omega$ is bounded and we can define:

$$A_4 = \sup\{\|p\| : p \in L_1 \cap \Omega\} < +\infty \qquad (\Longrightarrow A_4 > A_3).$$

Now take $M = 2A_4$ and *any natural number n* and consider the two radii $r = r(M, n+1)$ and $R = R(M, n+1)$ which are obtained applying Lemma 1 in the interval $[t_j, T]$. We set:

$$A_5 = \max\{\|z(T; t_j, p)\| : \|p\| \leq R\}$$

and call L_2 the vertical straight line $\{A_3 + A_5\} \times \mathbb{R}$. Now, Statement 2 in Lemma 3 guarantees that γ crosses at least one of the two vertical strips $[A_3, A_3 + A_5] \times \mathbb{R}$ and $[-A_3 - A_5, -A_3] \times \mathbb{R}$: assume that it crosses the first one (if it crosses the other one, one can argue in a similar way) and call it $S[L_1, L_2]$. By Lemma 2, the intersection of Ω with the vertical strip $S[L_1, L_2]$ is bounded, therefore:

$$A_6 = \sup\{\|p\| : p \in \Omega \cap [A_3, A_3 + A_5] \times \mathbb{R}\} < +\infty.$$

The curve γ, passing from L_1 to L_2, divides $\Omega \cap S[L_1, L_2]$ into two bounded regions. If $p \in S[L_1, L_2]$ belongs to the support of γ, then $\|z(T; 0, p)\| \leq A_2 < A_3 \leq \|p\|$; hence:

$$p \in \gamma(]\alpha, \beta[) \cap S[L_1, L_2] \Longrightarrow \phi(p) \leq 0.$$

On the other hand, if p lies in $\Omega \cap S[L_1, L_2]$ near $\partial\Omega$, then $\|p\|$ remains bounded by A_6, but $\|z(T; 0, p)\|$ can be made arbitrarily large, since p is near "bad" points with respect to the interval $[0, T]$; thus:

$$\phi(p) \to +\infty \quad \text{if } p \to \partial\Omega,\ p \in \Omega \cap S[L_1, L_2].$$

Therefore, on every curve σ contained in $\Omega \cap S[L_1, L_2]$ and such that it connects a point of γ with a point of $\partial\Omega$, we can find a point p in which $\phi(p) = 0$. This implies (but it is not a trivial topological fact) that there exists a continuum Γ_0 contained in $\Omega \cap S[L_1, L_2]$ and intersecting also L_1 and L_2 such that:

$$p \in \Gamma_0 \Longrightarrow \|z(T; 0, p)\| = \|p\|. \tag{6}$$

The set:

$$\Gamma_j = \{z(t_j; 0, p) : p \in \Gamma_0\}$$

is still a continuum since it is the image of Γ_0 through the continuous map $\Omega_0^T \ni p \mapsto z(t_j; 0, p)$. Γ_j does not intersect the y-axis, since, if $x(t_j; 0, p) = 0$, then one has $\|z(t_j; 0, p)\| \leq A_1$, by the definition of A_1, and $\|z(T; 0, p)\| =$

$\|z(T; t_j, z(t_j; 0, p))\| \le A_2 < A_3$, by the definition of A_2, while $\|z(T; 0, p)\| = \|p\| \ge A_3$ if $p \in \Gamma_0$. Moreover, fix $p \in L_1 \cap \Gamma_0$ and $q \in L_2 \cap \Gamma_0$; we have:

$$\|z(T; t_j, z(t_j; 0, p)\| = \|z(T; 0, p)\| = \|p\| \le A_4 < M \Longrightarrow \|z(t_j; 0, p)\| < r$$

by the definition of A_4 and Statement 1 in Lemma 1; and also:

$$\|z(T; t_j, z(t_j; 0, q)\| = \|z(T; 0, q)\| = \|q\| \ge A_3 + A_5 > A_5 \Longrightarrow \|z(t_j; 0, q)\| > R$$

by the definition of A_5. Thus, we have found one point of Γ_j inside the ball of radius r and another point of Γ_j outside that of radius R and we can say that Γ_j is a continuum crossing the annulus $A[r, R]$. Hence Γ_j fulfills all the requirements of Statement 2 in Lemma 1. In particular we have that the map $p \mapsto \arg z(T; 0, p)$ covers S^1 at least $n+1$ times as p ranges in Γ_0. Let us see what it means in terms of angles and of the function ψ. We can select a continuous angular coordinate $\theta : [0, T] \times \Gamma_0 \to \mathbb{R}$ such that:

1. $z(t; 0, p) = (\|z(t; 0, p)\| \cos\theta(t, p), \|z(t; 0, p)\| \sin\theta(t, p))$ for $(t, p) \in [0, T] \times \Gamma_0$;
2. $-\dfrac{\pi}{2} < \theta(0, p) < \dfrac{\pi}{2}$ for $p \in \Gamma_0$ (recall that Γ_0 is contained in the right half-plane).

With this choices, the function ψ can be written as:

$$\psi(p) = \theta(0, p) - \theta(T, p).$$

The fact that $\Gamma_0 \ni p \mapsto \arg z(T; 0, p)$ covers S^1 at least $n+1$ times, means that the image of $\theta(T, \cdot)$ contains a $2(n+1)\pi$-long interval. Since $\theta(0, \cdot)$ is forced in a π-long interval, we have that $\psi(p)$ reaches at least n successive integer multiples of 2π as p ranges in Γ_0. Therefore (1) has at least n T-periodic solutions, with n arbitrarily chosen.

$\square$

We have seen that the superlinear growth at infinity of the nonlinear term g in (1) leads to the blow-up of solutions in the intervals where q attains negative values. On the other hand, if g is sublinear around 0, there is the possibility of solutions reaching the origin in finite time, since the uniqueness of the zero solution is no more guaranteed. This case was studied by Butler in [23].

EXAMPLE 4. Let us consider the autonomous system (4) with a function g which is sublinear in zero, that is:

$$\lim_{x \to 0} \frac{g(x)}{x} = +\infty.$$

The uniqueness of the solution of Cauchy problems is still guaranteed, but, now, smaller solutions oscillate more and more.

On the other hand, if we look to the zero-energy solutions of (5), which satisfy $\frac{1}{2}\dot{x}^2 - G(x) = 0$, we found that the time that they take to reach $(0,0)$ from the value $x = x_0 > 0$ is given by the integral:

$$\int_0^{x_0} \frac{ds}{\sqrt{2G(s)}}$$

which is *finite* when $g(s)$ is a sublinear function like $s|s|^{\gamma-1}$, with $0 < \gamma < 1$. Therefore the uniqueness of the zero-solution holds no more.

In [48] Heidel gives conditions that prevent solutions of (1) from reaching the origin in finite time in the case of a nonnegative weight q. In particular assuming $q \in C^1$ and q being piecewise monotone around its zeros turns out to be sufficient to this aim and is what Butler needs in [23]. Indeed, Butler proves that, if g is sublinear around the origin and q is a T-periodic weight which changes sign and is enough regular, then (1) has infinitely many T-periodic solutions with an arbitrarily large number of small oscillations in the intervals of positivity of q.

On the other hand such solutions may be identically zero in some subintervals of the intervals of negativity of q. Indeed, let us consider a weight q_ϵ such that $q_\epsilon \equiv -1$ in $[0, 2[$, $q_\epsilon \equiv \epsilon > 0$ in $[2, 4[$ and which is 4-periodic. Then Butler shows that ϵ can be chosen sufficiently small in such a way that every solution of:

$$\ddot{x} + q_\epsilon(t)x^{\frac{1}{3}} = 0$$

which is nowhere trivial must be strictly monotone (and, hence, nonperiodic) on some half line.

4. Another possible approach: generalized Sturm–Liouville conditions

Let us consider a situation in which $q : [a, c] \to \mathbb{R}$ is such that:

$$q \geq 0 \text{ in } [a, b] \qquad \text{and} \qquad q \leq 0 \text{ in } [b, c],$$

and assume that g in (1) is superlinear at infinity in the sense that:

$$\lim_{c \to +\infty} \tau^+(c) = \lim_{c \to \pm\infty} \tau^-(c) = 0.$$

Let $Q_1 = [0, +\infty[\times [0, +\infty[$, $Q_2 =]-\infty, 0] \times [0, +\infty[$, $Q_3 =]-\infty, 0] \times]-\infty, 0]$ and $Q_4 = [0, +\infty[\times]-\infty, 0]$ be the four closed quadrants of the plane. Then we have the following result.

LEMMA 4. *There exists $R^* > 0$ (depending only on g and $q|_{[b,c]}$) such that, for every $R > 0$ there is a natural number $n^* = n^*_R$ with the property that for every natural numbers $n > n^*$ and $\delta \in \{0, 1\}$ and for any path $\gamma : [\alpha, \beta[\to [0, +\infty[\times \mathbb{R}$, with $\|\gamma(\alpha)\| \leq R$ and $\|\gamma(s)\| \to +\infty$ as $s \to \beta$, we can select an interval $I \subset]\alpha, \beta[$,* *with $I =]\alpha_n, \beta_n]$, if $\delta = 0$, and $I = [\beta_n, \alpha_n[$, if $\delta = 1$, in such a way that for each $s \in I$ we have:*

- *$z(c; a, \gamma(s))$ is defined*
- *$x(\cdot; a, \gamma(s))$ has exactly n zeros in $]a, b[$, δ zeros in $]b, c[$ and exactly $1 - \delta$ changes of sign of the derivative in $]b, c[$*
- *the curve $\gamma_n(s) = z(c; a, \gamma(s))$, $s \in I$, satisfies $\|\gamma_n(\beta_n)\| \le R^*$, $\|\gamma_n(s)\| \to +\infty$ as $s \to \alpha_n$ and its support lies either in Q_1 (if $n + \delta$ is even) or in Q_3 (if $n + \delta$ is odd).*

The same holds when the support of the curve γ lies in the left half plane $]-\infty, 0] \times \mathbb{R}$ by simply interchanging the role of Q_1 and Q_3.

Let us see how to use Lemma 4 in order to find multiple solutions of (1) satisfying the two-point boundary condition:

$$x(0) = x(T) = 0. \tag{7}$$

We assume that there are t_i, with $i = 0, \dots, 2j + 1$, such that $0 = t_0 < t_1 < \cdots < t_{2j+1} = T$ and:

$$q \ge 0, \quad q \not\equiv 0 \text{ in } [t_{2i-2}, t_{2i-1}] \qquad \text{and} \qquad q \le 0, \quad q \not\equiv 0 \text{ in } [t_{2i-1}, t_{2i}],$$

for $i = 1, \dots, j + 1$, so q is positive near both 0 and T. Let us apply Lemma 4 in the interval $[0, t_2]$ to the unbounded curve $\gamma_0(s) = (0, s)$, for $s \ge 0$, which parametrizes the positive y-axis in the phase plane: each solution x of (1)–(7) with $\dot{x}(0) > 0$ should start from the support of γ_0 at time $t = 0$. Let $R_1^* > 0$ and $n_1^* \in \mathbb{N}$ be respectively the numbers R^* and n_R^* given by Lemma 4 with an arbitrarily small $R > 0$ (since $\gamma_0(0) = (0, 0)$) and fix any $n_1 > n_1^*$ and $\delta_1 \in \{0, 1\}$: then, we obtain an interval $I_1 =]\alpha_1, \beta_1[\subset [0, +\infty[$ such that the solution of (1) starting at $t = 0$ from $\gamma_0(s)$ has nodal behavior in $[0, t_2]$ prescribed by the couple (n_1, δ_1), as in Lemma 4, if s belongs to I_1, and, moreover, the curve $\gamma_1(s) = z(t_2; 0, \gamma_0(s))$ is defined for $s \in I_1$, is contained either in the first or the third quadrant, it is unbounded when s tends to one of the endpoints of I_1, while it lies inside a circle of radius R_1^* for s belonging to a neighborhood of the other endpoint. Therefore we can apply Lemma 4 on the successive interval $[t_2, t_4]$ and to the curve γ_1 with the choice $R = R_1^*$.

After j successive applications of Lemma 4 to the intervals $[t_{2i-2}, t_{2i}]$, for $i = 1, \dots, j$, we get $R_j^* > 0$ and j positive integers $n_1^*, \dots, n_j^*$ such that, for every j-tuple $(n_1, \dots, n_j) \in \mathbb{N}^j$, with $n_i > n_i^*$, and for every j-tuple $(\delta_1, \dots, \delta_j) \in \{0, 1\}^j$, there is a final interval $I_j \subset [0, +\infty[$ with the following properties:

- the curve $\gamma(s) = z(t_{2j}; 0, (0, s))$ is defined for $s \in I_j$, lies in the first or in the third quadrant (it depends on the parity of $n_1 + \delta_1 + \cdots + n_j + \delta_j$), it is unbounded when s tends to one of the endpoints of I_j, while it is inside the circle of radius R_j^* if s belongs to a neighborhood of the other endpoint;

- if $s \in I_j$ and $i = 1, \ldots, j$, the solution $x(t; 0, (0, s))$ has a nodal behavior in $[t_{2i-2}, t_{2i}]$ which is described by the couple (n_i, δ_i) as in Lemma 4.

It remains to find some s in the interval I_j such that the solution starting at $t = t_{2j}$ from $\gamma(s)$ reaches the y-axis exactly at $t = T$ and this can be done by a result of Struwe [83], since the weight q is nonnegative in the interval $[t_{2j}, T]$ (see Example 2 for an idea of the argument).

Clearly another set of solutions can be found starting from the negative y-axis and it is not difficult to obtain the same kind of result if q is negative either near $t = 0$ or near $t = T$ or both. However, a more important fact is perhaps that we can adjust the technique explained above in order to find multiple solutions of more general boundary value problems for (1), namely all those problems whose boundary conditions can be expressed by:

$$(x(0), \dot{x}(0)) \in \Gamma_0 \qquad \text{and} \qquad (x(T), \dot{x}(T)) \in \Gamma_T,$$

where Γ_0 and Γ_T are suitable subsets of the phase plane. They are called “generalized” Sturm–Liouville boundary conditions (see [83]) since they coincide with the usual Sturm–Liouville conditions when Γ_0 and Γ_T are two straight lines. In particular, when q is positive near 0 and T, it is possible to adapt the technique to cover all the cases in which Γ_0 and Γ_T are two unbounded continua (i.e. connected, closed and unbounded sets) contained, for instance, in some half-planes: in fact, by approximating bounded portions of continua by means of supports of continuous curves, it is possible to prove a generalization of Lemma 4 which holds also when the path γ is substituted by an unbounded continuum Γ contained either in the right half plane or in the left one.

4.1. Application to homoclinic solutions

Assume that:

$$q(t) \leq 0 \qquad \forall\, t \in \,]-\infty, a] \cup [b, +\infty[$$

and that:

$$\int_{-\infty} q = \int^{+\infty} q = -\infty.$$

Then, using an argument similar to that employed by Conley in [29], it is possible to show that there are four unbounded continua $\Gamma_a^+ \subset Q_1$, $\Gamma_a^- \subset Q_3$, $\Gamma_b^+ \subset Q_4$ and $\Gamma_b^- \subset Q_2$ such that:

- $\lim_{t \to -\infty} z(t; a, p) = (0, 0)$ for every $p \in \Gamma_a^{\pm}$;
- $\lim_{t \to +\infty} z(t; b, p) = (0, 0)$ for every $p \in \Gamma_b^{\pm}$

(see Lemmas 5 and 7 in [72] for precise statements and proof). Therefore the problem of finding homoclinics solutions of (1) is reduced to that of determining solutions of (1) in $[a, b]$ which satisfy the generalized Sturm–Liouville boundary condition:

$$(x(a), \dot{x}(a)) \in \Gamma_a^{\pm} \qquad\qquad (x(b), \dot{x}(b)) \in \Gamma_b^{\pm}$$

and this can be done in the superlinear case by the technique already explained in this section (Lemma 4 plus Struwe's result [83]).

4.2. Application to blow-up solutions

In [61] (see also [62]) the problem of finding solutions of (1) which blow up at a precise time was considered when g has a superlinear growth at infinity and $q :]0,1[\to \mathbb{R}$ is a continuous weight such that q is nonpositive in some neighborhood of 0 and of 1 and both 0 and 1 are accumulation points of the set in which q is strictly negative. See the paper [27] for recent results about the analogous problem for partial differential equations.

To be precise, let us assume that q is nonpositive in $]0,a]$ and in $[b,1[$; then there are two unbounded continua Γ_0 and Γ_1 which are contained in the right half plane $x \geq 0$ and moreover:

1. there are $R > r > 0$ and $\epsilon > 0$ such that:

$$\begin{aligned} &\Gamma_0 \cap [0,r] \times \mathbb{R} \subset [0,r] \times]-\infty, -\epsilon] \\ &\Gamma_0 \cap [R,+\infty[\times \mathbb{R} \subset [R,+\infty[\times [\epsilon,+\infty[\\ &\Gamma_1 \cap [0,r] \times \mathbb{R} \subset [0,r] \times [\epsilon,+\infty[\\ &\Gamma_1 \cap [R,+\infty[\times \mathbb{R} \subset [R,+\infty[\times]-\infty,-\epsilon] \end{aligned}$$

2. $\lim\limits_{t\to 0} x(t;a,p) = \lim\limits_{t\to 1} x(t;b,q) = +\infty$ if $p \in \Gamma_0$ and $q \in \Gamma_1$.

If $q \leq 0$ in the whole $]0,1[$, then we can choose $a = b = 1/2$ and the localization properties in statement 1 imply that $\Gamma_0 \cap \Gamma_1 \neq \emptyset$ and this proves that there is a positive solution which blows up at 0 and 1.

On the other hand, if q changes sign a finite number of times inside $]0,1[$, we can consider the generalized Sturm–Liouville boundary value conditions:

$$(x(a), \dot{x}(a)) \in \Gamma_0 \qquad\qquad (x(b), \dot{x}(b)) \in \Gamma_1$$

and apply the procedure previously explained in order to find solutions of (1) in $]0,1[$ which blows up at 0 and 1 and have a prescribed nodal behavior inside the interval.

5. Chaotic-like dynamics

The chaotic features of (1) were studied in the papers [85] and [26] when g is superlinear at infinity. Here we would like to give an interpretation of chaos in the sense of "coin-tossing", as it is defined in [53] for the discrete dynamical system generated by the iterations of a continuous planar map ψ which is not required to be defined in the whole plane (like the Poincaré map associated to our equation (1) when g is superlinear at infinity and q is somewhere negative). To be more precise, consider the set X which is the union of two disjoint, nonempty and compact sets K_0 and K_1. We say that the discrete dynamical system generated by the iterates of a continuous mapping ψ is *chaotic in the sense of coin-tossing* if, for every doubly infinite sequence of binary digits $(\delta_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, there is a doubly infinite sequence $(p_i)_{i\in\mathbb{Z}}$ of points of X such that:

1. $\psi(p_i) = p_{i+1}$
2. $p_i \in K_{\delta_i}$

for every $i \in \mathbb{Z}$. The first condition states that the sequence $(p_i)_{i\in\mathbb{Z}}$ is an *orbit* of the dynamical system generated by ψ; the second one guarantees the possibility of finding orbits which touch at each time the prescribed component of X.

We remark that in this definition ψ is not necessarily defined in the whole X and it is not required to be 1-to-1. Actually we are interested in the case of planar maps, since we wish to study the Poincaré map associated to (1), and, in particular, we will consider compact sets K_i with a particular structure: we call an *oriented cell* a couple $(\mathcal{A}, \mathcal{A}^-)$ where $\mathcal{A} \subset \mathbb{R}^2$ is a two-dimensional cell (i.e., a subset of the plane homeomorphic to the unit square $Q = [-1,1]^2$) and $\mathcal{A}^- \subset \partial\mathcal{A}$ is the union of two disjoint compact arcs. The two components of $\mathcal{A}^-$ will be denoted by $\mathcal{A}_l^-$ and $\mathcal{A}_r^-$ and conventionally called the left and the right sides of $\mathcal{A}$. The order in which we make the choice of naming $\mathcal{A}_l^-$ and $\mathcal{A}_r^-$ is immaterial in what follows.

If ψ is a continuous map $\mathbb{R}^2 \supset \mathrm{Dom}(\psi) \to \mathbb{R}^2$ and $(\mathcal{A}, \mathcal{A}^-)$, $(\mathcal{B}, \mathcal{B}^-)$ are two oriented cells, we say that ψ *stretches* $(\mathcal{A}, \mathcal{A}^-)$ *to* $(\mathcal{B}, \mathcal{B}^-)$ and write:

$$\psi : (\mathcal{A}, \mathcal{A}^-) \Leftarrow\!\!\rightsquigarrow (\mathcal{B}, \mathcal{B}^-),$$

if:

- ψ is *proper* on $\mathcal{A}$, which means that $|\psi(p)| \to +\infty$ whenever $\mathrm{Dom}(\psi) \cap \mathcal{A} \ni p \to p_0 \in \partial\,\mathrm{Dom}(\psi) \cap \mathcal{A}$;
- for any path $\Gamma \subset \mathcal{A}$ such that $\Gamma \cap \mathcal{A}_l^- \neq \emptyset$ and $\Gamma \cap \mathcal{A}_r^- \neq \emptyset$, there is a path $\Gamma' \subset \Gamma \cap \mathrm{Dom}(\psi)$ such that:

$$\psi(\Gamma') \subset \mathcal{B}, \quad \psi(\Gamma') \cap \mathcal{B}_l^- \neq \emptyset, \quad \psi(\Gamma') \cap \mathcal{B}_r^- \neq \emptyset.$$

THEOREM 2. *If* $\psi : (\mathcal{A}, \mathcal{A}^-) \Leftarrow\!\!\rightsquigarrow (\mathcal{A}, \mathcal{A}^-)$, *then* ψ *has at least one fixed point in* $\mathcal{A}$.

Sketch of the proof. Let us consider just the case of $\mathcal{A} = [0,1]\times[0,1]$, with $\mathcal{A}_l^- = \{0\}\times[0,1]$ and $\mathcal{A}_r^- = \{1\}\times[0,1]$, and let $\psi(x_1,x_2) = (\psi_1(x_1,x_2), \psi_2(x_1,x_2))$. If $\Gamma \subset \mathcal{A}$ is a path joining the vertical sides of $\mathcal{A}$, let $\Gamma' \subset \Gamma$ be the subpath such that $\psi(\Gamma')$ is again a path in $\mathcal{A}$ which joins its vertical sides and, in particular, let $p = (p_1,p_2)$ and $q = (q_1,q_2)$ two points in Γ' such that $\psi(p) \in \{0\}\times[0,1]$ and $\psi(q) \in \{1\}\times[0,1]$. Therefore we have:

$$\psi_1(p_1,p_2) - p_1 = -p_1 \le 0 \qquad \text{and} \qquad \psi_1(q_1,q_2) - q_1 = 1 - q_1 \ge 0.$$

Hence, every path in $\mathcal{A}$ joining the vertical sides meets the closed set in which the function $\psi_1(x_1, x_2) - x_1$ vanishes and this implies that actually there is a whole continuum $\Gamma_1 \subset \mathcal{A}$ joining the *horizontal* sides of $\mathcal{A}$ such that $\psi_1(x_1, x_2) - x_1$ vanishes in Γ_1 and $\psi(\Gamma_1) \subset \mathcal{A}$ (see the argument to find Γ_0 in (6)). Again, this implies that the function $\psi_2(x_1, x_2) - x_2$ changes sign on Γ_1 : there is a point in Γ_1 where also $\psi_2(x_1, x_2) - x_2$ vanishes, and such a point is clearly a fixed point of ψ.

□

THEOREM 3. *Let $(\mathcal{A}_0, \mathcal{A}_0^-)$ and $(\mathcal{A}_1, \mathcal{A}_1^-)$ be two oriented cells. If ψ stretches each of them to itself and to the other one:*

$$\psi : (\mathcal{A}_i, \mathcal{A}_i^-) \Leftarrow\!\!\leadsto (\mathcal{A}_j, \mathcal{A}_j^-), \qquad \textit{for } (i, j) \in \{0, 1\}^2,$$

then ψ shows a chaotic dynamics of coin-tossing type.

These results can be applied, for instance, to the following situation:

$$\ddot{x} + [\alpha q^+(t) - \beta q^-(t)]g(x) = 0, \tag{8}$$

where α and β are positive constants, $q^+(t) = \max\{q(t), 0\}$ and $q^-(t) = \max\{-q(t), 0\}$ are respectively the positive and the negative part of a continuous and periodic function q which changes sign, and g is a nonlinear function such that:

$$0 < g'(0) \ll g'(\infty).$$

The parameter α regulates the twisting effect of the Poincaré map along the intervals of positivity of q, while β controls the stretching of the arcs along the intervals of negativity of q. Assume, for simplicity, that q is T-periodic with exactly one change of sign in $\tau \in]0, T[$ in such a way that:

$$q > 0 \quad \text{in }]0, \tau[\qquad \text{and} \qquad q < 0 \quad \text{in }]\tau, T[\,.$$

For every fixed $n \in \mathbb{N}$, using the theorems stated above, it is possible to find $\alpha_n > 0$ such that, for every $\alpha > \alpha_n$, there is $\beta_\alpha > 0$ such that for each $\beta > \beta_\alpha$ we have the following results (see Theorem 2.1 in [30]):

1. for any $m \in \mathbb{N}$ and any m-tuple of binary digits $(\delta_1, \ldots, \delta_m) \in \{0, 1\}^m$ such that $mn + \delta_1 + \cdots + \delta_m$ is an even number, there are at least two mT-periodic solutions x^+ and x^- of (8) which have exactly n zeros in $[(i-1)T, (i-1)T + \tau]$ and δ_i zeros in $[(i-1)T + \tau, iT]$, for each $i = 1, \ldots, m$; moreover $x^+(0) > 0$ and $x^-(0) < 0$;

2. for any doubly infinite sequence of binary digits $(\delta_i)_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, there is at least a globally defined solution x of (8) which has exactly n zeros in $[iT, iT + \tau]$ and δ_i zeros in $[iT + \tau, (i+1)T]$, for all $i \in \mathbb{Z}$.

6. Subharmonic solutions

Our aim here is to consider large solutions of equations like (1) in which the nonlinearity g is sublinear at infinity, as in Hill's equation (2) when $0 < \gamma < 1$. The results we are going to present are contained in a joint work with B. Liu [59] and are valid also for the forced version of (1):

$$\ddot{x} + q(t)g(x) = e(t).$$

Throughout this section we assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that:

- $g(0) = 0$;
- there is $R_0 \geq 0$ such that $g(s) \cdot s > 0$ and $g'(s) \geq 0$ if $|s| > R_0$ ($\Longrightarrow g(-\infty) < 0 < g(+\infty)$);
- $\displaystyle\lim_{s \to \pm\infty} \frac{g(s)}{s} = 0.$

The third condition is the so-called condition of sublinearity at infinity. Moreover we will suppose that q is a continuous and T-periodic function, even if continuity is not necessary: local integrability would be enough.

THEOREM 4. *Besides the assumptions stated above, suppose that:*

$$\overline{q} = \frac{1}{T}\int_0^T q(t)dt > 0. \tag{9}$$

Then for each integer $j \geq 1$ there is $m_j^* \in \mathbb{N}$ such that, for every $m \geq m_j^*$ equation (1) *has at least one mT-periodic solution $x_{j,m}$ which has exactly $2j$-zeros in* $[0, mT[$. *Moreover, for each $m \geq 1$ there is $M_m > 0$ such that any mT-periodic solution x of* (1) *satisfies:*

$$\|x\|_{C^1} \leq M_m;$$

on the other hand, for every fixed $j \geq 1$ we have:

$$\lim_{m \to +\infty} (|x_{j,m}(t)| + |\dot{x}_{j,m}(t)|) = +\infty,$$

uniformly with respect to $t \in \mathbb{R}$.

EXAMPLE 5. Theorem 4 holds, for instance, for the following Hill's equation:

$$\ddot{x} + [k + \cos(t + \theta)]|x|^{\gamma-1}x = 0,$$

where $0 < \gamma < 1$, $k > 0$ and $\theta \in \mathbb{R}$. The same is true if we substitute $|x|^{\gamma-1}x$ with another sublinear function like $x/(1 + |x|)$, for instance.

We remark that condition (9) was already considered by other authors dealing with the superlinear case (see for instance [76]). A partial converse, with respect to this assumption, holds in the case of Hill's equation (2) for $0 < \gamma < 1$; in this case, if $\overline{q} < 0$, there is a constant $B > 0$, such that every solution of (2) satisfying:

$$|x(0)| + |\dot{x}(0)| > B$$

is unbounded, that is:

$$\sup_{t\in\mathbb{R}}(|x(t)| + |\dot{x}(t)|) = +\infty.$$

REMARK 2. In the book [34, p. 129] it is pointed out that "the question is whether we can find *for each* $k \geq 2$ a subharmonic x_k [that is a kT-periodic solution] such that the x_k are pairwise distinct. No result is known in the subquadratic case". The same question was pointed out by Các and Lazer in [24]. Of course we deal with a scalar model, which is a very simple case of a Hamiltonian system.

The trick to study (1) is the introduction of the so-called "Riccati integral equation" associated to (1):

$$\frac{\dot{x}(t)}{g(x(t))} = \frac{\dot{x}(s)}{g(x(s))} - \int_s^t \left[\frac{\dot{x}(\xi)}{g(x(\xi))}\right]^2 g'(x(\xi))d\xi - \int_s^t q(\xi)d\xi,$$

which is easily deduced recalling that:

$$\frac{\ddot{x}(t)}{g(x(t))} = -q(t),$$

by equation (1). This integral equation was already used by people working in oscillation theory.

We use here a small variant of a notation already introduced. If z is a solution of (3), we denote by $\mathrm{rot}(z; t_1, t_2)$ the amplitude of the angle spanned by the vector $z(t)$ as t varies from t_1 to t_2, measured in clockwise sense. Thus we do not normalize any more by dividing by π, as we did in the previous sections.

Sketch of the proof. For simplicity we assume the uniqueness property for the Cauchy problems associated to (1) and divide the proof into several lemmas.

1. *The continuability of the solutions:* the sublinear growth of g at infinity implies that every maximal solution of (1) is defined on $\mathbb{R}$.

2. *There is $\nu > 1/2$ such that for every $R_1 > R_0$ there exists $R_2 > R_1$ such that, if $z(t) = (x(t), y(t))$ is any solution of* (3) *satisfying $\|z(t_1)\| = R_1$, $\|z(t_2)\| = R_2$ (or $\|z(t_2)\| = R_1$, $\|z(t_1)\| = R_2$) and $R_1 \leq \|z(t)\| \leq R_2$, for all $t \in [t_1, t_2]$, it follows that:*

$$rot(z; t_1, t_2) > \nu 2\pi.$$

   This lemma can be proved by arguments similar to those used in [44, 35, 32].

3. *Iteration of Step 2:* let us write $\nu = \delta + 1/2$, so that $\delta > 0$; we fix $R_1 > R_0$ and apply Step 2 obtaining $R_2 > R_1$; then we apply again Step 2 with R_2 in place of R_1 obtaining $R_3 > R_2$. Let z be a solution of (3) such that $\|z(t_1)\| = R_3$, $\|z(t_2)\| = R_1$ and $R_1 \le \|z(t)\| \le R_3$ for $t \in [t_1, t_2]$, and consider the first instant s_1 and the last instant s_2 in $[t_1, t_2]$ such that $\|z(s)\| = R_2$. Since the trajectories of (3) cross the positive y-axis from the left to the right hand side and the negative y-axis from the right to the left one, it is easy to see that actually $\mathrm{rot}(z; s_1, s_2) > -\pi$, therefore we obtain:

$$\begin{aligned} \mathrm{rot}(z; t_1, t_2) &= \mathrm{rot}(z; t_1, s_1) + \mathrm{rot}(z; s_1, s_2) + \mathrm{rot}(z; s_2, t_2) \\ &> \nu 2\pi - \pi + \nu 2\pi = \left(\frac{1}{2} + 2\delta\right) 2\pi. \end{aligned}$$

   Therefore, for every $j > 0$, it is possible to find sufficiently large annuli such that every solution which crosses them must rotate around the origin at least j times.

4. *If $\mathcal{A}$ is a sufficiently large annulus and z is a solution such that $z(t) \in \mathcal{A}$ for all $t \ge t_0$, then:*

$$\mathit{rot}(z; t_0, t) \to +\infty \qquad \textit{as } t \to +\infty,$$

   uniformly with respect to $t_0 \in [0, T]$.

5. *Large solutions rotate little:* using the sublinear condition at infinity it is possible to show that for every $L > 0$ there is $\widehat{R}_L > R_0$ such that, if $0 < t_1 - t_2 \le L$ and z is any solution satisfying $\|z(t)\| \ge \widehat{R}_L$ for all $t \in [t_1, t_2]$, then:

$$\mathrm{rot}(z; t_1, t_2) < 2\pi.$$

Now, let us fix j and, by Step 3, consider $R_0 < R_1 < R_2 < R_3$ such that each solution crossing either $B[R_2] \setminus B(R_1)$ or $B[R_3] \setminus B(R_2)$ turns at least $j + 1$ times around the origin. Let $\mathcal{A} = B[R_3] \setminus B(R_1)$. By Step 4, there is m_j^* such that:

$$m \ge m_j^* \qquad \Longrightarrow \qquad \mathrm{rot}(z; 0, mT) > j2\pi \quad \text{if } R_1 \le \|z(t)\| \le R_3 \quad \forall\, t \in [0, mT].$$

Consider any solution with $\|z(0)\| = R_2$: either $z(t)$ remains in $\mathcal{A}$ for all $t \in [0, mT]$ or there is a first instant $\hat{t}$ in which the solution z exits the annulus $\mathcal{A}$. In the former case we already know that $\mathrm{rot}(z; 0, mT) > j2\pi$; in the latter one we can select an interval $[t_1, t_2] \subset [0, mT]$ such that:

- either $\|z(t_1)\| = R_2$, $\|z(t_2)\| = R_1$ and $R_1 \le \|z(t)\| \le R_2$ for all $t \in [t_1, t_2]$
- or $\|z(t_1)\| = R_2$, $\|z(t_2)\| = R_3$ and $R_2 \le \|z(t)\| \le R_3$ for all $t \in [t_1, t_2]$.

In both these situations we can conclude that $\mathrm{rot}(z; t_1, t_2) > (j+1)2\pi$ by the choice of R_1, R_2 and R_3. Therefore, arguing as in Step 3, we conclude again that $\mathrm{rot}(z; 0, mT) > j2\pi$. We can summarize this by the following implication:

$$\|z(0)\| = R_2 \quad \Longrightarrow \quad \mathrm{rot}(z; 0, mT) > j2\pi.$$

Let us fix now $m \geq m_j^*$ and apply Step 5 with $L = mT$: we get $S_1 \geq R_2$ such that the conclusion of Step 5 holds if $\|z(t)\| \geq S_1$ for all $[0, mT]$. By the continuability of all the solutions of (3) (Step 1), it is possible to find $S_2 \geq S_1$ such that $\|z(t)\| \geq S_1$ for every $t \in [0, mT]$, if $\|z(0)\| = S_2$. Hence:

$$\|z(0)\| = S_2 \quad \Longrightarrow \quad \mathrm{rot}(z; 0, mT) < 2\pi.$$

Finally, consider the mT-Poincaré map:

$$B(S_2) \setminus B[R_2] \ni p \mapsto z(mT; 0, p)$$

whose fixed points are the mT-periodic solutions of (3). It turns out that the mT-Poincaré map satisfies the Poincaré–Birkhoff fixed point theorem by the discussion carried above, and, therefore, it has a fixed point such that the corresponding mT-periodic solution rotate exactly j times around the origin in $[0, mT]$ and, hence, has exactly $2j$ zeros in $[0, mT[$.

□

We remark that if, j and m are coprime numbers and x is the mT-periodic solution of (1) given by Theorem 4 with these choices, then it turns out that mT is actually the minimal period of x in the class of the integral multiples of T.

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