FROM THE YAMABE PROBLEM TO THE EQUIVARIANT YAMABE PROBLEM

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Abstract. The formulation and solution of the equivariant Yamabe problem are presented in this study. As a result, every compact Riemannian manifold distinct from the sphere posseses a conformal metric of constant scalar curvature which is also invariant under the action of the whole conformal group. This answers an old question of Lichnerowicz.

Résumé. Une étude du problème de Yamabe équivariant est présentée. En particulier, nous montrons que toute variété riemannienne compacte distincte de la sphère possède une métrique conforme à courbure scalaire constante dont le groupe d'isométries est le groupe conforme tout entier. Ceci répond à une question posée par Lichnerowicz.

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TABLE OF CONTENTS

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let (X, g) be a compact Riemannian manifold of dimension $n \geq 3$. The classical Yamabe problem can be stated as follows: "prove that there exists a metric conformal to g with constant scalar curvature". As is well known, it is equivalent to proving the existence of a positive solution $u \in C^\infty(X)$ of the equation

$$(E) \quad \Delta u + \frac{n-2}{4(n-1)} \mathrm{Scal}(g) u = C u^{(n+2)/(n-2)}$$

where $\Delta u = -g^{ij}(\partial_{ij} u - \Gamma_{ij}^k \partial_k u)$ in a local chart, and where $\mathrm{Scal}(g)$ is the scalar curvature of g.

Let J denote the functional defined on $W^{1,2}(X)/\{0\}$ by

$$J(u) = \frac{\int_X |\nabla u|^2 \, dv(g) + \frac{n-2}{4(n-1)} \int_X \mathrm{Scal}(g) \, u^2 \, dv(g)}{\left(\int_X |u|^{2n/(n-2)} \, dv(g) \right)^{(n-2)/n}} \; .$$

The positive critical points of J are smooth solutions of (E). We denote by ω_n the volume of the standard unit sphere S^n and $\mu(S^n) = \frac{1}{4} n(n-2) \, \omega_n^{2/n}$.

A positive answer to the problem was given by Yamabe [Y] in 1960, but his demonstration was incomplete as Trudinger [T] pointed out in 1968. Nevertheless :

(i) Trudinger [T] proved in 1968 that, if $\mathrm{Inf} J \leq 0$, then $\mathrm{Inf} J = \mathrm{Min} J$ and there exists a unique positive solution to (E) ;

(ii) T. Aubin [A1] proved in 1976 that, if $\mathrm{Inf} J < \mu(S^n)$, then again $\mathrm{Inf} J = \mathrm{Min} J$ and there exists a positive solution to (E). (When $\mathrm{Inf} J > 0$, many solutions may exist. See for instance [HV3] and [S2]). In addition, he proved that we always have $\mathrm{Inf} J < \mu(S^n)$ if (X, g) is a non locally conformally flat manifold of dimension $n \geq 6$.

(iii) Schoen [S1] proved in 1984 that $\mathrm{Inf} J < \mu(S^n)$ if $(X, [g]) \neq (S^n, [\mathrm{st.}])$ and $n = 3, 4, 5$ or (X, g) locally conformally flat. (Here, st. denotes the standard metric of S^n). As a consequence, the classical Yamabe problem is completely solved.

Let us now turn our attention to the equivariant Yamabe problem. Since we know that every compact Riemannian manifold has a conformal metric of constant scalar curvature, we will try to get some more precise geometric informations. As a matter of fact, we will ask to have a conformal metric with constant scalar curvature and prescribed isometry group. This new problem was first brought to our attention by Bérard-Bergery (UCLA, 1990). The precise statement of the problem is the following. "Given (X, g) a compact Riemannian manifold of dimension $n \geq 3$ and G a compact subgroup of the conformal group $C(X, g)$ of g, prove that there exists a conformal G-invariant metric to g which is of constant scalar curvature". We solved the problem in [HV2], namely

Theorem 1 (Hebey-Vaugon [HV2]). — *Let (X, g) be a compact Riemannian manifold and G a compact subgroup of $C(X, g)$. Then, there always exists a conformal G-invariant metric g' to g which is of constant scalar curvature. In addition, g' can be chosen such that it realizes the infimum of* $\mathrm{Vol}(\widetilde{g})^{(n-2)/n} \int_X \mathrm{Scal}(\widetilde{g}) dv(\widetilde{g})$ *over the G-invariant metrics conformal to g.*

In fact, we just have to prove the second point of the theorem, which can be restated as follows. Given (X, g) a compact Riemannian manifold and G a compact subgroup of $I(X, g)$, there exists $u \in C^\infty(X)$, $u > 0$ and G-invariant, which realizes $\mathrm{Inf}\, J(u)$ where the infimum is taken over the G-invariant functions of $W^{1,2}(X)/\{0\}$. Let us denote by $\mathrm{Inf}_G J(u)$ this infimum. A generalization of Aubin's result is needed here. Let $[g]$ be the conformal class of g and $O_G(x)$ be the G-orbit of $x \in X$. This generalization can be stated as follows.

Theorem 2 (Hebey-Vaugon [HV2]). — *If* $\mathrm{Inf}_G J(u) < \mu(S^n)\ (\mathrm{Inf}_{x \in X} \sharp O_G(x))^{2/n}\ (*)$, *then the infimum* $\mathrm{Inf}_G J(u)$ *is achieved and $[g]$ carries a G-invariant metric of constant scalar curvature. In addition, the non strict inequality always holds.*

($\sharp O_G(x) \in \mathbb{N}^* \cup \{\infty\}$ denotes the cardinal number of $O_G(x)$). As a consequence, the proof of Theorem 1 is straightforward if all the orbits of G are infinite. If not, the proof proceeds by choosing appropriate test functions.

This improvement of the classical Yamabe problem allows us to cover a conjecture of Lichnerowicz. This conjecture can be stated as follows: "$I_o(X, g) = C_o(X, g)$ as soon as $\mathrm{Scal}(g)$ is constant and $(X, [g]) \neq (S^n, [\mathrm{st.}])$" (where $I_o(X, g)$ and $C_o(X, g)$ are the connected components of the identity in the isometry group $I(X, g)$ of g and in the conformal group $C(X, g)$ of g). This statement is true when $\mathrm{Scal}(g)$ is nonpositive (since the metric of constant scalar curvature is unique), but can be false when $\mathrm{Scal}(g)$ is positive. One sees this by considering $S^1(T) \times S^{n-1}$ as $I_o(S^1(T)) \times I_o(S^{n-1})$ acts transitively on the product, which for T large possesses many conformal metrics of constant scalar curvature (see [HV3] and [S2]). In fact, the conjecture should be restated as follows: " Given (X, g) a compact Riemannian manifold, $(X, [g]) \neq (S^n, [\mathrm{st.}])$, there exists at least one g' in $[g]$ which has constant scalar curvature and which satisfies $I(X, g') = C(X, g)$". This is the best result possible and was proved in Hebey-Vaugon [HV2]. Using the work of Lelong-Ferrand [LF] (see also Schoen [S2]), this result can be seen as a corollary of Theorem 1. (Lelong-Ferrand proved that for any compact Riemannian manifold (X, g) distinct from the sphere, there exists $g' \in [g]$ such that $I(X, g') = C(X, g)$.)

Theorem 3 (Hebey-Vaugon [HV2]). — *Every compact Riemannian manifold (X, g), distinct from the sphere, possesses a conformal metric of constant scalar curvature which has $C(X, g)$ as isometry group.*

In the following, $R(g)$ denotes the Riemann curvature tensor of g, $\mathrm{Weyl}(g)$ denotes the Weyl tensor of g and $\mathrm{Ric}(g)$ denotes the Ricci tensor of g.

2. SOME WORDS ABOUT THE CLASSICAL YAMABE PROBLEM

We give here a new solution of the classical Yamabe problem which unifies the works of Aubin [A1] and Schoen [S1]. For completeness, we mention that other proofs have also been presented in [Ba], [BB], [LP], [S2] and [S3].

Proposition 4 (Hebey-Vaugon [HV1]). — *When $(X, [g]) \neq (S^n, [\mathrm{st.}])$, the test functions*

$$\begin{cases} u_{\varepsilon,x} = (\varepsilon + r^2)^{1-n/2} & \text{if } \; r \leq \delta, \; \delta > 0 \\ u_{\varepsilon,x} = (\varepsilon + \delta^2)^{1-n/2} & \text{if } \; r \geq \delta \end{cases}$$

give the strict inequality $\mathrm{Inf} J < \mu(S^n)$. *(Here, r is the distance from x fixed in X, δ and ε are small).Therefore,* $\mathrm{Inf} J(u)$ *is achieved and every compact Riemannian manifold carries, in its conformal class, a metric of constant scalar curvature.*

When the manifold is not locally conformally flat, the calculation of $J(u_{\varepsilon,x})$ is the same as that of Aubin [A1]. If $\mathrm{Ric}(g)(x) = 0$ (which is always possible to achieve by a conformal change of the metric), we get

$$J(u_{\varepsilon,x}) = \mu(S^n)\left(1 - \frac{\varepsilon^2}{12n(n-4)(n-6)}\left|\mathrm{Weyl}(g)(x)\right|^2 + o(\varepsilon^2))\right) \quad \text{when} \quad n > 6\ ,$$

$$J(u_{\varepsilon,x}) = \mu(S^n) + \frac{(n-2)(n-1)\omega_{n-1}}{60n(n+2)\omega_n^{1-2/n}}\left|\mathrm{Weyl}(g)(x)\right|^2 \varepsilon^2 \mathrm{Log}\varepsilon + o(\varepsilon^2 \mathrm{Log}\varepsilon) \quad \text{when} \quad n = 6\ .$$

The strict inequality $J(u_{\varepsilon,x}) < \mu(S^n)$ is then a consequence of the non nullity of the Weyl tensor at some point of X. When the manifold is locally conformally flat, with g Euclidean near x, we get for $n \geq 6$

$$J(u_{\varepsilon,x}) = \mu(S^n) + C\varepsilon^{n/2-1}\left(\frac{n-2}{4(n-1)}\int_X \mathrm{Scal}(g)\, dv(g) - \frac{(n-2)\omega_{n-1}\delta^n}{\varepsilon + \delta^2} + o(\varepsilon)\right)$$

where $C > 0$ is independent of ε. The functional characterization of the mass (FCM_1) (see §4) then shows that we can choose g such that $J(u_{\varepsilon,x}) < \mu(S^n)$ for $\varepsilon << 1$.

In dimensions 3, 4, 5 the argument works identically. If g is such that, in geodesic normal coordinates at x, $\det(g) = 1 + O(r^m)$, with $m >> 1$, we get

$$J(u_{\varepsilon,x}) \leq \mu(S^n) + C(\varepsilon,\delta)\,\varepsilon^{n/2-1}$$
$$\left[\int_X \mathrm{Scal}(g) dv(g) - 4(n-1)\frac{\omega_{n-1}\delta^n}{\varepsilon + \delta^2} + K\delta^{6-n}(\varepsilon+\delta^2)^{n-2}\right]$$

where K is a positive constant independent of ε and d, and where $C(\varepsilon,\delta)$ is always positive. Since $\alpha(x) > 0$ (see §4), the functional characterization (FCM_2) then shows that we can choose g such that $J(u_{\varepsilon,x}) < \mu(S^n)$ for $\varepsilon, \delta << 1$. This ends the proof of the proposition.

3. SOME WORDS ABOUT THE PROOF OF THEOREM 1

As already mentioned, the proof of Theorem 1, which is based on Theorem 2, is straightforward if all orbits of G are infinite. If not, the proof proceeds by choosing appropriate test functions. In dimensions 3, 4, 5 and in the locally conformally flat case, the difficulties are essentially technical. They come from the fact that we have to consider only G-invariant test functions. But the aim of the demonstration is quite the same as the one of the classical Yamabe problem.

Essential difficulties appear when we consider the non locally conformally flat case. The problem comes from the fact that the concentration points are points where $\sharp O_G(x)$ is minimal. Test functions we consider should then be centered on the minimal orbits of G, and although the manifold is not locally conformally flat, the Weyl tensor can vanish at those points. Thus, there is no chance to proceed as in the proof of the classical Yamabe problem.

In fact, if there exists a point x in a minimal G-orbit where $\mathrm{Weyl}(g)(x) \neq 0$, then we can conclude as in the classical Yamabe problem. If not, the Weyl tensor vanishes all along minimal G-orbits. Our test functions will then have to recover the first derivative of the Weyl tensor. We prove that, if $\mathrm{Weyl}(g)(x) = 0$ and $\nabla\mathrm{Weyl}(g)(x) \neq 0$ at a point of minimal G-orbit, then $(*)$ is true again. In the next step we prove that if $\mathrm{Weyl}(g)(x) = 0, \nabla\mathrm{Weyl}(g)(x) = 0$ and $\nabla^2\mathrm{Weyl}(g)(x) \neq 0$ at a point of minimal G-orbit, then $(*)$ is true once more.

On the other hand, if the Weyl tensor and its derivatives vanish up to order $\Lambda = \left[\frac{n-6}{2}\right]$ (the integral part of $(n-6)/2$), we are able to recover the mass of the asymptotically flat manifold $(X - \{x\}, G_x^{4/(n-2)}g)$, where G_x is the Green function at x of the conformal laplacian. So, here, the strong form of the positive mass theorem will be used (which was not the case for the classical Yamabe problem where only the weak form is used). The point is that if $\nabla^i\mathrm{Weyl}(g)(x) = 0$, for all $0 \leq i \leq \Lambda$, at a point of a minimal G-orbit, then positivity of the mass of the asymptotically flat manifold $(X - \{x\}, G_x^{4/(n-2)}g)$, which comes from the strong form of the positive mass theorem, gives the strict inequality $(*)$. (For more details on the positive mass theorems, see §4).

Here, it is essential to use at each step the geometric information given by the negation of the precedent step. The theorem which enables us to do this can be stated as follows.

Theorem 5 (Hebey-Vaugon [HV2]). — *Let (X, g) be a compact Riemannian manifold, G a subgroup of $I(X, g)$, x_o a point of X on a finite G-orbit, and $\omega \in \mathbb{N}$ such that $\nabla^i \mathrm{Weyl}(g)(x_0) = 0$, for all $i < \omega$. ($\omega = 0$ if $\mathrm{Weyl}(g)(x_o) \neq 0$). Then, there exists a conformal G-invariant metric g' to g such that in g'-geodesic normal coordinates at one of any $x \in O_G(x_o)$:*

(1) $\det g' = 1 + O(r^s)$, $s >> 1$ *(given in advance and arbitrary large) ;*

(2)
$$g'_{ij} = \delta_{ij} + \sum_{m=\omega+4}^{2\omega+5} \frac{2(m-3)}{(m-1)!} \sum_{p_j} \left(\nabla_{p_3 \cdots p_{m-2}} R(g')(x_o)_{ip_1p_2j}\right) x^{p_1} \cdots x^{p_{m-2}} +$$
$$+ \left[\frac{4(\omega+3)(2\omega+3)}{(2\omega+6)!} \sum_{p_j} \left(\nabla_{p_3 \cdots p_{2\omega+4}} R(g')(x_o)\right)_{ip_1p_2j}\right] x^{p_1} \cdots x^{p_{2\omega+4}}$$
$$+ \left(1 + \frac{\omega+3}{2\omega+5}\right) \frac{(\omega+1)^2}{(\omega+3)!^2}$$
$$\left[\sum_{q=1}^{n} \sum_{p_j} \left(\nabla_{p_3 \cdots p_{\omega+2}} R(g')(x_o)\right)_{ip_1p_2q} \left(\nabla_{p_{\omega+5} \cdots p_{2\omega+4}} R(g')(x_o)\right)_{jp_{\omega+3}p_{\omega+4}q}\right]$$
$$x^{p_1} \cdots x^{p_{2\omega+4}} + O(r^{2\omega+5}).$$

(3) $\nabla^i R(g')(x) = 0$, $\forall i < \omega$ *;*

(4) $\nabla_\alpha R(g')(x) = \partial_\alpha R(g')(x)$, $\nabla_\alpha \mathrm{Ric}(g')(x) = \partial_\alpha \mathrm{Ric}(g')(x)$ *and* $\nabla_\alpha \mathrm{Scal}(g')(x) = \partial_\alpha \mathrm{Scal}(g')(x)$ *for any multi-index a such that* $|\alpha| \leq 2\omega + 1$.

(5) $\mathrm{Sym}_{p_1 \cdots p_m} \left(\nabla_{p_3 \cdots p_m} \mathrm{Ric}(g')(x)\right)_{p_1p_2} = 0$ *for any* $\omega + 2 \leq m \leq 2\omega + 3$ *and* $\mathrm{Sym}_{p_1 \cdots p_{2\omega+4}} \left(\nabla_{p_3 \cdots p_{2\omega+4}} \mathrm{Ric}(g')(x)\right)_{p_1p_2} = -C(\omega)\ \mathrm{Sym}_{p_1 \cdots p_{2\omega+4}} \sum_{1 \leq i,j \leq n} \left(\nabla_{p_3 \cdots p_{\omega+2}} R(g')(x)\right)_{ip_1p_2j} \left(\nabla_{p_{\omega+5} \cdots p_{2\omega+4}} R(g')(x)\right)_{ip_{\omega+3}p_{\omega+4}j}$ *where* $C(\omega) = \frac{(\omega+1)^2(\omega+2)^2(2(\omega+1))!}{(\omega+3)!^2}$.

(6) $C(2,2)(\mathrm{Sym}_\alpha \nabla_\alpha \mathrm{Scal}(g')(x)) = 0$ *for any multi-index* α *such that* $|\alpha| \leq 2\omega + 1$.

Here: $\mathrm{Sym}_{p_1\cdots p_m} T_{p_1\cdots p_m} = \Sigma_{\{\sigma \text{permutation of } (1,\cdots,m)\}} T_{p_{\sigma(1)}\cdots p_{\sigma(m)}}$, $C(2,2)T_{p_1\cdots p_{2m}} = \Sigma_{p_j} T_{p_1 p_1 \cdots p_m p_m}$, $(C(2,2)T_{p_1\cdots p_{2m}k})_k = \Sigma_{p_j} T_{p_1 p_1\cdots p_m p_m k}$ and $\nabla iT = \nabla\cdots\nabla T$($i$ times). As an example, $C(2,2)T_{ijk\ell} = \Sigma_{i,j} T_{iijj}$ and $(C(2,2)T_{ijklm})_m = \Sigma_{i,j} T_{iijjm}$.

4. WEAK AND STRONG FORMS OF THE POSITIVE MASS THEOREM

The main references of this section are Bartnik [B], Lee-Parker [LP], Parker-Taubes [PT], Schoen [S], Schoen-Yau [SY 1,2,3] and Witten [W].

First of all, we need to define what we mean when we speak of asymptotically flat manifolds. These manifolds were originally introduced by physicists. They arose first in general relativity as solutions of the Einstein field equation $\mathrm{Ric}(g) - \frac{1}{2}\mathrm{Scal}(g)g = T$ (T an energy momentum tensor). This is the case for the Schwarzschild metric, a (singular) Lorentz metric on $\mathbb{R}^4$ which, when restricted to any constant-time three-plane, is asymptotically flat of order τ.

Definition 6. — *Let (X,g) be a Riemannian manifold. (X,g) is asymptotically flat of order 1, if there exists a decomposition $X = X_o \cup X_\infty$, with X_o compact, and if there exists a diffeomorphism from X_∞ to $\mathbb{R}^n - B_0(R)$, for some $R > 0$, the metric satisfying in the coordinates $\{z^i\}$ induced on X_∞ by the diffeomorphism*

$$g_{ij} = \delta_{ij} + O(r^{-\tau}),\ \partial_k g_{ij} = O(r^{-\tau-1}),\ \partial_{km} g_{ij} = O(r^{-\tau-2})\ .$$

The $\{z^i\}$ are called asymptotic coordinates.

This definition apparently depends on the choice of asymptotic coordinates. However, Bartnik [B] proved that the asymptotically flat structure is determined by the metric alone when $\tau > (n-2)/2$.

An important and simple remark one has to do here is that if (X,g) is a compact Riemannian manifold, if x is a point of X and if $\{x^i\}$ are normal coordinates at x, then $(X - \{x\}, r^{-4}g)$ is an asymptotically flat manifold with asymptotic coordinates $z^i = x^i/r^2$.

Physicists were then led to introduce the following geometric invariant.

Definition 7. — *Let (X,g) be an asymptotically flat manifold with asymptotic coordinates $\{z^i\}$. The mass $m(g)$ of (X,g) is defined by*

$$m(g)=\lim_{R\to\infty}\frac{1}{\omega_{n-1}}\int_{\partial B_0(R)}(H\rfloor dz)\qquad (\rfloor \textit{ the interior product})$$

where H is the mass-density vector field defined on X_∞ by $H=\Sigma_{i,j}(\partial_i g_{ij}-\partial_j g_{ii})\partial_j$.

Here again, it is possible to prove that if $\mathrm{Scal}(g)$ is a non negative function of $L_1(X)$ and if $\tau>(n-2)/2$, then $m(g)$ exists and depends only on the metric g.

Arnowitt, Deser and Misner then conjectured that in dimension 3, if $\mathrm{Scal}(g)\geq 0$, $m(g)$ is always non negative with equality to zero if and only if (X,g) is isometric to $\mathbb{R}^3$ with its Euclidean metric. The same conjecture was made in dimension 4, when $\mathrm{Scal}(g)=0$, by Gibbons, Hawking and Perry. The natural generalization of these conjectures (the strong form of the positive mass conjecture) is that an asymptotically flat manifold (X,g) of dimension $n\geq 3$ with non negative scalar curvature has $m(g)\geq 0$, with equality if and only if X is isometric to $\mathbb{R}^n$.

This conjecture was solved by Schoen-Yau and by Witten in the spinorial case. In fact, we have the following theorem.

Theorem 8. (Schoen-Yau. Strong form of the positive mass theorem.) — *Let (X,g) be an asymptotically flat manifold of dimension $n\geq 3$ and order $\tau>(n-2)/2$, with non negative scalar curvature belonging to $L_1(X)$. Its mass $m(g)$ is then non negative, and we have $m(g)=0$ if and only if (X,g) is isometric to $\mathbb{R}^n$ with its Euclidean metric.*

From now on, let (X,g) be a compact locally conformally flat Riemannian manifold of dimension $n\geq 3$ and scalar curvature satisfying $\int_X \mathrm{Scal}(g)dv(g)>0$ (this is equivalent to saying that $[g]$ posseses a metric of positive scalar curvature).

We define the conformal Laplacian, acting on functions, by $L(u)=\Delta u+\frac{n-2}{4(n-1)}\mathrm{Scal}(g)u$.

It is then easy to prove that L posseses a unique Green function G, and that if $g' = u^{4/(n-2)}g$ is conformal to g, we have $G'(P,Q) = \frac{G(P,Q)}{u(P)u(Q)}$. Moreover, if $x \in X$ and if g is Euclidean near x, the Green function G_x at x of L can be written (near x)

$$G_x = \frac{\text{Cte}}{r^{n-2}} + \alpha, \text{where } \alpha \text{ is a smooth function of } \ C^\infty(X)\ .$$

Here again, if $g' = u^{4/(n-2)}g$ is Euclidean near x, we get $\alpha'(x) = \frac{\alpha(x)}{u(x)^2}$.

Now, the weak form of the positive mass conjecture states that $\alpha(x)$ is always non negative, with equality to zero if and only if X is isometric to the standard unit sphere of $\mathbb{R}^{n+1}$. This weak form was proved by Schoen-Yau in [SY2].

Theorem 9 (Schoen-Yau). — *Suppose g is locally conformally flat and Euclidean near x, and let $G_x = Cr^{-n+2} + \alpha$, $\alpha \in C^\infty(X)$, be the Green function at x of the conformal Laplacian. Then, $\alpha(x)$ is always non negative, and we have $\alpha(x) = 0$ if and only if (X,g) is isometric to the standard unit sphere S^n.*

As a matter of fact, this theorem can be seen as a corollary of the strong form of the positive mass theorem, since it is possible to prove that $\alpha(x)$ is proportional (with positive coefficient) to the mass of the asymptotically flat manifold $(X - \{x\}, G_x^{4/(n-2)}g)$. This remark was first made by Schoen [S1]. Of course, the proof presented in [SY2] does not make use of this fact.

For our purpose, we need another characterization of $\alpha(x)$. In [HV1], we obtain the following result.

Proposition 10 (Hebey-Vaugon). — *If (X,g) is a compact locally conformally flat Riemannian manifold of dimension $n \geq 6$, with g Euclidean near x, then:*

$$(FCM_1) \qquad \alpha(x) = \text{Sup}\frac{4(n-1)}{(n-2)}\left(\frac{1}{\int_X \text{Scal}(g')dv(g')} - \frac{1}{4(n-1)\omega_{n-1}\rho^{n-2}}\right)$$

the supremum being taken over $\rho << 1$ and $g' \in [g]$, Euclidean on $B_x(\rho)$, which satisfy the normalization condition $g'(x) = g(x)$.

In dimensions 3, 4 and 5, for an arbitrary compact Riemannian manifold, one may also define $\alpha(x)$. In fact, if $x \in X$ and if, in geodesic normal coordinates at

$x, \det(g) = 1 + O(r^m)$, with $m >> 1$, the Green function at x of the conformal Laplacian has a good development and it is then possible to define the mass $\alpha(x)$. Here again, we can show that

$$(FCM_2) \qquad \alpha(x) = \overline{\lim_{\rho \to 0}} \left(\mathrm{Sup} \frac{4(n-1)}{(n-2)} \left[\frac{1}{\int_X \mathrm{Scal}(g')dv(g')} - \frac{1}{4(n-1)\omega_{n-1}\rho^{n-2}} \right] \right)$$

where the supremum is taken over the $g' \in [g]$ which satisfy $g' = g$ on $B_x(\rho)$.

The proof is the same as the the one done in [HV1] to prove Proposition 10. In the locally conformally flat case, the two characterizations coincide. Moreover, since $\alpha(x)$ is proportional (with positive coefficient) to the mass of the manifold $(X - \{x\}, G_x^{4/(n-2)}g)$, it is always positive, unless (X, g) is isometric to the standard unit sphere of $\mathbb{R}^{n+1}$.

5. THE EQUIVARIANT APPROACH. PROOF OF THEOREM 2

The proof of Theorem 2 is based on a detailed analysis of the concentration phenomena which may occur for minimizing subcritical sequences.

To be more precise, we first prove that for $1 < q < N = 2n/(n-2)$, there exists a G-invariant smooth function $u_q \in C^\infty(X)$ and there exists $\lambda_q > 0$ such that:

a) $\Delta u_q + \frac{n-2}{4(n-1)}\mathrm{Scal}(g)u_q = \lambda_q u_q^{q-1}$,

b) $\int_X u_q^q dv(g) = 1$,

c) $\overline{\lim_{q \to N}} \lambda_q \le \mathrm{Inf}_G J(u)$.

The existence of u_q is not difficult to obtain since the imbedding $W^{1,2}(X) \subset L_q(X)$ is compact for $1 < q < N$.

We then prove that if a subsequence of (u_q) converges as $q \to N$ to some $u \neq 0$ in $L_2(X)$, then $J(u) = \mathrm{Inf}_G J(u)$ and, therefore, $g' = u^{4/(n-2)}g$ is a G-invariant metric of constant scalar curvature. As a matter of fact, we may suppose that this subsequence converges to u strongly in $L_2(X) \cap L_{N-1}(X)$, almost everywhere and weakly in $W^{1,2}(X)$, with $\lim_{q\to N} \lambda_q = \lambda \le \mathrm{Inf}_G J(u)$ that exists. Classical arguments then prove that u is a smooth positive function of $C^\infty(X)$ which satisfies

$$(1) \qquad \Delta u + \frac{n-2}{4(n-1)} Scal(g)u = \lambda u^{(n+2)/(n-2)} \,.$$

Now, we have to prove that $\mathrm{Inf}_G J(u)$ and that $\int_X u^N dv(g) = 1$. But,

$$\begin{aligned}\int_X u^N dv(g) &= \lim_{q\to N} \int_X u_q^{N-1} u dv(g)\\ &\le \lim_{q\to N} \left(\int_X u_q^q dv(g)\right)^{(N-1)/q} \left(\int_X u^{q/(1+q-N)} dv(g)\right)^{(1+q-N)/q}\\ &\le \left(\int_X u^N dv(g)\right)^{1/N}\end{aligned}$$

and, therefore, $\int_X u^N dv(g) \le 1$. Independently, if we multiply (1) by u and if we integrate, we obtain $\mathrm{Inf}_G J(u) \le \lambda \left(\int_X u^N dv(g)\right)^{2/n}$. Since $\lambda \le \mathrm{Inf}_G J(u)$, we get what we wanted to prove, namely $\lambda = \mathrm{Inf}_G J(u)$ and $\int_X u^N dv(g) = 1$.

Now, we have to study the situation where all the subsequences of (u_q) which converge in a $L_p(X)$, $p \ge 2$, converge to zero. In this situation, it is possible to prove that there exists a finite number $\{x_1, ..., x_k\}$ of points of X such that:

d) $\mathrm{Inf}_G J(u) \left(\overline{\lim_{q\to N}} \int_{B_{x_i}(\delta)} u_q^q dv(g)\right)^{2/n} \ge \mu(S^n)$, for all $i = 1, ..., k$ and all $\delta > 0$,

e) for all $p \in \mathbb{N}$ and all compact $K \subset X - \{x_1, ..., x_k\}$, (u_q) converges to zero in $C^p(K)$.

We now use the fact that (u_q) is G-invariant and that $\int_X u_q^q dv(g) = 1$. First of all, we notice that $\sharp O_G(x_i) < \infty$, for all $i = 1, ..., k$. If not, for all $\varepsilon > 0$, we will find a $\delta > 0$ such that $\int_{B_{x_i}(\delta)} u_q^q dv(g) \le \varepsilon$. But if ε is small enough, this is in contradiction with d). In the same way, if $\sharp O_G(x_i) < \infty$, we can choose δ small enough such that $\int_{B_{x_i}(\delta)} u_q^q dv(g) \le \frac{1}{\sharp O_G(x_i)}$. Therefore, according to d), we obtain $\mathrm{In}_G J(u) \ge (\sharp O_G(x_i))^{2/n} \mu(S^n)$, $\forall i$. But this is impossible if $\mathrm{Inf}_G J(u) < \mu(S^n)$ $(\mathrm{Inf}_{x\in X} \sharp O_G(x))^{2/n}$.

As a consequence, under the hypothesis of theorem 2, the $\{x_1, ..., x_k\}$ do not exist. Therefore, there exists a subsequence of (u_q) which converges to $u \ne 0$ in $L_2(X)$. This ends the proof of the first part of the theorem.

Now, we have to prove that $\mathrm{Inf}_G J(u) \leq \mu(S^n)(\mathrm{Inf}_{x\in X}\sharp O_G(x))^{2/n}$. We may suppose that $\mathrm{Inf}_{x\in X}\sharp O_G(x) < \infty$. Let x_1 be a point of X of minimal G-orbit. If $O_G(x_1) = \{x_1, ..., x_k\}$ and if $\delta > 0$ is such that $B_{x_i}(\delta) \cap B_{x_j}(\delta) = \emptyset$ for $i \neq j$, we let (as in Aubin [A1]),

$$u_{i,\varepsilon}(x) = (\varepsilon + d(x_i, x)^2)^{1-n/2} - (\varepsilon + \delta^2)^{1-n/2} \text{ if } d(x_i, x) \leq \delta$$
$$u_{i,\varepsilon}(x) = 0 \text{ if } d(x_i, x) \geq \delta \ .$$

If $u_\varepsilon = \sum_i u_{i,\varepsilon}$, u_ε is G-invariant and we have $J(u_\varepsilon) = k^{2/n} J(u_{1,\varepsilon})$. Independently, it is possible to prove (cf. Aubin [A1]) that $\lim\limits_{\varepsilon\to 0} J(u_{1,\varepsilon}) = \mu(S^n)$.

Therefore, $\mathrm{Inf}_G J(u) \leq \lim\limits_{\varepsilon\to 0} J(u_\varepsilon) = \mu(S^n)\ (\mathrm{Inf}_{x\in X}\sharp O_G(x))^{2/n}$. This ends the proof of the theorem.

6. THE LOCALLY CONFORMALLY FLAT CASE

Let us start with the following two results (for details see [HV2]).

Lemma 1. — *Let $(S^n, \mathrm{st.})$ be the standard unit sphere of $\mathbb{R}^{n+1}$. If $x \in S^n$, we let $C_x(S^n, \mathrm{st.}) = \{\sigma \in C(S^n, \mathrm{st.})/\sigma(x) = x\}$ and $I_x(S^n, \mathrm{st.}) = \{\sigma \in I(S^n, \mathrm{st.})/\sigma(x) = x\}$. If g is a metric on S^n which is conformal to* st., *then there exists $\tau \in C_x(S^n, \mathrm{st.})$ such that $\tau^{-1} I_x(S^n, g)\tau \subset I_x(S^n, \mathrm{st.})$, where $I_x(S^n, g) = \{\sigma \in I(S^n, g)/\sigma(x) = x\}$.*

Lemma 2. — *Let (X, g) be a compact locally conformally flat manifold of dimension $n \geq 3$ and let G be a compact subgroup of $I(X, g)$. Then, for all $x \in X$ which has a finite G-orbit, there exists $g' \in [g]$ which is G-invariant and Euclidean in a neighbourhood of each $y \in O_G(x)$.*

Now, let (X, g) be a compact locally conformally flat manifold of dimension $n \geq 3$ and let G be a compact subgroup of $I(X, g)$. According to Theorem 2, we may restrict ourselves to the case where G posseses finite orbits.

Let $\{x_1, ..., x_k\}$ be a minimal G-orbit. With Lemma 2, we may suppose that g is Euclidean in a neighbourhood of each x_i. The Green function G_i at x_i of the conformal Laplacian can then be written (near x_i)

$$G_i(x) = \frac{1}{(n-2)\omega_{n-1} r_i^{n-2}} + A + \alpha_i(x) \ ,$$

where A is a constant and where $\alpha_i \in C^\infty(X)$ satisfies $\alpha_i(x_i) = 0$. ($r_i = d(x_i, x)$). According to the weak form of the positive mass theorem, we have $A > 0$ if $(X, [g]) \neq (S^n, [\text{st.}])$. Now, we consider (as Schoen in [S1]), the test functions $u^i_{\delta,\varepsilon}$ defined by

$$\begin{aligned} u^i_{\delta,\varepsilon}(x) &= \left(\frac{\varepsilon}{\varepsilon^2 + r_i^2}\right)^{(n-2)/2} \quad \text{if} \quad r_i \leq \delta \\ &= \varepsilon_0 (G_i(x) - \eta(x)\alpha_i(x)) \quad \text{if} \quad \delta \leq r_i \leq 2\delta \\ &= \varepsilon_0 G_i(x) \quad \text{if} \quad r_i \geq 2\delta \ , \end{aligned}$$

where $\delta > 0$ is chosen small enough that g is Euclidean on $B_{x_i}(2\delta)$ and such that $B_{x_i}(2\delta) \cap B_{x_j}(2\delta) = \emptyset$ if $i \neq j$, where η is a smooth radial function which satisfies $0 \leq \eta \leq 1$, $\eta(x) = 1$ if $r_i \leq \delta$, $\eta(x) = 0$ if $r_i \geq 2\delta$ and $|\nabla \eta| \leq \frac{2}{\delta}$, and where ε_0 satisfies

$$\left(\frac{\varepsilon}{\varepsilon^2 + \delta^2}\right)^{(n-2)/2} = \varepsilon_0 \left(A + \frac{1}{(n-2)\omega_{n-1}\delta^{n-2}}\right) .$$

We let $u_{\delta,\varepsilon} = \sum_{i=1}^k u^i_{\delta,\varepsilon}$. The function $u_{\delta,\varepsilon}$ is G-invariant and it is possible to prove that

$$J(u_{\delta,\varepsilon}) \leq k^{2/n} \mu(S^n) - \varepsilon_0^2 (C_0 A + C_1(k-1)) + \text{ terms in } \delta\varepsilon_0^2 \text{ and } o(\varepsilon_0^2) \ .$$

(We do not develop the calculations here. For more details, see [HV2]. The term $C_1(k-1)$, which does not appear in [S1], comes from the symmetrisation).

In fact, the same result holds also for manifolds of dimensions 3, 4 and 5, since for such manifolds we can choose g such that G_i still has a good development. (Here again, see [HV2].) In particular, according to this last inequality, we can find ε, δ small enough that $J(u_{\delta,\varepsilon}) < k^{2/n}\mu(S^n)$, if $C_0 A + C_1(k-1) > 0$. Therefore, with the weak form of the positive mass theorem, the strict inequality of Theorem 2 is satisfied by locally conformally flat manifolds and by manifolds of dimensions 3, 4 and 5, which are not conformally diffeomorphic to the standard sphere S^n. As already mentioned, this ends the proof of the theorems for such manifolds.

Moreover, the strict inequality of Theorem 2 is also satisfied by S^n when $\mathrm{Inf}_{x\in X}\sharp O_G(x) \geq 2$ (as $k-1>0$). Now, we have to deal with the case $(X,[g]) = (S^n,[\mathrm{st.}])$, $\mathrm{Inf}_{x\in X}\sharp O_G(x) = 1$. Let x be such that $\sharp O_G(x) = 1$. We then have $G \subset I_x(S^n,g)$ and, with Lemma 1, there exists $\tau \in C_x(S^n,\mathrm{st})$ such that $G \subset \tau^{-1}I_x(S^n,\mathrm{st})\tau$.

If $f>0$ is such that $(\tau^{-1})^*\mathrm{st} = f^{4/(n-2)}\mathrm{st}$, and if $\phi>0$ is such that $g = \phi^{4/(n-2)}\mathrm{st}$, we let

$$u(y) = \frac{1}{\phi(y)f(\tau(y))}, \qquad y \in S^n\ .$$

u is G-invariant. To see this, we consider $\sigma \in G$ and $i \in I_x(S^n,\mathrm{st})$ such that $\sigma = \tau^{-1}i\tau$. We then have

$$\begin{aligned}\sigma^*g &= \tau^*i^*(\tau^{-1})^*g\\ &= (\tau^*i^*)\left((\varphi\circ\tau^{-1})^{4/(n-2)}f^{4/(n-2)}\mathrm{st}\right)\\ &= ((\varphi\circ\sigma)(f\circ i\circ\tau))^{4/(n-2)}\,(f\circ\tau)^{-4/(n-2)}\mathrm{st}\ .\end{aligned}$$

Independently, $\sigma^*g = g$ implies $(\varphi\circ\sigma)\,((f\circ\tau)\circ\sigma) = \varphi(f\circ\tau)$. Therefore, $u\circ\sigma = u$, for all $\sigma\in G$.

Moreover, $J(u) = \mu(S^n)$ since $\phi u = \frac{1}{f\circ\tau}$ with $\tau^*\mathrm{st} = (f\circ\tau)^{-4/(n-2)}\mathrm{st}$. But, on $S^n, \mathrm{Inf}J(u) = \mu(S^n)$. Therefore u realizes $\mathrm{Inf}_G J(u)$. This ends the proof of Theorem 1 when $([X,[g]) = (S^n,[\mathrm{st.}])$.

7. CHOOSING AN APPROPRIATE REFERENCE METRIC

Let us start with the following result. This is the equivariant version of conformal normal coordinates. For more details on its proof, see [HV2].

Lemma 3. — *Let (X,g) be a compact Riemannian manifold of dimension $n\geq 3$ and let G be a compact subgroup of $I(X,g)$. If $x\in X$ is of finite G-orbit, then, for all*

$m \in \mathbb{N}$, there exists a G-invariant metric g', conformal to g, such that in g'-geodesic normal coordinates at each $y \in O_G(x)$, $\det g' = 1 + O(r^m)$ (where $r = d(y, \cdot)$, d the distance for g').

Now, we suppose that $\nabla^i R(g')(x_o) = 0$, $\forall i < \omega$. We will then prove that, in geodesic normal coordinates at x_o, g' can be written as in the relation (2) of Theorem 5. In fact, the exponential map at x_o allows us to study the problem in a neighbourhood of $0 \in \mathbb{R}^n$. Now, for $\tau, \xi \in \mathbb{R}^n$, we let $\gamma : \mathbb{R}\times\mathbb{R}\to\mathbb{R}^n$ be the map defined by $\gamma_s(t) = t(\tau+s\xi)$. In the same way, we let $T = \gamma_s'(t)$ and $X(\gamma_s(t)) = \partial/\partial s \gamma_s(t) = t\xi$.

If we derive the Jacobi relation $\nabla_T^2 X = R(g)(T, X)T$, we obtain for $r \geq 2$

$$\nabla_T^r X = \sum_{i=0}^{r-2} C_{r-2}^i \left(\nabla_T^{r-2-i} R(g)\right)(T, \nabla_T^i X)T \qquad (\text{as} \nabla_T T = 0) .$$

Therefore,

$$\nabla_T^r X(0) = 0 \text{ for } 2 \leq r \leq \omega + 2, \text{ and } X(0) = 0,\ \nabla X(0) = \xi .$$

Thus,

$$\nabla_T^r X(0) = (r-2)(\nabla_\tau^{r-3} R(g)(0))(I, \xi)I \text{ for } \omega + 3 \leq r \leq 2\omega + 4 ,$$

and

$$\nabla_T^{2\omega+5} X(0) = (2\omega+3)(\nabla_\tau^{2\omega+2} R(g)(0))(\tau,\xi)\tau + C_{2\omega+3}^{\omega}(\nabla_\tau^{\omega} R(g)(0))(\tau, \nabla_\tau^{\omega+3} X)\tau$$
$$= (2\omega+3)(\nabla_\tau^{2(\omega+1)} R(g)(0))(\tau,\xi)\tau + (\omega+1)C_{2\omega+3}^{\omega}(\nabla_\tau^{\omega} R(g)(0))(\tau, \nabla_\tau^{\omega} R(g)(0)(\tau,\xi)\tau)\tau .$$

Independently, if $f(t) = |X(\gamma_o(t))|^2$,

$$f^{(r)}(0) = (\nabla_T^r f)(0) = \nabla_T^r g(X, X)(0) = \sum_{i=0}^{r} C_r^i g(0)(\nabla_T^{r-i} X, \nabla(i, T)X) ,$$

and, therefore, $f(0) = 0$, $f'(0) = 0$, $f''(0) = 2g(0)(\xi, \xi)$ and

$$\begin{aligned} f^{(r)}(0) &= 0 \text{ for } 3 \leq r \leq \omega + 3 \\ &= 2r(r-3)g(0)(\nabla_\tau^{r-4} R(g)(0))(\tau,\xi)\tau,\xi) \text{ for } \omega + 4 \leq r \leq 2\omega + 5 \\ &= 4(\omega+3)(2\omega+3)g(0)(\nabla_\tau^{2(\omega+1)} R(g)(\tau, \nabla_\tau^{\omega} R(g)(\tau,\xi)\tau,\xi) \\ &\quad + 4(\omega+3)(\omega+1)C_{2\omega+3}^{\omega} g(0)(\nabla_\tau^{\omega} R(g)(\tau, \nabla_\tau^{\omega} R(g)(\tau,\xi)\tau)\tau,\xi) \\ &\quad + (\omega+1)^2 C_{2\omega+6}^{\omega+3} g(0)(\nabla_\tau^{\omega} R(g)(\tau,\xi)\tau, \nabla_\tau^{\omega} R(g)(\tau,\xi)\tau) \text{ for } r = 2\omega + 6 . \end{aligned}$$

We then obtain

$$g(t\tau)(\xi,\xi) = t^{-2}f(t) = g(0)(\xi,\xi) + \sum_{r=4}^{2\omega+5} \frac{2r(r-3)}{r!} t^{r-2} g(0)(\nabla_\tau^{r-4} R(g)(0))(\tau,\xi)\tau\ ,\xi)$$

$$+\frac{1}{(2\omega+6)!} f^{(2\omega+6)}(0) t^{2\omega+4} + O(t^{2\omega+5})\ .$$

But, if

$$\tau = \tau^i\partial_i, \nabla_\tau^r R(g)(\tau,\partial_p)\tau = (\nabla_{i_1\cdots ir} R^i_{mnp}(g)\partial_i)\tau^m\tau^n\tau^{i_1}\cdots\tau^{i_r}\ ,$$

and, therefore,

$$g(t\tau)(\partial_i,\partial_j) = \delta_{ij} + \sum_{m=\omega+4}^{2\omega+5} \frac{2(m-3)}{(m-1)!} t^{m-2} (\nabla_{p_3\cdots p_{m-2}} R(g)(0)_{ip_1p_2j})\tau^{p_1}\cdots\tau^{p_{m-2}}$$
$$+\left[\frac{4(\omega+3)(2\omega+3)}{(2\omega+6)!} t^{2\omega+4}\left(\nabla_{p_3\cdots p_{2\omega+4}} R(g)(0)_{ip_1p_2j}\right)\right]\tau^{p_1}\cdots\tau^{p_{2\omega+4}}$$
$$+(1+\frac{\omega+3}{2\omega+5})\frac{(\omega+1)^2}{(\omega+3)!^2}$$
$$t^{2\omega+4}\left[\sum_{q=1}^{n}\left(\nabla_{p_3\cdots p_{\omega+2}} R(g)(0)_{ip_1p_2q}\right)\left(\nabla_{p_{\omega+5}\cdots p_{2\omega+4}} R(g)(0)_{jp_{\omega+3}p_{\omega+4}q}\right)\right]\tau^{p_1}\cdots\tau^{p_{2\omega+4}}$$
$$+O(t^{2\omega+5})\ .$$

We then obtain the conclusion, i.e. relation (2) of Theorem 5, when we let $x = t\tau$.

From this relation, we get $\partial_\beta\Gamma^k_{ij} = 0$ for all $|\beta| \le \omega$. Since $\nabla^i R(g)(x_o) = 0$ for $i \le \omega - 1$, we obtain easily the point (4) of Theorem 5. To prove Point (5), we let (A_{ij}) be defined by $g_{ij} = \exp(A_{ij})$. We then obtain, since $\exp(A) = I + A + \frac{1}{2}A^2 + \cdots$,

$$A_{ij} = \sum_{m=\omega+4}^{2\omega+5} \frac{2(m-3)}{(m-1)!}\left(\nabla_{p_3\cdots p_{m-2}} R(g)(x_1)\right)_{ip_1p_2j} x^{p_1}\cdots x^{p_{m-2}}$$
$$+\left[\frac{4(\omega+3)(2\omega+3)}{(2\omega+6)!}\left(\nabla_{p_3\cdots p_{2\omega+4}} R(g)(x_1)\right)_{ip_1p_2j}\right] x^{p_1}\cdots x^{p_{2\omega+4}} - \frac{(\omega+1)^2(\omega+2)}{(2\omega+5)(\omega+3)!^2}$$
$$\left[\sum_{q=1}^{n}\sum_{p_j}\left(\nabla_{p_3\ldots p_{\omega+2}} R(g)(x_1)_{ip_1p_2q}\right)\left(\nabla_{p_{\omega+5}\cdots p_{2\omega+4}} R(g)(x_1)_{jp_{\omega+3}p_{\omega+4}q}\right)\right]$$
$$x^{p_1}\ldots x^{p_{2\omega+4}} + O(r^{2\omega+5})\ .$$

Point (5) of Theorem 5 is then a direct consequence of this relation, since $\det(g_{ij}) = \exp\,(\mathrm{trace}(A_{ij}))$. Moreover, the contraction of the first relations of this point (5), gives the point (6) of the theorem (i.e $C(2,2)\ \mathrm{Sym}_\alpha \nabla_\alpha \mathrm{Scal}(g)(x_o) = 0$, for $|\alpha| \leq 2\omega + 1$).

Finally, we have to prove that the two relations "$\det g = 1 + O(r^m)$, $m >> 1$" and "$\nabla^i \mathrm{Weyl}(g)(x_0) = 0$, $\forall i < \omega$" lead to "$\nabla^i R(g)(x_0) = 0$, $\forall i < \omega$". Here, the proof is by induction. If $\omega = 0$ or 1, the result is easily obtained. Thus, we have to prove that "$\det g = 1 + O(r^m)$, $m >> 1$" and "$\nabla^i \mathrm{Weyl}(g)(x_0) = 0$, $\forall i < \omega + 1$" lead to "$\nabla^\omega R(g)(x_0) = 0$". If $|\alpha| = \omega - 1$, we have (at the point x_0):

(a)

$$\nabla_{m\alpha} R(g)_{ijkl} - \frac{1}{(n-2)}(\nabla_{m\alpha}\mathrm{Ric}(g)_{ik}g_{j\ell} - \nabla_{m\alpha}\mathrm{Ric}(g)_{i\ell}g_{jk} + \nabla_{m\alpha}\mathrm{Ric}(g)_{j\ell}g_{ik}$$

$$-\nabla_{m\alpha}\mathrm{Ric}(g)_{jk}g_{i\ell}) + \frac{1}{(n-1)(n-2)}(\nabla_{m\alpha}\mathrm{Scal}(g))(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) = 0\ .$$

If we contract j and m, we then obtain

(b)

$$\frac{(n-3)}{(n-2)}(\nabla_{\ell\alpha}\mathrm{Ric}(g)_{ik} - \nabla_{k\alpha}\mathrm{Ric}(g)_{i\ell}) =$$

$$\frac{(n-3)}{2(n-1)(n-2)}(\nabla_{k\alpha}\mathrm{Scal}(g)g_{i\ell} - \nabla_{\ell\alpha}\mathrm{Scal}(g)g_{ik})\ .$$

Now, if $\alpha = m\beta$, $|\beta| = \omega - 2$, contraction of ℓ and m in (b) leads to

(c)

$$\frac{(n-3)}{(n-2)}\nabla_{mm\beta}\mathrm{Ric}(g)_{ik} = \frac{(n-3)}{2(n-1)}\nabla_{ik\beta}\mathrm{Scal}(g) - \frac{(n-3)}{2(n-1)(n-2)}\nabla_{mm\beta}\mathrm{Scal}(g)g_{ik}$$

and the relation $\mathrm{Sym}_{ik\ell m\beta}\nabla_{\ell m\beta}\mathrm{Ric}(g)(x_0)_{ikj} = 0$ (point (6) of the theorem) then allows us to prove that $\nabla_{ik\beta}\mathrm{Scal}(g)(x_0) = 0$.

According to (b) we then have $\nabla_{\ell\alpha}\mathrm{Ric}(g)(x_0)_{ik} = \nabla_{k\alpha}\mathrm{Ric}(g)(x_0)_{i\ell}$, and, therefore, we obtain $(\omega+2)!\nabla_{\ell\alpha}\mathrm{Ric}(g)(x_0) = 0$ since $\mathrm{Sym}_{ik\ell\alpha}\nabla_{\ell\alpha}\mathrm{Ric}(g)(x_0)_{ik} = 0$. Thus, according to (a), $\nabla_{m\alpha}R(g)(x_0)_{ijk\ell} = 0$. This ends the proof of the theorem.

8. THE NON LOCALLY CONFORMALLY FLAT CASE, n $\geq$ 6

Let $\Lambda = \left[\frac{n-6}{2}\right]$. G denotes a compact subgroup of $I(X,g)$ which possesses finite orbits and $\{x_1,...,x_k\}$ is a minimal G-orbit.

8.1. Let us first suppose that there exists $\omega \leq \Lambda$ such that $\nabla^i\mathrm{Weyl}(g)(x_1) = 0$, $\forall i < \omega$, with $\nabla^\omega\mathrm{Weyl}(g)(x_1) \neq 0$. In fact, we do not need to study this situation. A recent result of Schoen states that the concentration points of the sequence (u_q) introduced in Section 4 are points where the Weyl tensor and its derivatives vanish up to order Λ. According to this result we can directly study the situation described in the following subsection: there exists a minimal G-orbit $\{x_1,\cdots,x_k\}$ such that $\nabla^i\mathrm{Weyl}(g)(x_1) = 0$, $\forall i \leq \Lambda$. But we will loose strict inequality $\mathrm{Inf}_G J(u) < \mu(S^n)$ $(\mathrm{Inf}_{x\in X}\sharp O_G(x))^{2/n}$. Nevertheless, it is possible to prove that this strict inequality is true, at least when $\omega = 0$, 1 or 2. Therefore, according to what we have said before and according to the result of the next subsection, strict inequality will be satisfied by any manifold of dimension $n \leq 11$ (since $3 \leq \Lambda \Longrightarrow n \geq 12$). To summarize: we use Schoen's result, so that we just have to study the situation described in Subsection 8.2, when the manifolds are of dimension $n \geq 12$ and satisfy: for all x in a minimal G-orbit, $\exists\, 3 \leq \omega \leq \Lambda / \nabla^i\mathrm{Weyl}(g)(x) = 0$, $\forall i \leq \omega$ with $\nabla^\omega\mathrm{Weyl}(g)(x) \neq 0$. We have our own proof when $\omega = 0$, 1 or 2, and therefore, when the manifolds are of dimension $n \leq 11$. (In fact, this proof should work in all cases but we face important technical difficulties).

We treat here the cases $\omega = 0$ and $\omega = 1$. For details on the case $\omega = 2$, see [HV2]. If $\omega = 0$, i.e., if $\mathrm{Weyl}(g)(x_1) \neq 0$, according to Theorem 5 we may suppose that $\mathrm{Ric}(g)(x_1) = 0$. Here, we easily obtain the strict inequality $(*)$ of Theorem 2.

We proceed as in Aubin [A1]. We let $u_\varepsilon = \sum_{i=1}^k u_{i,\varepsilon}$ where

$$u_{i,\varepsilon} = \left(\varepsilon + r_i^2\right)^{1-n/2} - (\varepsilon + \delta^2)^{1-n/2} \text{ if } r_i \leq \delta$$
$$u_{i,\varepsilon} = 0 \text{ if } r_i \geq \delta, \delta > 0, r_i = d(xi, .) \ .$$

We then have (when δ is small enough), $J(u_\varepsilon) = k^{2/n} J(u_{1,\varepsilon})$ and

$$J(u_{1,\varepsilon}) = \mu(S^n)\left(1 - \frac{\varepsilon^2}{12n(n-4)(n-6)} \left|\mathrm{Weyl}(g)(x)\right|^2 + o(\varepsilon^2)\right) \text{ if } n > 6$$
$$J(u_{1,\varepsilon}) = \mu(S^n) + \frac{(n-2)(n-1)\omega_{n-1}}{60n(n+2)\omega_n^{1-2/n}} |\mathrm{Weyl}(g)(x)|^2 \varepsilon^2 \mathrm{Log}\varepsilon + o(\varepsilon^2 \mathrm{Log}\varepsilon) \text{ if } n = 6 \ .$$

Therefore, we can choose ε small enough such that $J(u_\varepsilon) < k^{2/n}\mu(S^n)$. We then obtain the strict inequality case $(*)$ of Theorem 2.

Now, let us study the case $\omega = 1$, i.e the case where $\mathrm{Weyl}(g)(x_1) = 0$, $\nabla\mathrm{Weyl}(g)(x_1) \neq 0$ (and $n \geq 8$). Here again, we may suppose that g satisfies all points (1) to (6) of Theorem 5 (with $\omega = 1$). If the u_ε are defined as before, it is then possible to prove (see [HV2]), that

$$J(u_{1,\varepsilon}) \leq \mu(S^n) + C\varepsilon^{(n/2)-1} + CA\varepsilon^3 + C\varepsilon^4 \text{ if } n > 8$$
$$J(u_{1,\varepsilon}) \leq \mu(S^n) - CA(\mathrm{Log}\varepsilon)\varepsilon^{(n/2)-1} + C\varepsilon^{(n/2)-1} \text{ if } n = 8 \ ,$$

where C is a positive constant and where $A = C(2,2)\ \mathrm{Sym}_{ijk\ell}\nabla_{ijk\ell}\mathrm{Scal}(g)(x_1)$.

Therefore, we will obtain the strict inequality case $(*)$ of Theorem 2, i.e., we will find ε small enough such that $J(u_\varepsilon) < k^{2/n}\mu(S^n)$, if $C(2,2)\ \mathrm{Sym}_{ijk\ell}\nabla_{ijk\ell}\mathrm{Scal}(g)(x_1) < 0$. But we have (Point (5) of Theorem 5)

$$\mathrm{Sym}_{ijk\ell mn}\nabla_{ijk\ell}\mathrm{Ric}(g)(x_1)_{mn} + \frac{3}{2}\mathrm{Sym}_{ijk\ell mn}\sum_{pq}(\nabla_m R(g)(x_1)_{pijq})(\nabla_n R(g)(x_1)_{pk\ell q}) = 0 \ ,$$

and if we take the $C(2,2)$ term of this relation, we obtain

$$C(2,2)\mathrm{Sym}_{ijk\ell}\nabla_{ijk\ell}\mathrm{Scal}(g)(x_1) + 12\sum_{ijk\ell m}(\nabla_m R(g)(x_1)_{ijk\ell})(\nabla_m R(g)(x_1)_{ijk\ell}) = 0 \ .$$

The calculation is quite simple. One must just use carefully the two Bianchi's identities and the relation $\sum_k \nabla_{kk}\mathrm{Ric}(g)(x_1)_{ij} = -3\nabla_{ij}\mathrm{Scal}(g)(x_1)$ (which comes from the contraction of the first relation of Point (5) of Theorem 5).

Therefore, since $\nabla R(g)(x_1) \neq 0$, we obtain

$$C(2,2)\mathrm{Sym}_{ijk\ell}\nabla_{ijk\ell}\mathrm{Scal}(g)(x_1) = -12|\nabla R(g)(x_1)|^2 < 0 \ .$$

The strict inequality case $(*)$ of Theorem 2 is then also satisfied when $\omega = 1$.

8.2. Let us now study the case where $\nabla^i \mathrm{Weyl}(g)(x_1) = 0$, $\forall i \leq \Lambda$, $X \neq S^n$. This includes the locally conformally flat case, where we were able to recover the weak form of the positive mass theorem. Here, in the general case, we recover the strong form of the positive mass theorem.

From now on, when we write $f = O''(r^s)$, we mean that $f = O(r^s)$, $\partial_i f = O(r^{s-1})$ and $\partial_{ij} f = O(r^{s-2})$. According to Theorem 5, we may suppose that g satisfies (in geodesic normal coordinates at each x_i) :

(A) $\det g = 1 + O(r_i^m)$, $m >> 1$, $\forall i = 1, \cdots, k$.

(B) $\nabla^s R(g)(x_i) = 0$, $\forall s \leq \Lambda$, $\forall i = 1, \cdots, k$. (In particular, $\mathrm{Scal}(g) = O(r_i^{\Lambda+1})$).

(C) $g_{ij} = \delta_{ij} + \sum_{\Lambda+1}^{2(\Lambda+1)+1} C_\alpha (\nabla_\alpha R(g)(x_s))_{ik\ell j} x^k x^\ell x^\alpha + O''(r_s^{2(\Lambda+3)})$, $\forall s = 1, \cdots, k$. (In particular, $g_{ij} = \delta_{ij} + O(r_s^{(\Lambda+3)})$).

(D) $\nabla_\alpha R(g)(x_i) = \partial_\alpha R(g)(x_i)$, $\nabla_\alpha \mathrm{Ric}(g)(x_i) = \partial_\alpha \mathrm{Ric}(g)(x_i)$ and $\nabla_\alpha \mathrm{Scal}(g)(x_i) = \partial_\alpha \mathrm{Scal}(g)(x_i)$, $\forall i = 1, \cdots, k$, $\forall |\alpha| \leq 2(\Lambda+1)+1$.

(E) $C(2,2)\mathrm{Sym}_\alpha \nabla_\alpha \mathrm{Scal}(g)(x_i) = 0$, $\forall i = 1, \cdots, k$, $\forall |\alpha| \leq 2(\Lambda+1)+1$.

It is then possible to prove that the Green function G_i at x_i of the conformal Laplacian can be written (near x_i)

$$\text{(F)} \qquad G_i = \frac{1}{(n-2)\omega_{n-1} r_i^{n-2}} \left(1 + \sum_{p=1}^{n} \Psi_p \right) + O''(1) \ ,$$

where the Ψ_p are homogeneous polynomials of order p, which are identically null for $1 \leq p \leq \Lambda + 2$, and which satisfy for $p \leq n-3$ and $r > 0$, $\int_{S^{n-1}(r)} \Psi_p(x) d\sigma_r(x) = 0$. (For details on this expension, see [HV2]).

Now, let $\gamma = r_1^{n-2} G_1$, $\widetilde{g} = G_1^{4/(n-2)} g$, and if $\{y^i\}$ is a geodesic normal coordinate system at x_1, let $z^i = r_1^{-2} y^i$ and $\rho_1 = |z| = r_1^{-1}$. In the coordinate system $\{z^i\}$ we then have $\widetilde{g}_{ij}(z) = \gamma(z)^{4/(n-2)}(\delta_{ij} + O''(\rho_1^{-\Lambda-3}))$. Therefore, according to the nullity of the Ψ_p for $p \leq \Lambda + 2$, $(X - \{x_1\}, \widetilde{g})$ is an asymptotically flat manifold of order $\Lambda + 3$. Its mass $m(\widetilde{g})$ is then well defined and always positive (since $\Lambda + 3 > \frac{(n-2)}{2}$).

We let $\Theta = \sum_{i=1}^k G_i$, $\gamma_i = r_i^{(n-2)}\Theta$ and $K = \sum_{i=2}^k G_i(x_1)$. Moreover, (as in Lee-Parker [LP]), we define the test functions $v_{\varepsilon,\delta}$ by

$$v_{\varepsilon,\delta} = \Theta r_i^{n-2}\left(\frac{\varepsilon}{\varepsilon^2 + r_i^2}\right)^{(n-2)/2} \quad \text{if } r_i \leq \delta$$

$$v_{\varepsilon,\delta} = \Theta \delta^{n-2}\left(\frac{\varepsilon}{\varepsilon^2 + \delta^2}\right)^{(n-2)/2} \quad \text{if } r_i \geq \delta,\ \delta > 0\ .$$

It is then possible to prove (see [HV2]) that

$$\int_X |\nabla v_{\varepsilon,\delta}|^2 dv(g) + \frac{n-2}{4(n-1)}\int_X \mathrm{Scal}(g) v_{\varepsilon,\delta}^2 dv(g)$$
$$\leq k^{2/n}\mu(S^n)\left(\int_X v_{\varepsilon,\delta}^{2n/(n-2)} dv(g)\right)^{(n-2)/n} - C\widetilde{\mu}\varepsilon^{n-2} + \varepsilon^{n-2}O(\delta) + O(\varepsilon^{n-1})\ ,$$

where C is a positive constant and where

$$\widetilde{\mu} = -\lim_{\rho_1\to\infty}\frac{1}{\omega_{n-1}}\int_{\partial B(x_1;\rho_1)}(\partial_{\rho_1}\gamma_1)d\sigma_{\rho_1} = -\lim_{\rho_1\to\infty}\rho_1^{n-1}\frac{1}{\omega_{\rho_1}}\int_{\partial B(x_1;\rho_1)}(\partial_{\rho_1}\gamma_1)d\sigma_{\rho_1}\ .$$

(According to (F), $\widetilde{\mu}$ is well defined since $\gamma_1 = \gamma + (K + O''(\rho_1^{-1}))\rho_1^{-n+2}$.)

Therefore, the strict inequality case $(*)$ of Theorem 2 will be satisfied if $\widetilde{\mu} > 0$. (i.e, we will find ε, δ small enough such that $J(v_{\varepsilon,\delta}) < k^{2/n}\mu(S^n)$ if $\widetilde{\mu} > 0$). But, we have

$$m(\widetilde{g}) = \lim_{\rho_1\to\infty}\frac{1}{\omega_{n-1}}\int_{\partial B(x_1;\rho_1)}\sum_m\left(\rho_1^{-2}z^i z^j\partial_m\widetilde{g}_{mi} - \partial_j\widetilde{g}_{mm}\right)\partial_j\rfloor dz$$
$$= \lim_{\rho_1\to\infty}\frac{1}{\omega_{n-1}}\int_{\partial B(x_1;\rho_1)}\sum_m\left(\partial_{\rho_1}(\widetilde{g}_{\rho_1\rho_1} - \widetilde{g}_{mm}) + \rho_1^{-1}\left(n\widetilde{g}_{\rho_1\rho_1} - \widetilde{g}_{mm}\right)\right)d\sigma_{\rho_1}\ ,$$

with, according to (C),

$$\widetilde{g}_{\rho_1\rho_1} = \widetilde{g}(\partial_{\rho_1}, \partial_{\rho_1}) = \rho_1^{-2}z^i z^j\widetilde{g}_{ij} = \rho_1^{-2}z^i z^j\gamma(z)^{4/(n-2)}\ .$$

Moreover, with (D), (E) and Bianchi, we get

$$\int_{\partial B(x_1,\rho_1)}\nabla_\alpha R(g)_{ik\ell j}(x_1)z^i z^j z^k z^\ell z^\alpha d\sigma_{\rho_1} \approx \int_{\partial B(x_1,\rho_1)}\nabla_\alpha \mathrm{Ric}(g)_{ij}(x_1)z^i z^j z^\alpha d\sigma_{\rho_1} = 0$$

for $|\alpha| \leq 2(\Lambda+1)+1$, while, according to (F),

$$\gamma(z) = \frac{1}{(n-2)\omega_{n-1}} + \sum_{p=\Lambda+3}^{n} \Psi_p(z) + O''(\rho_1^{-n+2})$$

where the Ψ_p are $O''(\rho_1^{-p})$ which satisfy $\int_{\partial B(x_1;\rho_1)} \Psi_p d\sigma_{\rho_1} = 0$ for $p \leq n-3$.

Therefore (remember that $2(\Lambda+3) \geq n-1$), we obtain

$$\begin{aligned} m(\widetilde{g}) &= -\frac{4(n-1)}{n-2} C_1^{-(n-6)/(n-2)} \lim_{\rho_1 \to \infty} \omega_{n-1}^{-1} \left[\int_{\partial B(x_1,\rho_1)} (\partial_{\rho_1}\gamma) d\sigma_{\rho_1} + \omega_{\rho_1} o(\rho_2^{-n+1}) \right] \\ &= -\frac{4(n-1)}{n-2} C_1^{(n-6)/(n-2)} \lim_{\rho_1 \to \infty} \omega_{n-1}^{-1} \int_{\partial B(x_1,\rho_1)} (\partial_{\rho_1}\gamma) d\sigma_{\rho_1} \ , \end{aligned}$$

where $C_1 = \frac{1}{(n-2)\omega_{n-1}}$.

Since $\gamma = \gamma_1 - K\rho_1^{-n+2} + O''(\rho_1^{-n+1})$, we finally get $\widetilde{\mu} = C_2 m(\widetilde{g}) + (n-2)K$, where C_2 is a positive constant. The constant $\widetilde{\mu}$ is then positive, and the strict inequality case of Theorem 2 is established.

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