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PRODUCING GOOD QUOTIENTS BY EMBEDDING INTO TORIC VARIETIES

by

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Abstract. — Let an algebraic torus T act effectively on a $\mathbb{Q}$-factorial algebraic variety X. Suppose that X has the A_2-property, that means any two points of X admit a common affine open neighbourhood in X. We prove the following embedding theorem: Let $U_1, \ldots, U_r \subset X$ be T-invariant open subsets with good quotients $U_i \to U_i /\!\!/ T$ such that the $U_i /\!\!/ T$ are A_2-varieties. Then there exists a T-equivariant closed embedding $X \hookrightarrow Z$ into a smooth toric variety Z on which T acts as a subtorus of the big torus such that each U_i is of the form $U_i = W_i \cap X$ with a toric open subset $W_i \subset Z$ admitting a good quotient $W_i \to W_i /\!\!/ T$. This result applies in particular to the family of open subsets $U \subset X$ that are maximal with respect to saturated inclusion among all open subsets admitting a good A_2-quotient space. In the appendix to this article we survey some general results on embeddings into toric varieties and prevarieties.

Introduction

This article deals with toric varieties as ambient spaces in algebraic geometry. We consider actions of algebraic tori T on a $\mathbb{Q}$-factorial (e.g. smooth) algebraic variety X and show that the problem of constructing good quotients for such an action extends to a purely toric problem of a suitable ambient toric variety of X, provided of course that X and the quotient varieties in question are embeddable into toric varieties.

Let us recall the basic notions and some background. A good quotient for the action of an algebraic torus T on a variety X is a T-invariant affine regular map $p\colon X \to X /\!\!/ T$ such that the natural homomorphism $\mathcal{O}_{X/\!\!/T} \to p_*(\mathcal{O}_X)^T$ is an isomorphism. In general, the whole X need not admit a good quotient, but there always exist nonempty open T-invariant subsets $U \subset X$ with a good quotient $U \to U /\!\!/ T$. It is one of the central problems in Geometric Invariant Theory to describe or even to construct all these open subsets.

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In the special case of toric varieties the above problem can be solved: Let Z be a toric variety, and let T be a subtorus of the big torus $T_Z \subset Z$. The description of Z in terms of its fan allows to figure out explicitly all the toric open $W \subset Z$ admitting a good quotient $W \to W/\!\!/T$, see [**9**], [**18**] and also Section 1. Moreover, every further open subset of Z admitting a good quotient by the action of T occurs as a saturated subset of one of these W. A different but also combinatorial approach for $Z = \mathbb{P}_n$ is presented in [**3**].

We shall show that in principle the general problem of constructing good quotients for torus actions can be reduced to the toric setting by means of embedding. Of course, in this approach one has to restrict oneself to embeddable spaces. In view of Włodarczyk's theorem [**20**], this amounts to considering spaces Y with the A_2-property: Any two points of Y admit a common affine open neighbourhood in Y. Our main result is the following, see Theorem 2.4:

Theorem. — *Let an algebraic torus T act effectively on a $\mathbb{Q}$-factorial A_2-variety X, and suppose that the T-invariant open subsets $U_1, \dots, U_r \subset X$ admit good quotients $U_i \to U_i/\!\!/T$ with A_2-varieties $U_i/\!\!/T$. Then there exists a T-equivariant closed embedding $X \hookrightarrow Z$ into a smooth toric variety Z on which T acts as a subtorus of the big torus such that each U_i is of the form $U_i = W_i \cap X$ with a toric open subset $W_i \subset Z$ admitting a good quotient $W_i \to W_i/\!\!/T$.*

This applies to the general problem: It suffices to consider the $(T, 2)$-maximal subsets of a given T-variety X, i.e., the invariant open subsets $U \subset X$ that admit a good quotient with an A_2-variety $U/\!\!/T$ and do not occur as a saturated subset of some properly larger U' having the same properties. Święcicka showed that the family of all $(T, 2)$-maximal subsets of X is finite [**19**]. Consequently, we obtain, see Corollary 2.6:

Corollary. — *Let an algebraic torus T act effectively on a $\mathbb{Q}$-factorial A_2-variety X. Then there exists a T-equivariant closed embedding $X \hookrightarrow Z$ into a smooth toric variety Z on which T acts as a subtorus of the big torus such that every $(T, 2)$-maximal open $U \subset X$ is of the form $U = W \cap X$ with a toric open subset $W \subset Z$ admitting a good quotient $W \to W/\!\!/T$.*

Note that this generalizes the following result due to Święcicka [**19**]: If the torus T acts on a smooth projective variety X with $\mathrm{Pic}(X) = \mathbb{Z}$ and $U \subset X$ is $(T, 2)$-maximal, then there is a T-equivariant embedding $X \subset \mathbb{P}_n$ such that $U = W \cap X$ with a $(T, 2)$-maximal and hence $\mathbb{T}_n$-invariant $W \subset \mathbb{P}_n$.

The present article is organized as follows: In Section 1 we introduce the basic notions and discuss some known results on good quotients for toric varieties. Section 2 is devoted to giving the precise formulation of our main result. In Section 3 we provide the techniques for the proof of our main result which is performed in Section 4. Finally, in the appendix, we survey some general results on embeddings into toric varieties and prevarieties.

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1. Good quotients of toric varieties

In this section we discuss some well-known results on good quotients for subtorus actions on toric varieties. As we shall need this later, we perform our fixing of terminology in the more general setting of possibly non separated prevarieties.

Throughout the whole article we work over an algebraically closed field $\mathbb{K}$. A *toric prevariety* is a normal (algebraic) prevariety X (over $\mathbb{K}$) together with an algebraic torus $T_X \subset X$ such that T_X is open in X and a regular action $T_X \times X \to X$ that extends the group structure of $T_X \subset X$. We refer to $T_X \subset X$ as the big torus of X. A *toric variety* is a separated toric prevariety.

A *toric morphism* of two toric prevarieties X, X' is a regular map $f\colon X \to X'$ that restricts to a group homomorphism $\varphi\colon T_X \to T_{X'}$ of the respective big tori satisfying $f(t{\cdot}x) = \varphi(t){\cdot}x$ for all $(t,x) \in T \times X$. Similarly to the separated case, the category of toric prevarieties can be described by certain combinatorial data, see [**1**].

A *good prequotient* for a regular action $G \times X \to X$ of a reductive group on a prevariety X is a G-invariant affine regular map $p\colon X \to X/\!\!/G$ of prevarieties such that the canonical map $\mathcal{O}_{X/\!\!/G} \to p_*(\mathcal{O}_X)^G$ is an isomorphism. A good prequotient $p\colon X \to X/\!\!/G$ is called *geometric*, if it separates orbits. If both spaces X and $X/\!\!/G$ are separated, then we speak of a *good* or a *geometric quotient*.

Now, let X be a toric prevariety. As announced above, we consider actions of subtori T of the big torus $T_X \subset X$. Concerning good prequotients of such subtorus actions, the first observation is, see e.g. [**1**, Corollary 6.5]:

Remark 1.1. — If the action of $T \subset T_X$ has a good prequotient $p\colon X \to X/\!\!/T$, then the quotient space $X/\!\!/T$ inherits the structure of a toric prevariety such that p becomes a toric morphism.

In our article the following property of varieties will play a central rôle: We say that a variety X has the *A_2-property*, if any two points $x, x' \in X$ admit a common affine open neighbourhood in X. This notion is due to J. Włodarczyk. In [**20**] he proves among other things that a normal variety X admits a closed embedding into a toric variety if and only if X is A_2.

The next statement is a simple, but useful toric version of [**4**, Theorem C]. It shows that the A_2-property is in a natural way connected with good quotients of toric varieties:

Proposition 1.2. — *Let X be a toric variety with big torus $T_X \subset X$. For every subtorus $T \subset T_X$ the following statements are equivalent:*

i) *The action of T on X has a good quotient $X \to X /\!\!/ T$.*

ii) *Any two points $x, x' \in X$ with closed T_X-orbit have a common T-invariant affine neighbourhood in X.*

Proof. — If the action of T on X admits a good quotient $X \to X/\!\!/T$, then the quotient space inherits the structure of a toric variety and hence has the A_2-property. Since the quotient map $X \to X/\!\!/T$ is affine and T-invariant, it follows that X fullfills Condition ii).

Now suppose that ii) holds. According to [**4**, Theorem C], we only have to show that any two points of X have a common affine T-invariant neighbourhood in X. So, given $z, z' \in X$, choose

$$x \in \overline{T_X \cdot z}, \qquad x' \in \overline{T_X \cdot z'}$$

such that the orbits $T_X \cdot x$ and $T_X \cdot x'$ are closed in X. By assumption, there exists a T-invariant affine open $U \subset X$ with $x, x' \in U$. Consider the sets

$$S := \{t \in T_X;\ t \cdot z \in U\}, \qquad S' := \{t \in T_X;\ t \cdot z' \in U\}.$$

These are non empty open subsets of T_X and hence we have $S \cap S' \neq \varnothing$. Let $t \in S \cap S'$. Then $t^{-1} \cdot U$ is the desired common affine neighbourhood of the points z and z'. □

Finally, we characterize existence of good quotients in terms of fans. For the terminology, see [**8**]. Let Δ be a fan in some lattice N, and let $L \subset N$ be a primitive sublattice. Then Δ defines a toric variety X, and L corresponds to a subtorus T of the big torus $T_X \subset X$.

Up to elementary convex geometry, the following statement is a reformulation of a well-known characterization obtained by J. Święcicka [**18**, Theorem 4.1] and, independently, by H. Hamm [**9**, Theorem 4.7]. For convenience, we present here a direct proof in our setting.

Proposition 1.3. — *The action of T on X admits a good quotient if and only if any two different maximal cones of Δ can be separated by an L-invariant linear form on N.*

Proof. — First suppose that the action of T has a good quotient $q\colon X \to X'$. Then X' inherits the structure of a toric variety such that q becomes a toric morphism. So we may assume that q arises from a map of fans $Q\colon N \to N'$ from Δ to a fan Δ' in a lattice N'.

Note that the sublattice $L \subset N$ is contained in $\ker(Q)$. Let $Q_{\mathbb{R}}\colon N_{\mathbb{R}} \to N'_{\mathbb{R}}$ be the linear map of real vector spaces associated to $Q\colon N \to N'$. We claim that there are bijections of the sets $\Delta^{\max}$ and $(\Delta')^{\max}$ of maximal cones:

$$(1) \qquad \Delta^{\max} \to (\Delta')^{\max}, \qquad \sigma \mapsto Q_{\mathbb{R}}(\sigma),$$

$$(2) \qquad (\Delta')^{\max} \to \Delta^{\max}, \qquad \sigma' \mapsto Q_{\mathbb{R}}^{-1}(\sigma') \cap |\Delta|$$

To check that the first map is well-defined, let $\sigma \in \Delta^{\max}$. Then the image $Q_{\mathbb{R}}(\sigma)$ is contained in some maximal cone $\sigma' \in \Delta'$. In particular, $q(X_\sigma) \subset X_{\sigma'}$ holds. Since

q is affine, the inverse image $q^{-1}(X_{\sigma'})$ is an affine invariant chart of X, and hence necessarily equals X_σ. Since q is in addition surjective, we must have $q(X_\sigma) = X_{\sigma'}$. This means $Q_{\mathbb{R}}(\sigma) = \sigma'$. So we see that (1) is well defined.

To see that also the second map is well defined, let $\sigma' \in (\Delta')^{\max}$. The inverse image of the associated affine chart $X_{\sigma'} \subset X'$ is given by the general formula

$$q^{-1}(X_{\sigma'}) = \bigcup_{\tau\in\Delta;Q_{\mathbb{R}}(\tau)\subset\sigma'} X_\tau.$$

Since q is affine, this inverse image is an affine invariant chart X_σ given by some cone $\sigma \in \Delta$. It follows that

$$\sigma = \operatorname{cone}(\tau \in \Delta;\ Q_{\mathbb{R}}(\tau) \subset \sigma') = Q_{\mathbb{R}}^{-1}(\sigma') \cap |\Delta|.$$

We still have to check that σ is maximal. By surjectivity of q, we see $Q_{\mathbb{R}}(\sigma) = \sigma'$ holds. Now assume, that $\sigma \subset \tau$ for some $\tau \in \Delta^{\max}$. As seen above, $Q_{\mathbb{R}}(\tau)$ is a maximal cone of Δ'. Since $Q_{\mathbb{R}}(\tau)$ contains the maximal cone σ', we get $Q_{\mathbb{R}}(\tau) = \sigma'$. By definition of σ, this implies $\tau = \sigma$. So, also (2) is well defined.

Obviously, the maps (1) and (2) are inverse to each other. We use them to find separating linear forms. Let σ_1, σ_2 be two different maximal cones. Then the maximal cones $\sigma_i' := Q_{\mathbb{R}}(\sigma_i)$ of Δ' can be separated by a linear form u' on N', i.e.,

$$u'|_{\sigma_1'} \geq 0, \qquad u'|_{\sigma_2'} \leq 0, \qquad (u')^\perp \cap \sigma_1' = (u')^\perp \cap \sigma_2' = \sigma_1' \cap \sigma_2'.$$

Now consider the linear form $u := u' \circ Q$. Then u is L-invariant, nonnegative on σ_1 and nonpositive on σ_2. Using (1) and (2) we obtain:

$$\begin{aligned}
u^\perp \cap \sigma_i &= Q_{\mathbb{R}}^{-1}((u')^\perp) \cap (Q_{\mathbb{R}}^{-1}(\sigma_i') \cap |\Delta|) \\
&= Q_{\mathbb{R}}^{-1}((u')^\perp \cap \sigma_i') \cap |\Delta| \\
&= Q_{\mathbb{R}}^{-1}(\sigma_1' \cap \sigma_2') \cap |\Delta| \\
&= (Q_{\mathbb{R}}^{-1}(\sigma_1') \cap |\Delta|) \cap (Q_{\mathbb{R}}^{-1}(\sigma_2') \cap |\Delta|) \\
&= \sigma_1 \cap \sigma_2.
\end{aligned}$$

Now suppose that any two different maximal cones of Δ can be separated by an L-invariant linear form on N. Let $P\colon N \to N/L$ denote the projection. We claim that the projected cones $P_{\mathbb{R}}(\sigma)$, where σ runs through the maximal cones of Δ, are the maximal cones of a quasifan Σ in N/L, i.e., this Σ behaves almost like a fan, merely its cones need not be strictly convex.

To verify this claim, we have to find for any two $\sigma_1' := P_{\mathbb{R}}(\sigma_1)$ and $\sigma_2' := P_{\mathbb{R}}(\sigma_2)$, where $\sigma_1, \sigma_2 \in \Delta^{\max}$, a separating linear form. By assumption, there is an L-invariant linear form u on N that separates σ_1 and σ_2. Let u' denote the linear form on N/L with $u = u' \circ P$. Then u' is nonnegative on σ_1' and nonpositive on σ_2'. Moreover, we have

$$\begin{aligned}(u')^{\perp} \cap \sigma'_i &= P_{\mathbb{R}}(P_{\mathbb{R}}^{-1}((u')^{\perp} \cap \sigma'_i)) \\ &= P_{\mathbb{R}}(u^{\perp} \cap (\sigma_i + L_{\mathbb{R}})) \\ &= P_{\mathbb{R}}((u^{\perp} \cap \sigma_i) + L_{\mathbb{R}}) \\ &= P_{\mathbb{R}}((\sigma_1 \cap \sigma_2) + L_{\mathbb{R}}) \\ &= P_{\mathbb{R}}(\sigma_1 \cap \sigma_2) \\ &\subset \sigma'_1 \cap \sigma'_2.\end{aligned}$$

Conversely, $\sigma'_1 \cap \sigma'_2$ is obviously contained in $(u')^{\perp} \cap \sigma'_i$. So we checked that u' separates the cones σ'_1 and σ'_2. Hence our claim is proved, and we know that Σ is indeed a quasifan.

To proceed we need a further observation. For a given maximal cone $\sigma' \in \Sigma$, we choose a maximal cone $\sigma \in \Delta$ with $\sigma' = P_{\mathbb{R}}(\sigma)$. We claim

$$P_{\mathbb{R}}^{-1}(\sigma') \cap |\Delta| \;=\; \bigcup_{\tau \in \Delta;\, P_{\mathbb{R}}(\tau) \subset \sigma'} \tau \;=\; \sigma.$$

Only the inclusion "$\subset$" of the last equation is not obvious. To obtain it, let $\tau \in \Delta$ with $P_{\mathbb{R}}(\tau) \subset \sigma'$. Then any L-invariant linear form on N that is nonnegative on σ is necessarily nonnegative on τ. Thus τ can not be separated from σ by an L-invariant linear form and hence is a face of σ. So the claim is proved.

Projecting Σ along its minimal face gives a fan Δ' in a lattice N' and a map $N \to N'$ of the fans Δ and Δ'. By the second claim, the associated toric morphism $q\colon X \to X'$ is affine. Now it is a standard conclusion that over the invariant affine charts of X', the map q is the classical invariant theory quotient for the action of T, see e.g. [**18**, Section 3]. □

2. Toric extension of good quotients

We come to the precise formulation of our main results. First consider the following setting: Let Z be a toric variety and let T be a subtorus of the big torus $T_Z \subset Z$. Assume that $X \subset Z$ is a T-invariant closed subvariety and that $U \subset X$ is an open T-invariant subset admitting a good quotient $p\colon U \to U /\!\!/ T$.

Definition 2.1. — A *toric extension* of $U \subset X$ is an open toric subvariety $Z_U \subset Z$ with a good quotient $Z_U \to Z_U /\!\!/ T$ such that $U = Z_U \cap X$ holds.

Remark 2.2. — Every toric extension $Z_U \subset Z$ of the subset $U \subset X$ gives rise to a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\subset"] \arrow[d, "/\!\!/T"'] & Z_U \arrow[d, "/\!\!/T"] \\ U/\!\!/T \arrow[r] & Z_U/\!\!/T \end{tikzcd}$$

where the lower horizontal map is a closed embedding. In particular, if such a toric extension exists, then $U/\!\!/T$ is an A_2-variety.

Even if the quotient variety $U/\!\!/T$ is A_2, one cannot expect that toric extensions always exist. In the following example, we realize $\mathbb{K}^2 \setminus \{0\}$ as an invariant subvariety of a 3-dimensional toric variety such that the geometric quotient $\mathbb{K}^2 \setminus \{0\} \to \mathbb{P}_1$ of the standard $\mathbb{K}^*$-action admits no toric extension:

Example 2.3. — The toric variety $X := \mathbb{K}^2 \setminus \{0\}$ is given by the fan in $\mathbb{Z}^2$ with maximal cones $\varrho_1 := \mathrm{cone}((1,0))$ and $\varrho_2 := \mathrm{cone}((0,1))$. The standard subtorus

$$T := \{(t,t);\ t \in \mathbb{K}^*\} \subset T_X$$

of X corresponds to the sublattice $\mathbb{Z}(1,1) \subset \mathbb{Z}^2$. The action of T on X has a geometric quotient $X \to \mathbb{P}_1$. We realize X as a T-invariant closed subvariety of a 3-dimensional toric variety Z. In $\mathbb{R}^3$, let

$$\tau_1 := \mathrm{cone}((1,0,-1),(0,1,1)), \qquad \tau_2 := \mathrm{cone}((2,0,1),(-3,0,-1)).$$

As they intersect in $\{0\}$, these cones are the maximal cones of a fan. Let Z denote the associated toric variety. The linear map $F\colon \mathbb{Z}^2 \to \mathbb{Z}^3$ given by

$$F(1,0) := (1,1,0), \qquad F(0,1) := (-1,0,0)$$

defines a toric embedding $X \to Z$. So we can regard X as a T_X-invariant subvariety of Z. Note that X intersects both closed orbits of the big torus $T_Z \subset Z$, and that the action of T on Z corresponds to the sublattice

$$F(\mathbb{Z}\cdot(1,1)) = \mathbb{Z}\cdot(0,1,0) \subset \mathbb{Z}^3$$

It follows from Proposition 1.3, that the action of T on Z does not admit a good quotient. In particular, there exists no toric extension of X.

In view of this example, the question is the following: Let U be an invariant open subset of an arbitrary variety X with an effective action of an algebraic torus T, and suppose that there is a good quotient $U \to U/\!\!/T$. Provided that X and $U/\!\!/T$ are A_2-varieties, can we realize X as a T-invariant closed subvariety of some toric variety such that U becomes torically extendible?

Our main result gives a positive answer to this question if X is $\mathbb{Q}$-factorial, i.e. if X is normal and for any Weil divisor on X some multiple is Cartier. In fact, we prove even more:

Theorem 2.4. — *Let X be a $\mathbb{Q}$-factorial A_2-variety with an effective regular algebraic torus action $T \times X \to X$, and suppose that $U_1, \ldots, U_r \subset X$ are T-invariant open subsets admitting good quotients $p_i \colon U_i \to U_i /\!\!/ T$ with A_2-varieties $U_i /\!\!/ T$. Then there exists a T-equivariant closed embedding $X \hookrightarrow Z$ into a smooth toric variety Z on which T acts as a subtorus of the big torus such that every $U_i \subset X$ has a toric extension.*

The proof of this theorem is performed in Section 4. Our motivation for the above statement is its application to maximal sets with good quotients. Let us recall briefly the basic notions, see [**19**, Definition 4.3]:

Let the algebraic torus T act on a variety X. An inclusion $U' \subset U$ of invariant open subsets $U', U \subset X$ is called *T-saturated* if for every $x \in U'$ the closure of the orbit $T{\cdot}x$ in U' is also closed in U. An open invariant subset $U \subset X$ is called *$(T,2)$-maximal* if it admits a good quotient with an A_2-variety $U /\!\!/ T$ and there is no open $U' \subset X$ with these properties containing U as a proper T-saturated subset.

Remark 2.5. — Every open subset $U' \subset X$ admitting a good quotient with $U' /\!\!/ T$ an A_2-variety is of the form $U' = p^{-1}(V)$, where $p \colon U \to U /\!\!/ T$ is the good quotient of some $(T,2)$-maximal open $U \subset X$ and $V \subset U /\!\!/ T$ is an open subset.

This observation reduces the study of good quotients with quotient spaces having the A_2-property to the study of $(T,2)$-maximal subsets. As an immediate consequence of Theorem 2.4, we obtain the following generalization of [**19**, Proposition 6.2]:

Corollary 2.6. — *Let X be a $\mathbb{Q}$-factorial A_2-variety with an effective regular algebraic torus action $T \times X \to X$. Then there exists a T-equivariant closed embedding $X \hookrightarrow Z$ into a smooth toric variety where T acts as a subtorus of the big torus such that every $(T,2)$-maximal $U \subset X$ admits a toric extension.*

Proof. — By [**19**, Theorem 4.4] there exist only finitely many $(T,2)$-maximal open subsets $U \subset X$. Thus the assertion follows from Theorem 2.4. $\square$

3. Ample groups and linearization

The proof of the embedding theorem 2.4 is based on the techniques introduced in [**11**, Section 2]. We recall in this section the basic notions and results adapted to our purposes and provide some additional details needed later on. We work here in terms of Cartier divisors instead of using line bundles as in [**11**]. The idea is to generalize the notion of an ample divisor to what we call an "ample group of divisors".

Let X be an arbitrary irreducible algebraic variety. Denote by $\mathrm{CDiv}(X)$ the group of Cartier divisors, and let $\Lambda \subset \mathrm{CDiv}(X)$ be a finitely generated free subgroup. For a divisor $D \in \Lambda$ let

$$\mathcal{A}_D := \mathcal{O}_X(D)$$

denote the associated invertible sheaf on X. Given two sections $f \in \mathcal{A}_D(U)$ and $f' \in \mathcal{A}_{D'}(U)$, we can multiply them as rational functions and get a section $ff' \in \mathcal{A}_{D+D'}(U)$. Extending this operation, we obtain a Λ-graded $\mathcal{O}_X$-algebra

$$\mathcal{A} = \bigoplus_{D \in \Lambda} \mathcal{A}_D.$$

Remark 3.1. — The algebra $\mathcal{A}$ is reduced and locally of finite type over $\mathcal{A}_0 = \mathcal{O}_X$.

Now, we can glue the canonical maps $\mathrm{Spec}(\mathcal{A}(U)) \to U$, where U ranges over small affine neighbourhoods $U \subset X$, to obtain a variety and a regular map

$$\widehat{X} := \mathrm{Spec}(\mathcal{A}), \qquad q\colon \widehat{X} \to X.$$

Note that $\mathcal{A} = q_*(\mathcal{O}_{\widehat{X}})$. We call $\widehat{X}$ the variety over X associated to the group $\Lambda \subset \mathrm{CDiv}(X)$. It comes along with a torus action: The Λ-grading of the $\mathcal{O}_X$-algebra $\mathcal{A}$ defines a regular action of the algebraic torus

$$H := \mathrm{Spec}(\mathbb{K}[\Lambda])$$

on $\widehat{X}$ such that for each affine open set $U \subset X$, the sections $\mathcal{A}_D(U)$ are precisely the functions of $q^{-1}(U)$ that are homogeneous with respect to the character $\chi^D \in \mathrm{Char}(H)$. The following is standard:

Remark 3.2. — H acts freely on $\widehat{X}$, and the map $q\colon \widehat{X} \to X$ is a geometric quotient for this action.

We turn to equivariant questions. Let G denote a connected linear algebraic group and assume that G acts by means of a regular map $G \times X \to X$ on the variety X. Recall that a G-sheaf on X is a sheaf $\mathcal{F}$ together with homomorphisms

$$\mathcal{F}(U) \to \mathcal{F}(g \cdot U), \qquad s \mapsto g \cdot s$$

that are compatible with restriction and fulfil $e_G \cdot s = s$ for the neutral element $e_G \in G$ as well as $g'g \cdot s = g' \cdot (g \cdot s)$ for any two $g, g' \in G$. The structure sheaf $\mathcal{O}_X$ becomes in a canonical way a G-sheaf of rings by setting

$$g \cdot f(x) := f(g^{-1} \cdot x).$$

Note that a G-sheaf structure on an $\mathcal{O}_X$-module or an $\mathcal{O}_X$-algebra requires by definition compatibility with the above canonical G-sheaf structure on the structure sheaf $\mathcal{O}_X$.

Definition 3.3. — A *G-linearization* of a finitely generated free subgroup $\Lambda \subset \mathrm{CDiv}(X)$ is a graded G-sheaf structure on the associated Λ-graded $\mathcal{O}_X$-algebra $\mathcal{A}$ such that for every G-invariant $U \subset X$ the representation of G on $\mathcal{A}(U)$ is rational.

Note that for any G-linearization of a subgroup $\Lambda \subset \mathrm{CDiv}(X)$ the induced G-sheaf structure on the (invariant) homogeneous component $\mathcal{A}_0 = \mathcal{O}_X$ of the associated Λ-graded $\mathcal{O}_X$-algebra $\mathcal{A}$ is the canonical one. We list below some statements on existence of G-linearizations.

Proposition 3.4. — *Suppose that X is smooth in codimension one and that the finitely generated free subgroup $\Lambda \subset \mathrm{CDiv}(X)$ consists of divisors with G-invariant support. Then there is a canonical G-linearization of Λ given by*

$$\mathcal{A}_D(U) \to \mathcal{A}_D(g\cdot U), \qquad g\cdot f(x) := f(g^{-1}\cdot x).$$

Proof. — By our assumption on X, we may view $\mathrm{CDiv}(X)$ as subgroup of the Weil divisors of X. Suppose that D is a G-invariant prime divisor, and let $x \in X$ be a smooth point. Fix $g \in G$ and let $\Phi\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,g\cdot x}$ denote the homomorphism of stalks induced by the map $X \to X$, $y \mapsto g^{-1}\cdot y$. For a germ $f_x \in \mathcal{O}_{X,x}$, we can compute the vanishing order of its translate:

$$\mathrm{ord}_{D,g\cdot x}(g\cdot f_x) = \mathrm{ord}_{D,g\cdot x}(\Phi(f_x)) = \mathrm{ord}_{g^{-1}\cdot D,g^{-1}g\cdot x}(\Phi^{-1}(\Phi(f_x))) = \mathrm{ord}_{D,x}(f_x).$$

As the prime cycles of a given $D \in \Lambda$ are G-invariant, it follows that for any section $f \in \mathcal{A}_D(U)$ the translate $g\cdot f$ is a section of $\mathcal{A}_D(g\cdot U)$. Moreover, setting $W := U \setminus \mathrm{Supp}(D)$ for a given G-invariant $U \subset X$, we have a G-equivariant injection $\mathcal{A}_D(U) \to \mathcal{O}_X(W)$. This implies that the representation of G on $\mathcal{A}(U)$ is rational, e.g. apply [**13**, Lemma p. 67] to the trivial bundle on W. $\square$

Proposition 3.5. — *Let X be normal, and let $\Lambda \subset \mathrm{CDiv}(X)$ be a finitely generated free subgroup. Then we have:*

i) *There exists a G-linearizable subgroup $\Lambda' \subset \Lambda$ of finite index.*
ii) *If G is factorial, then the group $\Lambda \subset \mathrm{CDiv}(X)$ is G-linearizable.*

Proof. — We begin with some general preparing observations. Consider an arbitrary $D \in \mathrm{CDiv}(X)$, and choose an open cover $\mathfrak{U} = (U_i)_{i\in I}$ of X such that D is represented on each U_i by some $f_i \in \mathbb{K}(X)$. This gives rise to a cocycle

$$\xi_{ij} := f_j/f_i \in Z^1(\mathfrak{U}, \mathcal{O}_X^*).$$

Let L_ξ be the line bundle over X defined by the cocycle ξ. We consider G-linearizations of L_ξ in the sense of [**13**, Section 2.1] and work with the following description of such G-linearizations in terms of local data: For $i, j \in I$, let

$$U_{(i,j)} := \{(g,x) \in G \times U_i;\ g\cdot x \in U_j\}.$$

Having in mind that L_ξ is the gluing of the products $U_i \times \mathbb{K}$, we see that a G-linearization of the line bundle L_ξ is locally of the form

$$U_{(i,j)} \times \mathbb{K} \to U_j \times \mathbb{K}, \qquad (g,x,t) \mapsto (g\cdot x, \alpha_{(i,j)}(g,x)t)$$

with certain functions $\alpha_{(i,j)} \in \mathcal{O}^*_{G\times X}(U_{(i,j)})$. These functions satisfy the following compatibility conditions:

$$\begin{aligned} \xi_{ik}(x)\alpha_{(k,l)}(g,x) &= \alpha_{(i,j)}(g,x)\xi_{jl}(g\cdot x), && \text{if } (g,x) \in U_{(i,j)} \cap U_{(k,l)}, \\ \alpha_{(i,k)}(g'g,x) &= \alpha_{(i,j)}(g,x)\alpha_{(j,k)}(g',g\cdot x), && \text{if } (g,x) \in U_{(i,j)},\ (g',g\cdot x) \in U_{(j,k)}. \end{aligned}$$

In fact, it turns out that the G-linearizations of L_ξ correspond to such families of functions. Now suppose that $E \in \mathrm{CDiv}(X)$ is a further Cartier divisor, defined on U_i by functions $h_i \in \mathbb{K}(X)$. Let $\eta \in Z^1(\mathfrak{U}, \mathcal{O}^*_X)$ be the associated cocycle.

Given families $\alpha_{(i,j)}$ and $\beta_{(i,j)}$ satisfying the above conditions with respect to the cocycles ξ and η respectively, the products $\alpha_{(i,j)}\beta_{(i,j)}$ define a G-linearization of the line bundle $L_\xi + L_\eta := L_{\xi\eta}$. Similarly, the family $\alpha^{-1}_{(i,j)}$ provides a G-linearization of $-L_\xi := L_{\xi^{-1}}$.

Now, the sheaf of sections of L_ξ identifies canonically to the sheaf $\mathcal{A}_D$ associated to D. Thus a G-linearization of the line bundle L_ξ induces a G-sheaf structure on $\mathcal{A}_D$, namely

$$(g\cdot f)(x) := g\cdot(f(g^{-1}\cdot x)).$$

This is compatible with products: given $f \in \mathcal{A}_D(U)$ and $h \in \mathcal{A}_E(U)$, we can use the local description of the G-linearizations in question in terms of families of functions to verify

$$g\cdot(fh) = (g\cdot f)(g\cdot h).$$

We prove now assertions i) and ii). Choose a basis $D_1, \dots, D_m$ of the group Λ. Then there is an open cover $\mathfrak{U} = (U_i)_{i\in I}$ such that all D_k are principal on the U_i. As above, we associate to each divisor D_k a cocycle $\xi_k \in Z^1(\mathfrak{U}, \mathcal{O}^*_X)$.

According to [**13**, Proposition 2.4], for some $n \in \mathbb{N}$, we can fix a G-linearization of every line bundle $L_k := L_{\xi_k^n}$. In the case of a factorial G, this can even be done with $n = 1$, see again [**13**, Remark p. 67]. As explained above, the respective products of the local data define a G-linearization of every linear combination $a_1L_1 + \cdots + a_mL_m$, where $a_i \in \mathbb{Z}$.

Let $\Lambda' \subset \Lambda$ be the subgroup generated by the divisors $D'_k := nD_k$, $k = 1, \dots, m$. The G-linearizations of the bundles $a_1L_1 + \cdots + a_mL_m$ carry over to G-sheaf structures of the homogeneous components $\mathcal{A}_{D'}$, $D' \in \Lambda'$. Note that on $\mathcal{A}_0 = \mathcal{O}_X$ we get back the canonical G-sheaf structure.

Using the fact that the G-sheaf structures of the $\mathcal{A}_{D'}$ are compatible with multiplication, we see that they make the Λ'-graded $\mathcal{O}_X$-algebra $\mathcal{A}'$ associated to Λ' into a G-sheaf. Finally, [**13**, Lemma p. 67] implies that for any G-invariant open $U \subset X$ the representation of G on $\mathcal{A}'(U)$ is in fact rational. ☐

We need a condition on a finitely generated free subgroup $\Lambda \subset \mathrm{CDiv}(X)$ ensuring that $\widehat{X} = \mathrm{Spec}(\mathcal{A})$ is quasiaffine. This is the following:

Definition 3.6. — We call a finitely generated free subgroup $\Lambda \subset \mathrm{CDiv}(X)$ *ample* if for each $x \in X$ there is a divisor $D \in \Lambda$ and a section $f \in \mathcal{A}_D(X)$ such that $X_f := X \setminus \mathrm{Supp}(D + \mathrm{div}(f))$ is an affine neighbourhood of x.

This generalizes the classical notion of an ample divisor in the sense that such a divisor generates an ample group.

Remark 3.7. — Suppose that a connected linear algebraic group acts on a normal variety X. Then every ample group $\Lambda \subset \mathrm{CDiv}(X)$ admits G-linearizable ample subgroups $\Lambda' \subset \Lambda$ of finite index.

By an *affine closure* of a quasiaffine variety Y we mean an affine variety $\overline{Y}$ containing Y as an open subvariety. The constructions and results of [**11**, Section 2] are subsumed in the following:

Theorem 3.8. — *Let G be a linear algebraic group and let X be a G-variety. Suppose that $\Lambda \subset \mathrm{CDiv}(X)$ is a G-linearized ample group and let $\widehat{X}$ denote the associated variety over X.*

i) *$\widehat{X}$ is quasiaffine and the representation of G on $\mathcal{O}(\widehat{X})$ induces a regular G-action on $\widehat{X}$ such that the actions of G and H commute and $q\colon \widehat{X} \to X$ becomes G-equivariant.*

ii) *For any collection $f_1, \ldots, f_r \in \mathcal{A}(X)$ satisfying the ampleness condition, there exists a $(G \times H)$-equivariant affine closure $\overline{X}$ of $\widehat{X}$ such that the f_i extend to regular functions on $\overline{X}$ and $q^{-1}(X_{f_i}) = \overline{X}_{f_i}$ holds.*

4. Proof of the main result

We come to the proof of Theorem 2.4. We shall need the following observation on linearizations:

Lemma 4.1. — *Let an algebraic torus T act regularly on a normal variety X, and suppose that $\Lambda = \Lambda_0 \oplus \Lambda_1$ is a finitely generated free subgroup of $\mathrm{CDiv}(X)$. If Λ_0 and Λ_1 are T-linearized, then these linearizations extend to a T-linearization of Λ.*

Proof. — Given $D \in \Lambda_0$ and $E \in \Lambda_1$, we make the $\mathcal{O}_X$-module $\mathcal{A}_{D+E}$ associated to $D + E$ into a T-sheaf: On small open sets $U \subset X$, each section f of $\mathcal{A}_{D+E}$ is of the form $f|_U = f_0 f_1$ with $f_0 \in \mathcal{A}_D(U)$ and $f_1 \in \mathcal{A}_E(U)$. For $t \in T$ set

$$t\cdot f|_U := (t\cdot f_0)(t\cdot f_1).$$

Then the local translates $t\cdot f|_U$ patch together to a well defined translate $t\cdot f$. This makes the $\mathcal{O}_X$-module $\mathcal{A}_{D+E}$ into a T-sheaf. Note that these structures extend to a T-sheaf structure of the graded $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to Λ.

We still have to show that for a given T-invariant open $U \subset X$ the representation of T on $\mathcal{A}(U)$ is rational. For this, choose a non-empty affine T-invariant open subset

$V \subset U$, use e.g. [**16**, Corollary 2]. Since the restriction $\mathcal{A}(U) \to \mathcal{A}(V)$ is injective, it suffices to verify that the T-representation on $\mathcal{A}(V)$ is rational.

But this follows from the facts that for any $D \in \Lambda_0$ and any $E \in \Lambda_1$ we have an equivariant isomorphism

$$\mathcal{A}_D(V) \otimes_{\mathcal{O}(V)} \mathcal{A}_E(V) \to \mathcal{A}_{D+E}(V), \qquad f_0 \otimes f_1 \mapsto f_0 f_1$$

and the tensor product of two rational representations is again a rational representation. □

Proof of Theorem 2.4. — For each index i, cover the quotient space $Y_i := U_i /\!\!/ T$ by open affine subsets $V_{i1}, \ldots, V_{in_i}$ such that every pair $y, y' \in Y_i$ is contained in some common V_{ij}. Let $D_{ij} \in \mathrm{CDiv}(X)$ be effective Cartier divisors with

$$U_{ij} := p_i^{-1}(V_{ij}) = X \setminus \mathrm{Supp}(D_{ij}).$$

Let $\Lambda_0 \subset \mathrm{CDiv}(X)$ denote the subgroup generated by the D_{ij}. Then we find a finitely generated group $\Lambda_1 \subset \mathrm{CDiv}(X)$ such that $\Lambda_0 \cap \Lambda_1 = 0$ holds and for any two $x, x' \in X$ there is an effective $D \in \Lambda := \Lambda_0 \oplus \Lambda_1$ such that $X \setminus \mathrm{Supp}(D)$ is a common affine neighbourhood of x and x'.

Let $\mathcal{A}$ denote the graded $\mathcal{O}_X$-algebra associated to Λ and let $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ be the associated variety over X. Recall that the map $q_X \colon \widehat{X} \to X$ is a geometric quotient for the action of $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$ on $\widehat{X}$. Since Λ is in particular ample, the variety $\widehat{X}$ is quasiaffine, see Theorem 3.8.

We use Propositions 3.4 and 3.5 to linearize the group $\Lambda = \Lambda_0 \oplus \Lambda_1$. Every $D \in \Lambda_0$ has T-invariant support and hence its sheaf $\mathcal{A}_D = \mathcal{O}_X(D)$ is T-linearized by $t \cdot f(x) := f(t^{-1} \cdot x)$. Since T is factorial, we can choose a T-linearization of Λ_1. By Lemma 4.1 these linearizations extend to a T-linearization of Λ.

According to Theorem 3.8, the representation of T on $\mathcal{O}(\widehat{X}) = \mathcal{A}(X)$ induces a regular T-action on $\widehat{X}$ that commutes with the H-action and makes $q_X \colon \widehat{X} \to X$ a T-equivariant map. Our next task is to construct an appropriate $(T \times H)$-equivariant affine closure $\overline{X}$ of $\widehat{X}$.

Viewed as regular functions on $\widehat{X}$, the canonical sections $f_{ij} := 1 \in \mathcal{O}_{D_{ij}}(X)$ are T-invariant and H-homogeneous. Choose effective $E_1, \ldots, E_m \in \Lambda$ such that every pair $x, x' \in X$ has a common affine neighbourhood of the form $V_l := X \setminus \mathrm{Supp}(E_l)$. Then every $g_l := 1 \in \mathcal{O}_{E_l}(X)$ is a H-homogeneous regular function on $\widehat{X}$.

According to Theorem 3.8, we find a $(T \times H)$-equivariant affine closure $\overline{X}$ of $\widehat{X}$ such that the above functions f_{ij} and g_l extend regularly to $\overline{X}$ and we have

$$\widehat{U}_{ij} := q_X^{-1}(U_{ij}) = \overline{X}_{f_{ij}}, \qquad \widehat{V}_l := q_X^{-1}(V_l) = \overline{X}_{g_l}.$$

Now choose $(T \times H)$-homogeneous generators $h_1, \ldots, h_s$ of the algebra $\mathcal{O}(\overline{X})$. Thereby make sure that the first $h_1, \ldots, h_k$ generate the ideal of the complement $\overline{X} \setminus \widehat{X}$. Then we have a $(T \times H)$-equivariant closed embedding

$$\overline{X} \to \mathbb{K}^n, \qquad x \mapsto (h_1(x), \ldots, h_s(x), f_{11}(x), \ldots, f_{rn_r}(x)),$$

where $n := s + n_1 + \cdots + n_r$ and $T \times H$ acts diagonally on $\mathbb{K}^n$. In the sequel we regard $\overline{X}$ as a closed subvariety of $\mathbb{K}^n$. Then the functions h_i and f_{ij} are just the restrictions of the respective coordinate functions z_i and z_{ij}. Set

$$\widehat{Z} := \bigcup_{\overline{\mathbb{T}^n \cdot z} \cap \widehat{X} \neq \varnothing} \mathbb{T}^n \cdot z.$$

This $\widehat{Z}$ is the minimal open toric subvariety of $\mathbb{K}^n$ containing $\widehat{X}$. Moreover, the set $\widehat{X}$ is closed in $\widehat{Z}$. This follows immediately from the fact that by our choice of $\widehat{Z}$, $\overline{X}$ and the embedding $\overline{X} \to \mathbb{K}^n$ we have:

$$\widehat{X} = \overline{X} \cap (\mathbb{K}^n_{z_1} \cup \cdots \cup \mathbb{K}^n_{z_k}), \qquad \widehat{Z} \subset \mathbb{K}^n_{z_1} \cup \cdots \cup \mathbb{K}^n_{z_k}.$$

Note that the torus H acts freely on $\widehat{Z}$. Consequently this action has a geometric prequotient $q_Z \colon \widehat{Z} \to Z$ with a smooth orbit space $Z := \widehat{Z}/H$. Remark 1.1 tells us that Z is again a toric prevariety and q_Z is a toric morphism. Moreover, the properties of a geometric prequotient yield a commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "\subset"] \arrow[d, "q_X"'] & \widehat{Z} \arrow[d, "q_Z"] \\ X \arrow[r] & Z \end{tikzcd}$$

where the lower horizontal map is a closed embedding. Note that T acts on Z as a subtorus of the big torus $T_Z \subset Z$ making this embedding equivariant. In the sequel we regard X as a subvariety of Z, and show that Z is the desired ambient space.

The first thing to check is that Z is separated. To verify this, it suffices to construct for any two closed orbits B, B' of the big torus $T_Z \subset Z$ an affine open subset $W \subset Z$ which intersects both orbits B and B' non trivially, see e.g. [**11**, Corollary 4.4].

So, let $B, B' \subset Z$ be closed T_Z-orbits. Since $q_Z \colon \widehat{Z} \to Z$ is a geometric prequotient, the inverse images $q_Z^{-1}(B)$ and $q_Z^{-1}(B')$ are closed $\mathbb{T}^n$-orbits of $\widehat{Z}$. By definition of $\widehat{Z}$, we find points $z, z' \in \widehat{X}$ with $q_Z(z) \in B$ and $q_Z(z') \in B'$.

For one of the above functions $g_l \in \mathcal{O}(\overline{X})$, we have $z, z' \in \overline{X}_{g_l}$. This g_l is the restriction of some H-homogeneous polynomial $\overline{g}_l \in \mathbb{K}[T_1, \ldots, T_n]$. Let

$$\widehat{W_l} := \mathbb{K}^n_{\overline{g}_l}, \qquad A := \widehat{W_l} \setminus \widehat{Z}.$$

Then $\widehat{W_l}$ is an open affine H-invariant subset of $\mathbb{K}^n$ containing $\widehat{V}_l = \overline{X}_{g_l}$ as a closed subset. Since also A is a closed H-invariant subset of $\widehat{W_l}$ and we have $A \cap \widehat{V}_l = \varnothing$, the good quotient $\widehat{W_l} \to \widehat{W_l} /\!\!/ H$ separates A and $\widehat{V}_l$.

Thus we find an H-invariant function $h \in \mathcal{O}(\widehat{W_l})$ that vanishes along A but satisfies $h(z) = h(z') = 1$. Consequently, removing the zeroes of this function h from $\widehat{W_l}$ yields an H-invariant open affine neighbourhood $\widehat{W} \subset \widehat{Z}$ of z and z'. Now, $W := q_Z(\widehat{W})$ is as wanted and our claim is verified.

To complete the proof we still have to show that all the open subsets $U_i \subset X$ admit toric extensions. For this, let $\widehat{U}_i := q_X^{-1}(U_i)$ and define

$$\widehat{Z}_i := \bigcup_{\overline{\mathbb{T}^n \cdot z} \cap \widehat{U}_i \neq \varnothing} \mathbb{T}^n \cdot z.$$

Then each $\widehat{Z}_i$ is an open $\mathbb{T}^n$-invariant subset of $\widehat{Z}$. Moreover, we have $\widehat{U}_i = \widehat{X} \cap \widehat{Z}_i$. Again this holds because by our choice of $\widehat{Z}_i$, $\overline{X}$ and the embedding $\overline{X} \to \mathbb{K}^n$ we have

$$\widehat{U}_i = \overline{X} \cap (\mathbb{K}^n_{z_{i1}} \cup \cdots \cup \mathbb{K}^n_{z_{in_i}}), \qquad \widehat{Z}_i \subset \mathbb{K}^n_{z_{i1}} \cup \cdots \cup \mathbb{K}^n_{z_{in_i}}.$$

We shall show that $\widehat{Z}_i$ admits a good quotient for the action of $T \times H$. Once this is settled, the proof is complete: The image $Z_i := q_Z(\widehat{Z}_i)$ is an open toric subvariety of Z. Moreover, $U_i = X \cap Z_i$ holds, and there is a commutative diagram

$$\begin{tikzcd}
\widehat{Z}_i \arrow[rr] \arrow[dr, "q_Z"'] & & \widehat{Z}_i /\!\!/ (T \times H) \\
& Z_i \arrow[ur] &
\end{tikzcd}$$

One easily checks that the induced map $Z_i \to \widehat{Z}_i /\!\!/ (T \times H)$ is a good quotient for the action of T on Z_i. That means that $Z_i \subset Z$ fulfills the desired conditions of a toric extension of the open subset $U_i \subset X$.

Thus, the remaining task is to show that the action of $T \times H$ on $\widehat{Z}_i$ has a good quotient. For this we use Proposition 1.2: It suffices to verify that any two points of $\widehat{Z}_i$ having closed $\mathbb{T}^n$-orbits in $\widehat{Z}_i$ admit a common $(T \times H)$-invariant affine open neighbourhood in $\widehat{Z}_i$.

So, let $z, z' \in \widehat{Z}_i$ with $\mathbb{T}^n \cdot z$ and $\mathbb{T}^n \cdot z'$ closed in $\widehat{Z}_i$. By the definition of $\widehat{Z}_i$ there exist elements $t, t' \in \mathbb{T}^n$ and an index j such that

$$t \cdot z \in \widehat{U}_{ij}, \quad t' \cdot z' \in \widehat{U}_{ij}.$$

The set $\widehat{W}_{ij} := \mathbb{K}^n_{z_{ij}}$ is an affine $\mathbb{T}^n$-invariant neighbourhood of z and z'. Moreover $\widehat{U}_{ij}$ equals $\overline{X} \cap \widehat{W}_{ij}$ and hence is closed in $\widehat{W}_{ij}$. The complement $A := \widehat{W}_{ij} \setminus \widehat{Z}_i$ is a closed $\mathbb{T}^n$-invariant subset of $\widehat{W}_{ij}$ with $A \cap \widehat{U}_{ij} = \varnothing$. Consider the good quotient

$$o \colon \widehat{W}_{ij} \to \widehat{W}_{ij} /\!\!/ (T \times H).$$

This is a toric morphism of affine toric varieties, see e.g. Remark 1.1. The images $o(\widehat{U}_{ij})$ and $o(A)$ are disjoint closed subsets of $\widehat{W}_{ij} /\!\!/ (T \times H)$. In particular, it follows

$$o(t \cdot z) \notin o(A), \qquad o(t' \cdot z') \notin o(A).$$

Since A is $\mathbb{T}^n$-invariant, we see that neither $o(z)$ nor $o(z')$ lie in $o(A)$. Consequently, there exists a $(T \times H)$-invariant regular function $g \in \mathcal{O}(\widehat{W}_{ij})$ such that

$$g|_A = 0, \qquad g(z) = g(z') = 1$$

holds. Thus, removing the zero set of g from $\widehat{W}_{ij}$ yields the desired common $(T \times H)$-invariant affine open neighbourhood $\widehat{W} \subset \widehat{Z}_i$ of the points z and z'. ☐

Appendix: A little survey on embedding theorems

This appendix is independent from the previous sections. We collect some general results concerning embeddings into toric varieties and prevarieties. The little survey begins with two classical statements on embeddings into the projective space $\mathbb{P}_n$.

Let X be an irreducible algebraic variety over an algebraically closed field $\mathbb{K}$. For a Cartier divisor D on X, we denote by $\mathcal{O}_D$ its associated invertible sheaf. The set of zeroes of a section $f \in \mathcal{O}_D(X)$ is

$$Z(f) := \mathrm{Supp}(\mathrm{div}(f) + D).$$

Following [**6**, Section 4.5], we call an effective Cartier divisor D on X *ample* if every $x \in X$ has an affine neighbourhood of the form $X_f := X \setminus Z(f)$ with a section $f \in \mathcal{O}_{nD}(X)$ where $n \geq 0$.

Theorem A.1 **([6,** Théorème 4.5.2**])**. — *For a variety X the following statements are equivalent:*

i) *There exists an ample Cartier divisor on X.*
ii) *X admits a locally closed embedding into a projective space $\mathbb{P}_n$.*

Given a quasiprojective variety X, it is often important to find embeddings $X \to \mathbb{P}_n$ that are compatible with respect to further structure on X. We concentrate here on regular actions

$$G \times X \to X, \qquad (g, x) \mapsto g \cdot x$$

of algebraic groups G. A G-action on the projective space $\mathbb{P}_n$ is called linear if in homogeneous coordinates it is given by $g \cdot [z] = [\varrho(g)z]$ with a regular representation $\varrho \colon G \to \mathrm{GL}(n+1, \mathbb{K})$.

Theorem A.2 **([14], [16,** Theorem 1**])**. — *Suppose that a connected linear algebraic group G acts regularly on a normal quasiprojective variety X. Then X admits a G-equivariant locally closed embedding into some $\mathbb{P}_n$ where G acts linearly.*

Our intention is to present generalizations of these two classical results to non quasiprojective varieties X. So, in this setting, the ambient space $\mathbb{P}_n$ has to be replaced with more general objects:

A *toric prevariety* is a normal (possibly non separated) algebraic prevariety Z over $\mathbb{K}$ together with regular action $T_Z \times Z \to Z$ of an algebraic torus T_Z such that for some $z_0 \in Z$ the orbit map $T \to Z$, $t \mapsto t \cdot z_0$ is an open embedding.

Theorem A.3 **([20, Theorem C])**. — *Every normal variety X admits a closed embedding into a toric prevariety Z.*

Similarly to the separated case, the category of toric prevarieties can be completely described in terms of combinatorial data. For an introduction to this we refer to [**1**]. Concerning embbedings into separated ambient spaces we have:

Theorem A.4 **([20, Theorem A])**. — *For a normal variety X the following statements are equivalent:*

i) *Any two points of X have a common affine neighbourhood in X.*
ii) *X admits a closed embedding into a toric variety Z.*

We call a variety X with Property A.4 i) for short an *A_2-variety.* There exist examples of normal varieties that don't have this property and hence cannot be embedded into separated toric varieties:

Remark A.5

i) The normal surfaces discussed in [**12**] are not A_2.
ii) The Hironaka twist is a smooth threefold that is not A_2, see e.g. [**15**, p. 83].

For several algebro-geometric constructions it is convenient to embed an arbitrary singular variety X into a smooth ambient space. This requires some condition on X, namely divisoriality in the sense of Borelli [**5**]:

Theorem A.6 **([11, Theorem 3.2])**. — *For an irreducible variety X the following statements are equivalent:*

i) *X is divisorial, i.e., every $x \in X$ has an affine neighbourhood of the form $X \setminus \operatorname{Supp}(D)$ with an effective Cartier divisor D on X.*
ii) *X admits a closed embedding into a smooth toric prevariety Z having an affine diagonal map $Z \to Z \times Z$.*

Here the last condition on Z means just that for any two affine open subsets $U, U' \subset X$ their intersection $U \cap U'$ is again affine. We say that a prevariety having this property is *of affine intersection.* The nonseparatedness of such a space is rather mild.

Remark A.7

i) There exist three dimensional toric varieties that don't admit nontrivial effective Cartier divisors and hence cannot be embedded into smooth toric varieties, see e.g. [**7**, Example 3.5].
ii) The normal surfaces discussed in [**12**] admit neither embeddings into $\mathbb{Q}$-factorial toric prevarieties nor into toric prevarieties of affine intersection.

Again, it is interesting to know, when one can reckon on a separated ambient space. The criterion is related to Włodarczyk's A_2-property:

Theorem A.8 **([11, Corollary 5.4])**. — *For an irreducible variety X, the following statements are equivalent:*

i) *Any two points $x, x' \in X$ have a common affine neighbourhood of the form $X \setminus \mathrm{Supp}(D)$ with an effective Cartier divisor D on X.*

ii) *X admits a closed embedding into a smooth toric variety.*

We say that a variety satisfying Condition A.8 i) is 2-*divisorial*. Simlarly, one can define k-divisoriality also for $k > 2$ and obtains analogous embedding results, see [**11**, Theorem 5.3].

Remark A.9. — There exists a toric variety that is divisorial but not 2-divisorial, see [**2**, Proposition 4.1].

Now we turn to equivariant embeddings, i.e. generalizations of Theorem A.2. Let G be an algebraic group. First we have to generalize the notion of a linear G-action on the ambient space:

Suppose that a subtorus $H \subset (\mathbb{K}^*)^r$ acts freely on an open toric subvariety $U \subset \mathbb{K}^r$. Then there is a geometric prequotient $q\colon U \to Z$ for this action, and the quotient space Z is a toric prevariety of affine intersection. Conversely every smooth toric prevariety of affine intersection is of this form, see [**1**, Section 8].

Assume moreover, that an algebraic group G acts on $\mathbb{K}^r$ by means of a representation $\varrho\colon G \to \mathrm{GL}(r, \mathbb{K})$ such that U is G-invariant and the actions G and H commute. Then we call the induced G-action on Z *linear*.

Theorem A.10 **([11, Theorem 3.4 and Corollary 5.7])**. — *Suppose that a connected linear algebraic group G acts regularly on a normal variety X.*

i) *If X is divisorial then it admits a G-equivariant closed embedding into a smooth toric prevariety Z of affine intersection where G acts linearly.*

ii) *If X is 2-divisorial then it admits a G-equivariant closed embedding into a smooth toric variety Z where G acts linearly.*

An application of this result is that every $\mathbb{Q}$-factorial toric variety admits a toric embedding into a smooth one, see [**11**, Corollary 5.8].

Remark A.11

i) The Hironaka twist [**15**, p. 83] shows that Theorem A.10 does in general not hold for disconnected groups G.

ii) The standard $\mathbb{K}^*$-action on the complete rational curve with a node shows that some condition like normality on X is necessary in Theorem A.10.

Of course it would also be interesting to have equivariant versions of Theorems A.3 and A.4. The only statement in this direction I know so far is:

Theorem A.12 **([10,** Theorem 9.1]). — *Every normal* $\mathbb{K}^*$*-variety admits a* $\mathbb{K}^*$*-equivariant closed embedding into a toric prevariety where* $\mathbb{K}^*$ *acts as a subgroup of the big torus.*

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