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# THE CHERN NUMBERS OF THE NORMALIZATION OF AN ALGEBRAIC THREEFOLD WITH ORDINARY SINGULARITIES

*by*

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***Abstract.*** — By a classical formula due to Enriques, the Chern numbers of the non-singular normalization $X$ of an algebraic surface $S$ with ordinary singularities in $\mathbb{P}^3(\mathbb{C})$ are given by $\int_X c_1^2 = n(n-4)^2 - (3n-16)m + 3t - \gamma$, $\int_X c_2 = n(n^2 - 4n + 6) - (3n-8)m + 3t - 2\gamma$, where $n =$ the degree of $S$, $m =$ the degree of the double curve (singular locus) $D_S$ of $S$, $t =$ the cardinal number of the triple points of $S$, and $\gamma$=the cardinal number of the cuspidal points of $S$. In this article we shall give similar formulas for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ (Theorem 1.15, Theorem 2.1, Theorem 3.2). As a by-product, we obtain a numerical formula for the Euler-Poincaré characteristic $\chi(X, \mathcal{T}_X)$ with coefficient in the sheaf $\mathcal{T}_X$ of holomorphic vector fields on the non-singular normalization $X$ of $\overline{X}$ (Theorem 4.1).

***Résumé*** **(Les nombres de Chern de la normalisée d'une variété algébrique de dimension 3 à points singuliers ordinaires)**

Par une formule classique due à Enriques, les nombres de Chern de la normalisation non singulière $X$ de la surface algébrique $S$ avec singularités ordinaires dans $\mathbb{P}^3(\mathbb{C})$ sont donnés par $\int_X c_1^2 = n(n-4)^2 - (3n-16)m + 3t - \gamma$, $\int_X c_2 = n(n^2 - 4n + 6) - (3n-8)m + 3t - 2\gamma$, où $n$ est le degré de $S$, $m$ est le degré de la courbe double (lieu singulier) $D_S$ de $S$, $t$ est le nombre de points triples de $S$, et $\gamma$ est le nombre de points cuspidaux de $S$. Dans cet article nous donnons des formules similaires pour une "threefold" algébrique $\overline{X}$ avec singularités ordinaires dans $\mathbb{P}^4(\mathbb{C})$ (Théorème 1.15, Théorème 2.1, Théorème 3.2). Comme application, nous obtenons une formule numérique pour la caractéristique d'Euler-Poincaré $\chi(X, \mathcal{T}_X)$ à coefficients dans le faisceau $\mathcal{T}_X$ de champs de vecteurs holomorphes de la normalisation non singulière $X$ de $\overline{X}$ (Théorème 4.1).

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## Introduction

An irreducible hypersurface $\overline{X}$ in the complex projective 4-space $\mathbb{P}^4(\mathbb{C})$ is called an *algebraic threefold with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 4-space $\mathbb{C}^4$ at every point of $\overline{X}$:

$$(0.1)\qquad \begin{cases} \text{(i)} & w=0 \quad \text{(simple point)}\\ \text{(ii)} & zw=0 \quad \text{(ordinary double point)}\\ \text{(iii)} & yzw=0 \quad \text{(ordinary triple point)}\\ \text{(iv)} & xyzw=0 \quad \text{(ordinary quadruple point)}\\ \text{(v)} & xy^2-z^2=0 \quad \text{(cuspidal point)}\\ \text{(vi)} & w(xy^2-z^2)=0 \quad \text{(stationary point)} \end{cases}$$

where $(x,y,z,w)$ is the coordinate on $\mathbb{C}^4$. These singularities arise if we project a non-singular threefold embedded in a sufficiently higher dimensional complex projective space to its four dimensional linear subspace by a *generic* linear projection ([**R**]), though the singularities (iv) and (vi) above do not occur in the surface case. This fact can also be proved by use of the classification theory of multi-germs of *locally stable* holomorphic maps ([**M-3**], [**T-1**]). Indeed, in the threefold case, the pair of dimensions of the source and target manifolds belongs to the so-called *nice range*([**M-2**]). Hence the multi-germ of a *generic* linear projection at the inverse image of any point of $\overline{X}$ is *stable*, *i.e.*, stable under small deformations ([**M-4**]).

In [**T-2**] we have proved, for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ which is free from quadruple points, a formula expressing the Euler number $\chi(X)$ of the non-singular normalization $X$ of $\overline{X}$ in terms of numerical characteristics of $\overline{X}$ and its singular loci. Note that, by the Gauss-Bonnet formula, the Euler number $\chi(X)$ is equal to the Chern number $\int_X c_3$, where $c_3$ denotes the top Chern class of $X$. In § 1 we shall extend this formula to the general case where $\overline{X}$ admits quadruple points. In this general case, we need to blow up $\overline{X}$ twice. First, along the quadruple point locus, and secondly, along the triple point locus. It turns out that the existence of quadruple points adds only the term $4\#\Sigma\overline{q}$ to the formula, where $\#\Sigma\overline{q}$ denotes the cardinal number of the quadruple point locus $\Sigma\overline{q}$. Using Fulton-MacPherson's intersection theory, especially, the *excess intersection formula* ([**F**], Theorem 6.3, p. 102), the *blow-up formula* (ibid., Theorem 6.7, p. 116), the *double point formula* (ibid., Theorem 9.3, p. 166) and the *ramification formula* (ibid., Example 3.2.20, p. 62), we compute the *push-forwards* $f_*[D]^2$ and $f_*[D]^3$ for $D$ the inverse image of the singular locus of $\overline{X}$ by the normalization map in order to know the *Segre classes* $s(\overline{J},\overline{X})_i$ $(0\leqslant i\leqslant 2)$ of the singular subscheme $\overline{J}$ defined by the Jacobian ideal of $\overline{X}$.

In § 2 we shall give a formula for the Chern number $\int_X c_1^3 = -[K_X]^3$, where $[K_X]$ is the canonical class of $X$. The expressions for $f_*[D]^2$ and $f_*[D]^3$ obtained in § 1 enable us to compute it, because $[K_X] = f^*[\overline{X}+K_Y]-[D]$ by the *double point formula*, where

$K_Y$ is the canonical divisor of $\mathbb{P}^4(\mathbb{C})$. In § 3 we shall give a formula for the Chern number $\int_X c_1c_2$. In fact, we shall calculate the Euler-Poincaré characteristic $\chi(X, K_X)$ with coefficient in the canonical line bundle of $X$, which is equal to $-(1/24)\int_X c_1c_2$ by the Riemann-Roch theorem. In § 4, as a by-product, we shall give a numerical formula for the Euler-Poincaré characteristic $\chi(X, \mathcal{T}_X)$ with coefficient in the sheaf $\mathcal{T}_X$ of holomorphic tangent vector fields on $X$.

## Notation and Terminology

Throughout this article we fix the notation as follows:

$Y := \mathbb{P}^4(\mathbb{C})$: the complex projective 4-space,
$\overline{X}$: an algebraic threefold with ordinary singularities in $Y$,
$\overline{J}$: the singular subscheme of $\overline{X}$ defined by the Jacobian ideal of $\overline{X}$,
$\overline{D}$: the singular locus of $\overline{X}$,
$\overline{T}$: the triple point locus of $\overline{X}$, which is equal to the singular locus of $\overline{D}$,
$\overline{C}$: the cuspidal point locus of $\overline{X}$, precisely, its closure, since we always consider $\overline{C}$ contains the stationary points,
$\Sigma\overline{q}$: the quadruple point locus of $\overline{X}$,
$\Sigma\overline{s}$: the stationary point locus of $\overline{X}$,
$n_{\overline{X}} : X \to \overline{X}$: the normalization of $\overline{X}$,
$f : X \to Y$: the composite of the normalization map $n_{\overline{X}}$ and the inclusion $\overline{\imath} : \overline{X} \hookrightarrow Y$,
$J$: the scheme-theoretic inverse of $\overline{J}$ by $f$,
$D$, $T$, $C$ and $\Sigma q$: the inverse images of $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\Sigma\overline{q}$ by $f$, respectively,
$\Sigma s = T \cap C$: the intersection of $T$ and $C$.

We put

$$n := \deg \overline{X} \text{ (the degree of } \overline{X} \text{ in } \mathbb{P}^4(\mathbb{C})), \quad m := \deg \overline{D}, \quad t := \deg \overline{T}, \quad \gamma := \deg \overline{C}.$$

Note that $\overline{T}$ and $\overline{C}$ are non-singular curves, intersecting transversely at $\Sigma\overline{s}$, and that the normalization $X$ of $\overline{X}$ is also non-singular. Calculating by use of local coordinates, we can easily see the following:

(i) $J$ contains $D$, and the *residual scheme* to $D$ in $J$ is the reduced scheme $C$, *i.e.*, $\mathcal{I}_J = \mathcal{I}_D \otimes_{\mathcal{I}_X} \mathcal{I}_C$, where $\mathcal{I}_J$, $\mathcal{I}_D$, $\mathcal{I}_C$ are the ideal sheaves of $J$, $D$ and $C$, respectively (*cf.* [**F**], Definition 9.2.1, p. 160);

(ii) $D$ is a surface with ordinary singularities, whose singular locus is $T$,

(iii) $D$ is the *double point* locus of the map $f : X \to Y$, *i.e.*, the closure of $\{q \in X \mid \#f^{-1}(f(q)) \geqslant 2\}$;

(iv) the map $f_{|D} : D \to \overline{D}$ is generically two to one, simply ramified at $C$;

(v) the map $f_{|T} : T \to \overline{T}$ is generically three to one, simply ramified at $\Sigma s$.

Furthermore, we need the following diagram consisting of two fiber squares:

$$
\begin{array}{ccc}
X'' & \xrightarrow{f''} & Y'' \\
\tau_{T'}\downarrow & & \downarrow\sigma_{\overline{T}'} \\
X' & \xrightarrow{f'} & Y' \\
\tau_{\Sigma q}\downarrow & & \downarrow\sigma_{\Sigma\overline{q}} \\
X & \xrightarrow[f]{} & Y,
\end{array}
\tag{0.2}
$$

which is defined as follows:

$\sigma_{\Sigma\overline{q}} : Y' \to Y$: the blowing-up of $Y$ along the quadruple point locus $\Sigma\overline{q}$ of $\overline{X}$,

$\overline{X}'$: the proper inverse image of $\overline{X}$ by $\sigma_{\Sigma\overline{q}}$,

$X' := X \times_{\overline{X}} \overline{X}'$: the fiber product of $X$ and $\overline{X}'$ over $\overline{X}$,

$n_{\overline{X}'} : X' \to \overline{X}'$: the projection to the second factor of $X \times_{\overline{X}} \overline{X}'$, which is nothing but the normalization of $\overline{X}'$,

$f' : X' \to Y'$: the composite of the normalization map $n_{\overline{X}'}$ and the inclusion $\overline{\iota}' : \overline{X}' \hookrightarrow Y'$,

$\Sigma q$: the inverse image of the quadruple point locus $\Sigma\overline{q}$ of $\overline{X}$ by $f$,

$\tau_{\Sigma q} : X' \to X$: the projection to the first factor of $X \times_{\overline{X}} \overline{X}'$, which is nothing but the blowing-up of $X$ along $\Sigma q$,

$\overline{D}'$, $\overline{T}'$, $\overline{C}'$ and $\Sigma\overline{s}'$: the proper inverse images of $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\Sigma\overline{s}$ by $\sigma_{\Sigma\overline{q}}$, respectively.

$D'$, $T'$ and $C'$: the proper inverse images of $D$, $T$ and $C$ by $\tau_{\Sigma q}$, which are equal to the inverse images of $\overline{D}'$, $\overline{T}'$ and $\overline{C}'$ by $f'$, respectively,

$\Sigma s'$: the inverse image of $\Sigma s$ by $\tau_{\Sigma q}$, which is equal to $T' \cap C'$,

$\sigma_{\overline{T}'} : Y'' \to Y'$: the blowing-up of $Y'$ along $\overline{T}'$,

$\overline{X}''$: the proper inverse image of $\overline{X}'$ by $\sigma_{\overline{T}'}$,

$X'' := X' \times_{\overline{X}'} \overline{X}''$: the fiber product of $X'$ and $\overline{X}''$ over $\overline{X}'$,

$n_{\overline{X}''} : X'' \to \overline{X}''$: the projection to the second factor of $X' \times_{\overline{X}'} \overline{X}''$, which is nothing but the normalization of $\overline{X}''$

$f'' : X'' \to Y''$: the composite of the normalization map $n_{\overline{X}''}$ and the inclusion $\iota'' : \overline{X}'' \hookrightarrow Y''$,

$\tau_{T'} : X'' \to X'$: the projection to the first factor of $X' \times_{\overline{X}'} \overline{X}''$, which is nothing but the blowing-up of $X'$ along $T'$,

$\overline{D}''$, $\overline{T}''$, $\overline{C}''$ and $\Sigma\overline{s}''$: the proper inverse images of $\overline{D}'$, $\overline{T}'$, $\overline{C}'$ and $\Sigma\overline{s}'$ by $\sigma_{\overline{T}}$, respectively,

$D''$, $T''$ and $C''$: the proper inverse images of $D'$, $T'$ and $C'$ by $\tau_{T'}$, which are equal to the inverse images of $\overline{D}''$, $\overline{T}''$ and $\overline{C}''$ by $f''$, respectively,

$\Sigma s''$: the inverse image of $\Sigma s'$ by $\tau_{T'}$, which is equal to $T'' \cap C''$.

We also use the following notation throughout this article:

$[\alpha]$: the rational equivalence class of an algebraic cycle $\alpha$,

$\alpha \cdot \beta$: the intersection class of two algebraic cycle classes $[\alpha]$ and $[\beta]$.

Finally, we give the definitions of *regular embeddings* and *local complete intersection morphisms* of schemes.

***Definition 0.1***. — We say a closed embedding $\iota : X \to Y$ of schemes is a *regular embedding of codimension $d$* if every point in $X$ has an affine neighborhood $U$ in $Y$, such that if $A$ is the coordinate ring of $U$, $I$ the ideal of $A$ defining $X$, then $I$ is generated by a regular sequence of length $d$.

If this is the case, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of $X$ in $Y$, is a locally free sheaf of rank $d$. The *normal bundle* to $X$ in $Y$, denoted by $N_XY$, is the vector bundle on $X$ whose sheaf of sections is dual to $\mathcal{I}/\mathcal{I}^2$. Note that the normal bundle $N_XY$ is canonically isomorphic to the normal cone $C_XY$ for a (closed) regular embedding $\iota : X \to Y$ since the canonical map from $\mathrm{Sym}(\mathcal{I}/\mathcal{I}^2)$ to $S^{\cdot} := \Sigma_{k=0}^{\infty} \mathcal{I}^k/\mathcal{I}^{k+1}$ is an isomorphism (*cf.* [**F**], Appendix B, B.7).

***Definition 0.2***. — A morphism $f : X \to Y$ is called a *local complete intersection morphism of codimension $d$* if $f$ factors into a (closed) regular embedding $\iota : X \to P$ of some constant codimension $e$, followed by a smooth morphism $p : P \to Y$ of constant relative dimension $d + e$.

## 1. The computation of $\int_X c_3$

In [**T-2**] we have proved, for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ which is free from quadruple points, a formula expressing the the Euler number $\chi(X)$ of the non-singular normalization $X$ of $\overline{X}$ in terms of numerical characteristics of $\overline{X}$ and its singular loci. We recall its proof briefly. We have first proved the following:

***Theorem 1.1* ([T-2], Theorem 2.1)**. — *We have a linear pencil $\overline{\mathcal{L}} := \bigcup_{\lambda\in\mathbb{P}^1} \overline{X_\lambda}$ on $\overline{X}$, consisting of hyperplane sections $\overline{X_\lambda}$ of $\overline{X}$ in $\mathbb{P}^4(\mathbb{C})$, whose pull-back $\mathcal{L} := \bigcup_{\lambda\in\mathbb{P}^1} X_\lambda$ to $X$ by the normalization map $f : X \to \overline{X}$ has the following properties: There exists a finite set $\{\lambda_1, \dots, \lambda_c\}$ of points of $\mathbb{P}^1$ such that*

(i) *$X_\lambda$ is non-singular for $\lambda$ with $\lambda \neq \lambda_i$ $(1 \leqslant i \leqslant c)$, and*

(ii) *$X_{\lambda_i}$ is a surface with only one isolated ordinary double point which is contained in $X \smallsetminus f^{-1}(\overline{C_\infty})$ for any $i$ with $1 \leqslant i \leqslant c$,*

*where $c$ is the* class *of $\overline{X}$, i.e., the degree of the top polar class $[M_3]$ of $\overline{X}$ in $\mathbb{P}^4(\mathbb{C})$ (cf. [**P**]), and $\overline{C_\infty}$ the base point locus of the linear pencil $\overline{\mathcal{L}}$, which is an irreducible curve with $m$ $(= \deg \overline{D})$ ordinary double points in $\mathbb{P}^2(\mathbb{C})$ whose degree is equal to $n$ $(= \deg \overline{X})$.*

Let $\sigma : \widehat{X} \to X$ be the blowing-up along $C_\infty := f^{-1}(\overline{C_\infty})$, and $\widehat{\mathcal{L}} := \bigcup_{\lambda\in\mathbb{P}^1} \widehat{X_\lambda}$ the proper inverse of $\mathcal{L} := \bigcup_{\lambda\in P} X_\lambda$. Then $\widehat{\mathcal{L}}$ gives a fibering of $\widehat{X}$ over $\mathbb{P}^1(\mathbb{C})$. Hence the Euler number $\chi(\widehat{X})$ of $\widehat{X}$ is given by

$$\begin{aligned}\chi(\widehat{X}) &= \chi(\mathbb{P}^1(\mathbb{C}))\chi(\widehat{X_\lambda}) + \Sigma_{j=1}^{c}(\chi(\widehat{X_{\lambda_j}}) - \chi(\widehat{X_\lambda}))\\ &= 2\chi(\widehat{X_\lambda}) - c\end{aligned}$$

where $\widehat{X_\lambda}$ denotes a generic fiber of the fiber space $\widehat{X} \to \mathbb{P}^1$. The second equality above follows from the fact that a topological 2-cycle vanishes when $\lambda \to \lambda_j$ for $j = 1, \ldots, c$. We put $\widehat{E} := \sigma^{-1}(C_\infty)$. Then, since $\widehat{X} \setminus \widehat{E} \simeq X \setminus C_\infty$,

$$\begin{aligned}\chi(\widehat{X}) - \chi(X) &= \chi(\widehat{E}) - \chi(C_\infty)\\ &= \chi(\mathbb{P}^1(\mathbb{C}))\chi(C_\infty) - \chi(C_\infty)\\ &= \chi(C_\infty)\end{aligned}$$

Hence,

$$\begin{aligned}\chi(X) = \chi(\widehat{X}) - \chi(C_\infty) &= 2\chi(\widehat{X_\lambda}) - \chi(C_\infty) - c\\ &= 2\chi(X_\lambda) - \chi(C_\infty) - c.\end{aligned} \tag{1.1}$$

Since $\overline{C_\infty}$ is a curve whose degree is equal to $n$ with $m$ ordinary double points in $\mathbb{P}^2(\mathbb{C})$, the genus $g(C_\infty)$ is given by

$$g(C_\infty) = \frac{1}{2}(n-1)(n-2) - m.$$

Hence,

$$\chi(C_\infty) = 2 - 2g(C_\infty) = 2 - (n-1)(n-2) + 2m. \tag{1.2}$$

Note that $\overline{X_\lambda}$ is a surface with ordinary singularities in a hyperplane $H_\lambda \simeq \mathbb{P}^3(\mathbb{C})$ of degree $n$, whose numerical characteristics related to its singularities are as follows:

- the degree of its double curve $\overline{D_\lambda} = m$
- $\#\{\text{triple points of } \overline{X_\lambda}\} = t$,
- $\#\{\text{cuspidal points of } \overline{X_\lambda}\} = \gamma$.

Therefore, by the classical formula,

$$\chi(X_\lambda) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma \tag{1.3}$$

By (1.1), (1.2) and (1.3), we have the following:

***Proposition 1.2*** **([T-2], Proposition 2.2)**

$$\begin{aligned}\chi(X) &= 2n(n^2 - 4n + 6) - 2(3n - 8)m + 6t - 4\gamma\\ &\qquad - 2 + (n-1)(n-2) - 2m - c\\ &= n(2n^2 - 7n + 9) - 2(3n - 7)m + 6t - 4\gamma - c\end{aligned} \tag{1.4}$$

Even if $\overline{X}$ admits quadruple points, Theorem 1.1 and Proposition 1.2 above can be proved without change of their proofs in [**T-2**]. Hence what we have to do is compute the *class* $c$ of $\overline{X}$, *i.e.*, the degree of the top polar class $[M_3]$ of $\overline{X}$ in $\mathbb{P}^4(\mathbb{C})$. By the result due to R. Piene ([**P**], Theorem (2.3)), the top polar class $[M_3]$ of $\overline{X}$ is given by

$$[M_3] = (n-1)^3 h^3 - 3(n-1)^2 h^2 \cap s_2 - 3(n-1) h \cap s_1 - s_0, \tag{1.5}$$

where $h$ denotes the hyperplane section class and $s_i$ $i$-th Segre class $s(\overline{J}, \overline{X})_i$ $(0 \leqslant i \leqslant 2)$. Since $f_* s(J, X)_i = s(\overline{J}, \overline{X})_i$ $(0 \leqslant i \leqslant 2)$, it suffices to compute the Segre classes $s(J, X)_i$ and their push-forwards by $f$. To compute the Segre class $s(J, X)_i$, the following proposition is useful.

***Proposition 1.3* ([F], Proposition 9.2, p. 161)**. — *Let $D \subset W \subset V$ be closed embeddings of schemes, with $V$ a $k$-dimensional variety, and $D$ a Cartier divisor on $V$. Let $R$ be the residual scheme to $D$ in $W$. Then, for all $m$,*

$$s(W, V)_m = s(D, V)_m + \Sigma_{j=0}^{k-m} \binom{k-m}{j} [-D]^j \cdot s(R, V)_{m+j}$$

*in $A_m(W)$, the $m$-th rational equivalence class group of algebraic cycles on $W$.*

In our case, since $D = f^{-1}(\overline{D})$ is a Cartier divisor, its normal cone $C_D X$ to $D$ in $X$ is isomorphic to $\mathcal{O}_X(D)_{|D}$, the restriction to $D$ of the line bundle $\mathcal{O}_X(D)$ associated to $D$. Therefore, the total Segre class $s(D, X)$ of $D$ in $X$ is given as follows:

$$\begin{aligned} s(D, X) &= c(\mathcal{O}_X(D)_{|D})^{-1} \cap [D] \\ &= [D] - c_1(\mathcal{O}_X(D)_{|D}) \cap [D] + c_1(\mathcal{O}_X(D)_{|D})^2 \cap [D] \\ &= [D] - [D]^2 + [D]^3. \end{aligned}$$

Since $C$ is non-singular,

$$c(N_{C/X})^{-1} \cap [C] = [C] - c_1(N_{C/X}) \cap [C].$$

Hence, applying Proposition 1.3 above to $W = J$, $D = f^{-1}(\overline{D})$ and $R = C$, we have

$$\begin{cases} s(J, X)_2 = [D] \\ s(J, X)_1 = -[D]^2 + [C] \\ s(J, X)_0 = [D]^3 - c_1(N_{C/X}) \cap [C] - 3D \cdot C \end{cases} \tag{1.6}$$

where $N_{C/X}$ is the normal bundle of $C$ in $X$. Since $f_*[D] = 2[\overline{D}]$, it follows from the first identity in (1.6) that

$$s(\overline{J}, \overline{X})_2 = 2[\overline{D}].$$

In what follows we use the notation in the diagram (0.2) freely without mention.

***Lemma 1.4***

$$\sigma^*_{\Sigma\overline{q}}[\overline{D}] = [\overline{D'}] + 6\overline{j}'_* \sum_{\overline{q}} [H'_{\overline{q}}], \tag{1.7}$$

*where* $H'_{\overline{q}}$ *is a hyperplane of* $E_{\overline{q}} := \sigma^{-1}_{\Sigma\overline{q}}(\overline{q}) \simeq \mathbb{P}^3(\mathbb{C})$ *for each quadruple point* $\overline{q}$, *and* $\overline{j}'$ *the inclusion map* $\Sigma_{\overline{q}} E_{\overline{q}} \hookrightarrow Y'$.

*Proof.* — Since the multiplicity of $\overline{D}$ at each quadruple point $\overline{q}$ of $\overline{X}$ is 6, (1.7) follows from the *blow-up formula* ([**F**], Theorem 6.7, p. 116 and Corollary 6.7.1, p. 117). ☐

We consider the following fiber square:

$$\begin{array}{ccc} E_{\overline{T}'} & \xrightarrow{\overline{j}''} & Y'' \\ {\scriptstyle \overline{p}''}\downarrow & & \downarrow{\scriptstyle \sigma_{\overline{T}'}} \\ \overline{T}' & \xrightarrow[\overline{\iota}']{} & Y', \end{array} \tag{1.8}$$

where $E_{\overline{T}'} = P(N_{\overline{T}'}Y')$ is the exceptional divisor of the blowing-up $\sigma_{\overline{T}'}$, which is a $\mathbb{P}^2(\mathbb{C})$-bundle over $\overline{T}'$, and $\overline{p}'' : E_{\overline{T}'} \to \overline{T}'$ is the projection to the base space of this bundle. We denote by $\mathcal{O}_{N_{\overline{T}'}Y'}(1)$ the *canonical line bundle* on $E_{\overline{T}'}$, and by $\mathcal{O}_{N_{\overline{T}'}Y'}(-1)$ its dual, or the *tautological line bundle* on $E_{\overline{T}'}$.

***Lemma 1.5***. — $\sigma^*_{\overline{T}'}[\overline{D}']$ *is expressed as*

$$\sigma^*_{\overline{T}'}[\overline{D}'] = [\overline{D}''] + 3\overline{j}''_*[\xi_{\overline{T}'}] + \overline{j}''_*\overline{p}''^*[\alpha_0] \tag{1.9}$$

*where* $[\xi_{\overline{T}'}] = c_1(\mathcal{O}_{N_{\overline{T}'}Y'}(1)) \cap [E_{\overline{T}'}]$ *and* $[\alpha_0]$ *an algebraic* 0*-cycle class on* $\overline{T}'$.

*Proof.* — By the *blow-up formula*,

$$\sigma^*_{\overline{T}'}[\overline{D}'] = [\overline{D}''] + \overline{j}''_*\{c(E'') \cap \overline{p}''^* s(\overline{T}', \overline{D}')\}_2 \tag{1.10}$$

where $E'' = \overline{p}''^* N_{\overline{T}'}Y'/N_{E_{\overline{T}'}}$, $Y'' = \overline{p}''^* N_{\overline{T}'}Y'/\mathcal{O}_{N_{\overline{T}'}Y'}(-1)$ and $s(\overline{T}', \overline{D}')$ is the total Segre class of $\overline{T}'$ in $\overline{D}'$. Since

$$c_1(E'') = \overline{p}''^* c_1(N_{\overline{T}'}Y') - c_1(\mathcal{O}_{N_{\overline{T}'}Y'}(-1)) = \overline{p}''^* c_1(N_{\overline{T}'}Y') + c_1(\mathcal{O}_{N_{\overline{T}'}Y'}(1)),$$

we have

$$\begin{aligned} \{c(E'') \cap s(\overline{T}', \overline{D}')\}_2 &= \overline{p}''^* s_0(\overline{T}', \overline{D}') + c_1(E'') \cap \overline{p}''^* s_1(\overline{T}', \overline{D}') \\ &= \overline{p}''^*\{s_0(\overline{T}', \overline{D}') + c_1(N_{\overline{T}'}Y') \cap s_1(\overline{T}', \overline{D}')\} \\ &\qquad + c_1(\mathcal{O}_{N_{\overline{T}'}Y'}(1)) \cap \overline{p}''^* s_1(\overline{T}', \overline{D}') \end{aligned} \tag{1.11}$$

To compute $s(\overline{T}', \overline{D}')$, we consider the normalization map $n_{\overline{D}'} : \overline{D}'^* \to \overline{D}'$. $\overline{D}'^*$ is non-singular. Hence, if we put $\overline{T}'^* := n^{-1}_{\overline{D}'}(\overline{T}')$, we have

$$\begin{aligned} s(\overline{T}'^*, \overline{D}'^*) &= c(N_{\overline{T}'^*}\overline{D}'^*)^{-1} \cap [\overline{T}'^*] \\ &= (1 - c_1(N_{\overline{T}'^*}\overline{D}'^*)) \cap [\overline{T}'^*] \\ &= [\overline{T}'^*] - \overline{T}'^* \cdot \overline{T}'^*. \end{aligned}$$

Therefore,

$$s(\overline{T}', \overline{D}') = n_{\overline{D}'*} s(\overline{T}'^*, \overline{D}'^*) = 3[\overline{T}'] - n_{\overline{D}'*}(\overline{T}'^* \cdot \overline{T}'^*),$$

and so,

$$\begin{cases} s_0(\overline{T}', \overline{D}') = -n_{\overline{D}'_*}(\overline{T}'^* \cdot \overline{T}'^*) \\ s_1(\overline{T}', \overline{D}') = 3[\overline{T}'] \end{cases} \tag{1.12}$$

By (1.11) and (1.12), if we put $[\alpha_0] := -n_{\overline{D}'_*}(\overline{T}'^* \cdot \overline{T}'^*) + 3c_1(N_{\overline{T}'}Y') \cap [\overline{T}']$,

$$\{c(E'') \cap s(\overline{T}', \overline{D}')\}_2 = \overline{p}''^*[\alpha_0] + 3[\xi_{\overline{T}'}].$$

Consequently, by (1.10), we obtain (1.9). □

By Lemma 1.4 and Lemma 1.5 we have the following:

***Lemma 1.6***

$$\sigma^*_{\overline{T}'}\sigma^*_{\Sigma\overline{q}}[\overline{D}] = [\overline{D}''] + 3\overline{j}''_*[\xi_{\overline{T}'}] + \overline{j}''_*\overline{p}''^*[\alpha_0] + 6\overline{\ell}''_* \sum_{\overline{q}}[H''_{\overline{q}}], \tag{1.13}$$

*where* $[\xi_{\overline{T}'}] = c_1(\mathcal{O}_{N_{\overline{T}'}Y'}(1)) \cap [E_{\overline{T}'}]$ *and* $[\alpha_0]$ *an algebraic* 0*-cycle class on* $\overline{T}'$, $H''_{\overline{q}}$ *the proper inverse image of* $H'_{\overline{q}}$ *by* $\sigma_{\overline{T}'}$, *and* $\overline{\ell}''$ *the inclusion map* $\Sigma_{\overline{q}}E'_{\overline{q}} \hookrightarrow Y''$ *where* $E'_{\overline{q}}$ *is the proper inverse image of* $E_{\overline{q}}$ *by* $\sigma_{\overline{T}'}$.

***Proposition 1.7***

$$f''^*[\overline{D}''] = f''^*[\overline{X}''] \cdot D'' - [D'']^2 - [C''] \tag{1.14}$$

*Proof*. — Since $\overline{D}''$ is regularly embedded in $Y''$, *i.e.*, $C_{\overline{D}''}Y'' \simeq N_{\overline{D}''}Y''$, while $\overline{D}'$ is not, we can apply the *excess intersection formula* ([**F**], Theorem 6.3, p. 102) to $\overline{D}''$. Then, denoting the tangent bundle of a non-singular algebraic variety, say $Z$, by $\mathcal{T}_Z$, we have

$$\begin{aligned} f''^*[\overline{D}''] &= c_1(f''^*N_{\overline{D}''}Y''/N_{D''}X'') \cap [D''] \\ &= \{c_1(f''^*\mathcal{T}_{Y''}) - c_1(f''^*\mathcal{T}_{\overline{D}''}) - c_1(\mathcal{T}_{X''}) + c_1(\mathcal{T}_{D''})\} \cap [D''] \\ &= \{c_1(f''^*\mathcal{T}_{Y''}) - c_1(\mathcal{T}_{X''})\} \cap [D''] - C'', \end{aligned} \tag{1.15}$$

where the last equality follows from the *ramification formula* ([**F**], Example 3.2.20, p. 62). On the other hand, by the *double point formula* ([**F**], Theorem 9.3, p. 166, Example 9.3.4, p. 167),

$$[D''] = f''^*[\overline{X}''] - \{c_1(f''^*\mathcal{T}_{Y''}) - c_1(\mathcal{T}_{X''})\} \cap [X'']. \tag{1.16}$$

By (1.15) and (1.16), we obtain (1.14). □

***Proposition 1.8***

$$f'^*\sigma^*_{\Sigma\overline{q}}[\overline{D}] = f'^*[\overline{X}'] \cdot D' - [D']^2 - [C'] + [T'] + 6k'_* \sum_q [H'_q], \tag{1.17}$$

*where* $H'_q$ *is a hyperplane of* $\tau^{-1}_{\Sigma q}(q) := E_q \simeq \mathbb{P}^2(\mathbb{C})$ *for each point* $q$ *of* $\Sigma q$, *and* $k'$ *the inclusion map* $\Sigma_q E_q \hookrightarrow X'$.

*Proof.* — We first note that

$$f'^{*}\sigma^{*}_{\Sigma\overline{q}}[\overline{D}] = \tau_{T'*}\tau^{*}_{T'}f'^{*}\sigma^{*}_{\Sigma\overline{q}}[\overline{D}] \qquad (1.18)$$
$$= \tau_{T'*}f''^{*}\sigma^{*}_{\overline{T}'}\sigma^{*}_{\Sigma\overline{q}}[\overline{D}]$$

The first equality above follows from the fact that $\tau_{T'}$ is the blowing-up of $X'$ along $T'$, and the second one from the commutativity of the upper fiber square in (0.2). Therefore, it suffices to compute the image of each term on the right hand side in (1.13) by $\tau_{T'*}f''^{*}$. First, we will compute the image by $f''^{*}$. $f''^{*}[\overline{D}'']$ is given by (1.14). To compute $f''^{*}(3\overline{j}''_{*}[\xi'_{\overline{T}}] + \overline{j}''_{*}\overline{p}''^{*}[\alpha_0])$, we consider the following fiber square:

$$\begin{array}{ccc} E_{T'} & \xrightarrow{j''} & X'' \\ {\scriptstyle p''}\downarrow & & \downarrow{\scriptstyle \tau_{T'}} \\ T' & \xrightarrow[\iota']{} & X', \end{array} \qquad (1.19)$$

where $E_{T'} = P(N_{T'}X')$ is the exceptional divisor of the blowing-up $\tau_{T'}$, which is a $\mathbb{P}^1(\mathbb{C})$-bundle over $T'$, and $p'' : E_{T'} \to T'$ is the projection to the base space of this bundle. There is a set of morphisms from the diagram in (1.19) to the one in (1.8) induced by those in the upper fiber square in (0.2). We denote by $g'$ and $g''$ the restriction of $f' : X' \to Y'$ to $T'$ and that of $f'' : X'' \to Y''$ to $E_{T'}$, respectively. Note that the morphism $g'' : E_{T'} \to E_{\overline{T}'}$ maps each fiber of $p'' : E_{T'} \to T'$ to that of $\overline{p}'' : E_{\overline{T}'} \to \overline{T}'$, and so $g''^{*}[\xi_{\overline{T}'}] = [\xi_{T'}]$, where $[\xi_{T'}] = c_1(\mathcal{O}_{N_{T'}X'}(1)) \cap [E_{T'}]$. Since $f'' : X'' \to Y''$ and $g'' : E_{T'} \to E_{\overline{T}'}$ are *local complete intersection morphisms* of the same codimension, we can apply Proposition 6.6, (c) in [**F**] (p. 113) to the fiber square

$$\begin{array}{ccc} E_{T'} & \xrightarrow{g''} & E_{\overline{T}'} \\ {\scriptstyle j''}\downarrow & & \downarrow{\scriptstyle \overline{j}''} \\ X'' & \xrightarrow[f'']{} & Y''. \end{array} \qquad (1.20)$$

Then,

$$f''^{*}\overline{j}''_{*}[\xi_{\overline{T}'}] = j''_{*}g''^{*}[\xi_{\overline{T}'}] = j''_{*}[\xi_{T'}], \quad \text{and} \qquad (1.21)$$

$$f''^{*}\overline{j}''_{*}\overline{p}''^{*}[\alpha_0] = \overline{j}''_{*}g''^{*}\overline{p}''^{*}[\alpha_0] = j''_{*}p''^{*}g'^{*}[\alpha_0]. \qquad (1.22)$$

To compute $f''^{*}(6\overline{\ell}''_{*}\Sigma_{\overline{q}}[H''_{\overline{q}}])$, we consider the following fiber squares:

$$\begin{array}{ccc} \Sigma_q E'_q & \xrightarrow{\ell''} & X'' \\ {\scriptstyle q''}\downarrow & & \downarrow{\scriptstyle \tau_{T'}} \\ \Sigma_q E_q & \xrightarrow[k']{} & X', \end{array} \qquad \begin{array}{ccc} \Sigma_{\overline{q}} E'_{\overline{q}} & \xrightarrow{\overline{\ell}''} & Y'' \\ {\scriptstyle \overline{q}''}\downarrow & & \downarrow{\scriptstyle \sigma_{\overline{T}'}} \\ \Sigma_{\overline{q}} E_{\overline{q}} & \xrightarrow[\overline{k}']{} & Y'. \end{array} \qquad (1.23)$$

As before there is a set of morphisms from the diagram on the left to the one on the right in (1.23) by those in the upper fiber square in (0.2). We denote by $h'$ and $h''$ the restriction of $f'$ to $\Sigma_q E_q$ and that of $f'' : X'' \to Y''$ to $\Sigma_q E'_q$, respectively. Since

$f'' : X'' \to Y''$ and $h'' : \Sigma_q E'_q \to \Sigma_{\overline{q}} E'_{\overline{q}}$ are *local complete intersection morphisms* of the same codimension, we have

$$f''^{*}\overline{\ell}''_{*}[H''_{\overline{q}}] = \ell''_{*}h''^{*}[H''_{\overline{q}}]. \tag{1.24}$$

Similarly, applying the same arguments for $f' : X' \to Y'$ and $h' : \Sigma_q E_q \to \Sigma_{\overline{q}} E_{\overline{q}}$, we have

$$f'^{*}\overline{k}'_{*}[H'_{\overline{q}}] = k'_{*}h'^{*}[H'_{\overline{q}}] = k'_{*}[H'_{q}]. \tag{1.25}$$

Since $h''^{*}\overline{q}''^{*} = q''^{*}h'^{*}$ and $[H''_{\overline{q}}] = \overline{q}''^{*}[H'_{\overline{q}}]$,

$$\begin{aligned} \ell''_{*}h''^{*}[H''_{\overline{q}}] &= \ell''_{*}h''^{*}\overline{q}''^{*}[H'_{\overline{q}}] \\ &= \ell''_{*}q''^{*}h'^{*}[H'_{\overline{q}}] \\ &= \ell''_{*}q''^{*}[H'_{q}]. \end{aligned} \tag{1.26}$$

Further, since $\tau_{T'} : X'' \to X'$ and $g'' : \Sigma_q E'_q \to \Sigma E_q$ are *local complete intersection morphisms* of the same codimension,

$$\ell''_{*}q''^{*}[H'_{q}] = \tau^{*}_{T'}k'_{*}[H'_{q}]. \tag{1.27}$$

Therefore, by (1.24), (1.26) and (1.27),

$$f''^{*}\overline{\ell}''_{*}[H''_{\overline{q}}] = \tau^{*}_{T'}k'_{*}[H'_{q}]. \tag{1.28}$$

Consequently, by (1.13), (1.14), (1.21), (1.22) and (1.28),

$$f''^{*}\sigma^{*}_{\overline{T}'}\sigma^{*}_{\Sigma\overline{q}}[\overline{D}] = f''^{*}[\overline{X}''] \cdot D'' - [D'']^2 - [C''] + 3\, j''_{*}[\xi_{T'}] + j''_{*}p''^{*}g'^{*}[\alpha_0] + 6\tau^{*}_{T'}k'_{*}\Sigma_q[H'_q].$$

Since $\tau_{T'*}[C''] = [C']$, $\tau_{T'*}j''_{*}[\xi_{T'}] = [T']$, $\tau_{T'*}j''_{*}p''^{*}g'^{*}[\alpha_0] = 0$ and $\tau_{T'*}\tau^{*}_{T'}k'_{*}[H'_q] = k'_{*}[H'_q]$, by (1.18) and the equality above, we have

$$f'^{*}\sigma^{*}_{\overline{q}}[\overline{D}] = \tau_{T'*}(f''^{*}[\overline{X}''] \cdot D'']) - \tau_{T'*}[D'']^2 - [C'] + 3[T'] + 6k'_{*}\Sigma_q[H'_q]. \tag{1.29}$$

Since $\tau^{*}_{T'}[D'] = [D''] + 2[E_{T'}]$,

$$\tau_{T'*}(f''^{*}[\overline{X}''] \cdot D'') = \tau_{T'*}(f''^{*}[\overline{X}''] \cdot \tau^{*}_{T'}[D'] - 2f''^{*}[\overline{X}''] \cdot E_{T'}). \tag{1.30}$$

On the other hand, since $\sigma^{*}_{\overline{T}'}[\overline{X}'] = [\overline{X}''] + 3[E_{\overline{T}'}]$,

$$f''^{*}[\overline{X}''] = f''^{*}\sigma^{*}_{\overline{T}'}[\overline{X}'] - 3[E_{T'}].$$

Hence,

$$\begin{aligned} \tau_{T'*}(f''^{*}[\overline{X}''] \cdot \tau^{*}_{T'}D') &= \tau_{T'*}(f''^{*}[\overline{X}'']) \cdot D' \\ &= \tau_{T'*}(f''^{*}\sigma^{*}_{\overline{T}'}[\overline{X}'] - 3[E_{T'}]) \cdot D' \\ &= \tau_{T'*}(f''^{*}\sigma^{*}_{\overline{T}'}[\overline{X}']) \cdot D' \\ &= \tau_{T'*}(\tau^{*}_{T'}f'^{*}[\overline{X}']) \cdot D' \\ &= f'^{*}[\overline{X}'] \cdot D', \end{aligned} \tag{1.31}$$

and

$$\begin{aligned}\tau_{T'*}(f''^*[\overline{X}''] \cdot E_{T'}) &= \tau_{T'*}((f''^*\sigma^*_{\overline{T}'}[\overline{X}']) \cdot E_{T'} - 3[E_{T'}]^2) \\ &= \tau_{T'*}(\tau^*_{T'} f'^*[\overline{X}'] \cdot E_{T'}) + 3\tau_{T'*} j''_*[\xi_{T'}] \\ &= f'^*[\overline{X}'] \cdot \tau_{T'*}[E_{T'}] + 3\iota'_*[T'] = 3[T'].\end{aligned} \tag{1.32}$$

Therefore, by (1.30), (1.31) and (1.32),

$$\tau_{T'*}(f''^*[\overline{X}''] \cdot D'') = f'^*[\overline{X}'] \cdot D' - 6[T']. \tag{1.33}$$

Furthermore, we have

$$\begin{aligned}\tau_{T'*}[D'']^2 &= \tau_{T'*}((\tau^*_{T'}[D'] - 2[E_{T'}])^2) \\ &= \tau_{T'*}((\tau^*_{T'}[D'])^2 - 4\tau^*_{T'}[D'] \cdot [E_{T'}] + 4[E_{T'}]^2) \\ &= (\tau_{T'*}\tau^*_{T'}[D']) \cdot [D'] - 4[D'] \cdot \tau_{T'*}[E_{T'}] - 4\tau_{T'*}[\xi_{T'}] \\ &= [D']^2 - 4[T'].\end{aligned} \tag{1.34}$$

Consequently, by (1.29), (1.33) and (1.34), we obtain (1.17). □

***Proposition 1.9***

$$f^*[\overline{D}] = f^*[\overline{X}] \cdot D - [D]^2 - [C] + [T] \tag{1.35}$$

*Proof.* — Since $\tau_{\Sigma q*} f'^* \sigma^*_{\Sigma\overline{q}}[\overline{D}] = \tau_{\Sigma q*}\tau^*_{\Sigma q} f^*[\overline{D}] = f^*[\overline{D}]$, $\tau_{\Sigma q*}[C'] = [C]$, $\tau_{\Sigma q*}[T'] = [T]$ and $\tau_{\Sigma q*}[H'_q] = 0$, by Proposition 1.8, we have

$$f^*[\overline{D}] = \tau_{\Sigma q*}(f'^*[\overline{X}] \cdot D') - \tau_{\Sigma q*}[D']^2 - [C] + [T]. \tag{1.36}$$

Since $\tau^*_{\Sigma q}[D] = [D'] + 3[\Sigma_q E_q]$, we have

$$\begin{aligned}\tau_{\Sigma q*}[D']^2 &= \tau_{\Sigma q*}((\tau^*_{\Sigma q}[D] - 3[\Sigma_q E_q])^2) \\ &= \tau_{\Sigma q*}(\tau^*_{\Sigma q}[D])^2 - 6\tau_{\Sigma q*}(\tau^*_{\Sigma q} D \cdot \Sigma_q E_q) + 9\,\tau_{\Sigma q*}[\Sigma_q E_q]^2 \\ &= [D]^2 - 6\,D \cdot \tau_{\Sigma q*}[E_{\Sigma q}] - 9k'_*\tau_{\Sigma q*}[\Sigma_q H'_q] \\ &= [D]^2,\end{aligned} \tag{1.37}$$

where $H'_q$ is a hyperplane of $E_q \simeq \mathbb{P}^2(\mathbb{C})$, and

$$\tau_{\Sigma q*}(f'^*[\overline{X}'] \cdot D') = \tau_{\Sigma q*}(f'^*[\overline{X}'] \cdot \tau^*_{\Sigma q}[D]) - 3\,\tau_{\Sigma q*}(f'^*[\overline{X}'] \cdot \Sigma_q E_q). \tag{1.38}$$

On the other hand, since $\sigma_{\Sigma\overline{q}}[\overline{X}] = [\overline{X}'] + 4[\Sigma_{\overline{q}} E_{\overline{q}}]$,

$$f'^*[\overline{X}'] = f'^*\sigma^*_{\Sigma\overline{q}}[\overline{X}] - 4\,\Sigma_q E_q$$

Hence,

$$\begin{aligned}\tau_{\Sigma q*}(f'^*[\overline{X}'] \cdot \tau^*_{\Sigma q}[D]) &= \tau_{\Sigma q*}(f'^*\sigma^*_{\Sigma\overline{q}}[\overline{X}] \cdot \tau^*_{\Sigma q}[D]) - 4\,\tau_{\Sigma q*}(\Sigma_q E_q \cdot \tau^*_{\Sigma q}[D]) \\ &= \tau_{\Sigma q*}(f'^*\sigma^*_{\Sigma\overline{q}}[\overline{X}]) \cdot [D] - 4\,\tau_{\Sigma q*}(\Sigma_q E_q) \cdot [D] \\ &= \tau_{\Sigma q*}\tau^*_{\Sigma q} f^*[\overline{X}] \cdot [D] = f^*[\overline{X}] \cdot [D],\end{aligned} \tag{1.39}$$

and

$$
\begin{aligned}
\tau_{\Sigma q*}(f'^*[\overline{X}'] \cdot \Sigma_q E_q) &= \tau_{\Sigma q*}(f'^* \sigma^*_{\Sigma \overline{q}}[\overline{X}] \cdot \Sigma_q E_q) - 4\,\tau_{\Sigma q*}[\Sigma_q E_q]^2 \\
&= \tau_{\Sigma q*}(\tau^*_{\Sigma q} f^*[\overline{X}] \cdot \Sigma_q E_q) + 4\,\tau_{\Sigma q*}(k'_* \Sigma_q H'_q) \\
&= f^*[\overline{X}] \cdot \tau_{\Sigma q*}[\Sigma_q E_q] = 0.
\end{aligned} \tag{1.40}
$$

Therefore, by (1.38), (1.39) and (1.40),

$$
\tau_{\Sigma q*}(f'^*[\overline{X}'] \cdot D') = f^*[\overline{X}] \cdot [D]. \tag{1.41}
$$

Consequently, by (1.36), (1.37) and (1.41), we obtain (1.35). □

Since $f_*[X] = [\overline{X}]$, $f_*[D] = 2[\overline{D}]$, $f_*[T] = 3[\overline{T}]$ and $f_*[C] = [\overline{C}]$, by Proposition 1.9, we have the following:

***Corollary 1.10***

$$
f_*[D]^2 = [\overline{X}] \cdot [\overline{D}] + 3[\overline{T}] - [\overline{C}]
$$

By Proposition 1.9,

$$
[D]^2 = f^*[\overline{X}] \cdot D - f^*[\overline{D}] - [C] + [T]. \tag{1.42}
$$

Hence, by the second equality in (1.6),

$$
s(J,X)_1 = -f^*[\overline{X}] \cdot D + f^*[\overline{D}] + 2[C] - [T],
$$

and so, by the *projection formula*,

$$
s(\overline{J},\overline{X})_1 = -\overline{X} \cdot \overline{D} - 3[\overline{T}] + 2[\overline{C}].
$$

We are now going to compute $s(\overline{J},\overline{X})_0$. Since $s(\overline{J},\overline{X})_0 = f_* s(J,X)_0$, it suffices to know the push-forward of each term of the right hand side of the last identity in (1.6). By (1.42),

$$
[D]^3 = f^*[\overline{X}] \cdot [D]^2 - f^*[\overline{D}] \cdot D - D \cdot C + D \cdot T. \tag{1.43}
$$

To realize $f_*[D]^3$, we compute the push-forward of each term on the right hand side of (1.43). By the *projection formula* and Corollary 1.10,

$$
\begin{aligned}
f_*(f^*[\overline{X}] \cdot [D]^2) &= [\overline{X}] \cdot f_*[D]^2 \\
&= [\overline{X}]^2 \cdot [\overline{D}] + 3[\overline{X}] \cdot \overline{T} - [\overline{X}] \cdot \overline{C}.
\end{aligned} \tag{1.44}
$$

Since $f_*[D] = 2[\overline{D}]$, by the *projection formula*,

$$
\begin{aligned}
f_*(f^*[\overline{D}] \cdot D) &= [\overline{D}] \cdot f_*[D] \\
&= 2[\overline{D}]^2.
\end{aligned} \tag{1.45}
$$

To realize $f_*[D \cdot C]$, we compute $f^*[\overline{C}]$. Since $\overline{C}$ is *regularly embedded* in $Y$, we can apply the *excess intersection formula* to it. Then,

$$
\begin{aligned}
f^*[\overline{C}] &= c_1(f^* N_{\overline{C}} Y / N_C X) \cap [C] \\
&= \{c_1(f^* \mathcal{T}_Y) - c_1(f^* \mathcal{T}_{\overline{C}}) - c_1(\mathcal{T}_X) + c_1(\mathcal{T}_C)\} \cap [C] \\
&= \{c_1(f^* \mathcal{T}_Y) - c_1(\mathcal{T}_X)\} \cap [C] \\
&= f^*[\overline{X}] \cdot C - D \cdot C,
\end{aligned} \tag{1.46}
$$

where the last equality but one follows from the fact $\overline{C} \simeq C$ and the last equality from the *double point formula* for $f : X \to Y$. Therefore, by (1.46) and the *projection formula*, we have

$$f_*(D \cdot C) = \overline{X} \cdot f_*[C] - \overline{C} \cdot f_*[X] = \overline{X} \cdot \overline{C} - \overline{C} \cdot \overline{X} = 0 \tag{1.47}$$

To realize $f_*(D \cdot T)$, we compute $f^*[\overline{T}]$. Since $\overline{T}$ is not *regularly embedded* in $Y$, we cannot apply the *excess intersection formula* to $\overline{T}$. But, since $\overline{T}'$ is *regularly embedded* in $Y'$, we can apply it to $\overline{T}'$. Then, by the same way as in the case of $\overline{C}$,

$$f'^*[\overline{T}'] = f'^*[\overline{X}'] \cdot T' - D' \cdot T' - [\Sigma s'] \tag{1.48}$$

Here the term $[\Sigma s']$ comes from $\{c_1(f'^*\mathcal{T}_{\overline{T}'}) - c_1(\mathcal{T}_{T'})\} \cap [T'] = [\Sigma s']$, which is the *ramification formula* for $f'_{|T'}$.

***Lemma 1.11***

(i) $\sigma^*_{\Sigma\overline{q}}[\overline{T}] = [\overline{T}'] + 4\Sigma_{\overline{q}}[H'_{\overline{q}}]^2$,
(ii) $\tau^*_{\Sigma q}[T] = [T'] + 3\Sigma_q[H'_q]$,

*where $H'_{\overline{q}}$ is a hyperplane of $E_{\overline{q}} := \sigma^{-1}_{\Sigma\overline{q}}(\overline{q}) \simeq \mathbb{P}^3(\mathbb{C})$ for each quadruple point $\overline{q}$ and $H'_q$ that of $E_q := \tau^{-1}_{\Sigma q}(q) \simeq \mathbb{P}^2(\mathbb{C})$ for each point $q$ of $f^{-1}(\Sigma\overline{q})$.*

*Proof.* — Since the multiplicity of $\overline{T}$ (resp. $T$) at each quadruple point $\overline{q}$ (resp. at each point $q$ of $f^{-1}(\Sigma\overline{q})$) is 4 (resp. 3), (i) (resp. (ii)) follows from the *blow-up formula* ([**F**], Theorem 6.7, p. 116, and Corollary 6.7.1, p. 117). □

***Proposition 1.12***

$$f^*[\overline{T}] = f^*[\overline{X}] \cdot T - D \cdot T - [\Sigma s] + [\Sigma q] \tag{1.49}$$

*Proof.* — Since $f'^*(4\,\Sigma_{\overline{q}}[H'_{\overline{q}}]^2) = 4\Sigma_q[H'_q]^2$, by Lemma 1.11, (i) and (1.48),

$$\begin{aligned} f'^*\sigma^*_{\Sigma\overline{q}}[\overline{T}] &= f'^*[\overline{T}'] + 4\,\Sigma_q[H'_q]^2 \\ &= f'^*[\overline{X}'] \cdot T' - D' \cdot T' - [\Sigma s'] + 4\,\Sigma_q[H'_q]^2. \end{aligned} \tag{1.50}$$

Since

$$f^*[\overline{T}] = \tau_{\Sigma q*}\tau^*_{\Sigma q} f^*[\overline{T}] = \tau_{\Sigma q*} f'^*\sigma^*_{\Sigma\overline{q}}[\overline{T}], \tag{1.51}$$

it suffices to compute the push-forward of each term on the right hand side in (1.50) by $\tau_{\Sigma q*}$ in order to know $f^*[\overline{T}]$. Since $\sigma^*_{\Sigma\overline{q}}[\overline{X}] = [\overline{X}'] + 4[\Sigma_{\overline{q}} E_{\overline{q}}]$ and $f'^*[\Sigma_{\overline{q}} E_{\overline{q}}] = [\Sigma_q E_q]$, by Lemma 1.11, (ii),

$$\begin{aligned} \tau_{\Sigma q*}(f'^*[\overline{X}'] \cdot T') &= \tau_{\Sigma q*}((f'^*\sigma^*_{\Sigma\overline{q}}[\overline{X}] - 4[\Sigma_q E_q]) \cdot (\tau^*_{\Sigma q}[T] - 3[\Sigma_q H'_q])) \\ &= \tau_{\Sigma q*}((\tau^*_{\Sigma q} f^*[\overline{X}]) \cdot \tau^*_{\Sigma q}[T] - 4[\Sigma_q E_q] \cdot \tau^*_{\Sigma q}[T] \\ &\qquad -3\tau^*_{\Sigma q} f^*[\overline{X}] \cdot [\Sigma_q H'_q] + 12[\Sigma_q E_q] \cdot [\Sigma_q H'_q]) \\ &= f^*[\overline{X}] \cdot T - 12[\Sigma q]. \end{aligned} \tag{1.52}$$

Here the second equality follows from the commutativity of the lower fiber square in (0.2) and the third one from the *projection formula* and the following facts:

$$(1.53)\qquad \begin{cases} \tau_{\Sigma q*}[\Sigma_q E_q] = 0, \\ \tau_{\Sigma q*}[\Sigma_q H'_q] = 0, \\ [\Sigma_q E_q]\cdot[\Sigma_q H'_q] = -\Sigma_q [H'_q]^2, \\ \tau_{\Sigma q*}(\Sigma_q [H'_q]^2) = [\Sigma q]. \end{cases}$$

Since $\tau^*_{\Sigma q}[D] = [D'] + 3[\Sigma_q E_q]$, by Lemma 1.11, (ii),

$$\begin{aligned} D'\cdot T' &= (\tau^*_{\Sigma q}[D] - 3[\Sigma_q E_q])\cdot(\tau^*_{\Sigma q}[T] - 3\Sigma_q[H'_q]) \\ &= \tau^*_{\Sigma q}[D]\cdot\tau^*_{\Sigma q}[T] - 3(\tau^*_{\Sigma q}[D]\cdot\Sigma_q[H'_q]) - 3([E_{\Sigma q}]\cdot\tau^*_{\Sigma q}[T]) + 9[E_{\Sigma q}]\cdot\Sigma_q[H'_q] \end{aligned}$$

Hence, by the *projection formula* and (1.53),

$$(1.54)\qquad \tau_{\Sigma q*}(D'\cdot T') = D\cdot T - 9[\Sigma q]$$

Consequently, by (1.51), (1.50), (1.52), (1.54) and the fourth equality in (1.53),

$$\begin{aligned} f^*[\overline{T}] &= f^*[\overline{X}]\cdot T - 12[\Sigma q] - D\cdot T + 9[\Sigma q] - [\Sigma s] + 4[\Sigma q] \\ &= f^*[\overline{X}]\cdot T - D\cdot T - [\Sigma s] + [\Sigma q]. \end{aligned}$$ □

***Corollary 1.13***

$$(1.55)\qquad f_*[D]^3 = [\overline{X}]^2\cdot[\overline{D}] - 2[\overline{D}]^2 + 5\overline{X}\cdot\overline{T} - \overline{X}\cdot\overline{C} - [\Sigma\overline{s}] + 4[\Sigma\overline{q}].$$

*Proof*. — By Proposition 1.12,

$$D\cdot T = f^*[\overline{X}]\cdot T - f^*[\overline{T}] - [\Sigma s] + [\Sigma q].$$

Hence,

$$(1.56)\qquad \begin{aligned} f_*(D\cdot T) &= 3\overline{X}\cdot\overline{T} - \overline{X}\cdot\overline{T} - [\Sigma\overline{s}] + 4[\Sigma\overline{q}] \\ &= 2\overline{X}\cdot\overline{T} - [\Sigma\overline{s}] + 4[\Sigma\overline{q}] \end{aligned}$$

By (1.43), (1.47), (1.56) and Corollary 1.10,

$$\begin{aligned} f_*[D]^3 &= [\overline{X}]\cdot f_*[D]^2 - 2[\overline{D}]^2 + 2\overline{X}\cdot\overline{T} - [\Sigma\overline{s}] + 4[\Sigma\overline{q}] \\ &= [\overline{X}]^2\cdot\overline{D} + 3\overline{X}\cdot\overline{T} - \overline{X}\cdot\overline{C} - 2[\overline{D}]^2 + 2\overline{X}\cdot\overline{T} - [\Sigma\overline{s}] + 4[\Sigma\overline{q}] \\ &= [\overline{X}]^2\cdot\overline{D} + 5\overline{X}\cdot\overline{T} - \overline{X}\cdot\overline{C} - 2[\overline{D}]^2 - [\Sigma\overline{s}] + 4[\Sigma\overline{q}] \end{aligned}$$ □

Since

$$(1.57)\qquad \begin{aligned} s(\overline{J},\overline{X})_0 &= f_* s(J,X)_0 \\ &= f_*[D]^3 - f_* c_1(N_{C/X})\cap[C] - 3f_*(D\cdot C) \\ &= f_*[D]^3 - f_* c_1(N_{C/X})\cap[C]\ \ (\textit{cf.}\ \ (1.47)), \end{aligned}$$

what remains is to compute $f_*c_1(N_{C/X}) \cap [C]$ in order to know $s(\overline{J}, \overline{X})_0$. By the *adjunction formula*, the *double point formula* for $f : X \to Y$ and (1.46),

$$\begin{aligned} c_1(N_C X) \cap [C] &= -K_X \cdot C + [k_C] \\ &= (-f^*[\overline{X} + K_Y] + D) \cdot C + [k_C] \\ &= -f^*[K_Y] \cdot C - f^*[\overline{C}] + [k_C], \end{aligned}$$

where $K_Y$, $K_X$ and $k_C$ are the canonical divisors of $Y$, $X$ and $C$, respectively. Therefore, by the *projection formula* and the fact $C \simeq \overline{C}$,

$$f_*(c_1(N_C X) \cap [C]) = -K_Y \cdot \overline{C} - \overline{X} \cdot \overline{C} + [k_{\overline{C}}] \tag{1.58}$$

Substituting (1.55) and (1.58) into (1.57), we have

$$s(\overline{J}, \overline{X})_0 = [\overline{X}]^2 \cdot \overline{D} - 2[\overline{D}]^2 + 5\overline{X} \cdot \overline{T} + K_Y \cdot \overline{C} - [k_{\overline{C}}] - [\Sigma\overline{s}] + 4[\Sigma\overline{q}].$$

We collect the results concerning the Segre classes of $\overline{X}$ obtained up to this point in the following proposition:

***Proposition 1.14***. — *The Segre classes of the singular subscheme $\overline{J}$, defined by the Jacobian ideal, of an algebraic threefold $\overline{X}$ with ordinary singularities in the four dimensional projective space $Y = \mathbb{P}^4(\mathbb{C})$ are given as follows:*

$$\begin{cases} s(\overline{J}, \overline{X})_2 = 2[\overline{D}] \\ s(\overline{J}, \overline{X})_1 = -\overline{X} \cdot \overline{D} - 3\overline{T} + 2\overline{C} \\ s(\overline{J}, \overline{X})_0 = [\overline{X}]^2 \cdot \overline{D} - 2[\overline{D}]^2 + 5\overline{X} \cdot \overline{T} + K_Y \cdot \overline{C} - [k_{\overline{C}}] - [\Sigma\overline{s}] + 4[\Sigma\overline{q}]. \end{cases}$$

*Here $\overline{D}$, $\overline{T}$, $\overline{C}$, $\Sigma\overline{s}$ and $\Sigma\overline{q}$ are the* singular locus*,* triple point locus*,* cuspidal point locus*,* stationary point locus *and* quadruple point locus *of $\overline{X}$, respectively. $K_Y$ is the canonical divisor of the projective 4-space $Y$, and $k_{\overline{C}}$ that of $\overline{C}$.*

Note that the effect of the existence of quadruple points of $\overline{X}$ is only the term $4[\Sigma\overline{q}]$ in the expression of $s(\overline{J}, \overline{X})_0$.

Then, by Proposition 1.14,

$$\begin{cases} \deg s_2 = 2m \\ \deg s_1 = -nm + 2\gamma - 3t \\ \deg s_0 = n^2 m - 2m^2 + 5nt - 5\gamma - \#\Sigma\overline{s} - \deg k_{\overline{C}} + 4\#\Sigma\overline{q}, \end{cases}$$

where $n = \deg \overline{X}$ (the degree of $\overline{X}$ in $Y$), $m = \deg \overline{D}$, $t = \deg \overline{T}$, $\gamma = \deg \overline{C}$, and $\#\Sigma\overline{s}$ = the cardinal number of $\Sigma\overline{s}$, and $\#\Sigma\overline{q}$ = the cardinal number of $\Sigma\overline{q}$. Consequently, by (1.5), the *class* $c$ of $\overline{X}$ is given by

$$\begin{aligned} c = \deg[M_3] &= (n-1)^3 \deg \overline{X} - 3(n-1)^2 \deg s_2 - 3(n-1) \deg s_1 - \deg s_0 \\ &= (n-1)^3 n - (4n^2 - 9n - 2m + 6)m + (4n - 9)t - (6n - 11)\gamma \\ &\qquad + \#\Sigma\overline{s} + \deg k_{\overline{C}} - 4\#\Sigma\overline{q}. \end{aligned}$$

By this formula together with Proposition 1.2, we have the following:

***Theorem 1.15***. — *The Euler number $\chi(X)$ of the non-singular normalization $X$ of an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathrm{P}^4(\mathbb{C})$ is given by*

$$\int_X c_3 = \chi(X) = -n(n^3-5n^2+10n-10)+(4n^2-15n-2m+20)m-(4n-15)t \\ +(6n-15)\gamma - \#\Sigma\overline{s} - \deg k_{\overline{C}} + 4\#\Sigma\overline{q}.$$

*Here $n=\deg\overline{X}$, $m=\deg\overline{D}$, $t=\deg\overline{T}$ and $\gamma=\deg\overline{C}$ are the degrees of $\overline{X}$, the singular locus, the triple point locus and the cuspidal point locus, respectively. $\#\Sigma\overline{s}$ is the cardinal number of the stationary point locus $\Sigma\overline{s}$, $\deg k_{\overline{C}}$ the degree of the canonical divisor of the cuspidal point locus $\overline{C}$, and $\#\Sigma\overline{q}$ the cardinal number of the quadruple point locus $\Sigma\overline{q}$.*

## 2. The computation of $\int_X c_1^3$

By the *double point formula*, the canonical class $[K_X]$ of $X$ is given by

$$[K_X] = f^*[\overline{X}+K_Y]-[D] = f^*[(n-5)H]-[D]. \tag{2.1}$$

where $H$ is a hyperplane in $Y=\mathbb{P}^4(\mathbb{C})$. Therefore,

$$\begin{aligned}[K_X]^3 &= (f^*[(n-5)H]-[D])^3 \\ &= (f^*[(n-5)H])^3 - 3f^*[(n-5)H])^2\cdot D + 3f^*[(n-5)H]\cdot[D]^2-[D]^3 \\ &= f^*((n-5)^3[H]^3) - 3f^*[(n-5)^2[H]^2]\cdot D + 3f^*[(n-5)H]\cdot[D]^2-[D]^3.\end{aligned}$$

Hence, by the projection formula, Corollary 1.10 and Corollary 1.13,

$$\begin{aligned}f_*[K_X]^3 &= (n-5)^3[H]\cdot\overline{X} - 6[(n-5)^2[H]^2\cdot\overline{D}+3(n-5)[H])\cdot f_*[D]^2 - f_*[D]^3 \\ &= (n-5)^3[H]^3\cdot\overline{X} - 6(n-5)^2[H]^2\cdot\overline{D} + 3(n-5)[H]\cdot(\overline{X}\cdot\overline{D}+3[\overline{T}]-[\overline{C}]) \\ &\qquad - [\overline{X}]^2\cdot\overline{D} + 2[\overline{D}]^2 - 5\overline{X}\cdot\overline{T} + \overline{X}\cdot\overline{C} + [\Sigma\overline{s}] - 4[\Sigma\overline{q}].\end{aligned}$$

Consequently,

$$\begin{aligned}\deg\,[K_X]^3 &= \deg f_*[K_X]^3 \\ &= n(n-5)^3 - 6(n-5)^2m + 3(n-5)(nm+3t-\gamma) \\ &\qquad - n^2m + 2m^2 - 5nt + n\gamma + \#\Sigma\overline{s} - 4\#\Sigma\overline{q}.\end{aligned}$$

Since $\int_X c_1^3 = -\deg[K_X]^3$, we have the following proposition.

***Theorem 2.1***. — *The Chern number $\int_X c_1^3$ of the normalization $X$ of an algebraic threefold $\overline{X}$ in $\mathbb{P}^4(\mathbb{C})$ is given by the following formula:*

$$\int_X c_1^3 = -n(n-5)^3 + 6(n-5)^2m - 3(n-5)(nm+3t-\gamma) \\ + n^2m - 2m^2 + 5nt - n\gamma - \#\Sigma\overline{s} + 4\#\Sigma\overline{q}.$$

## 3. The computation of $\int_X c_1c_2$

By the Riemann-Roch theorem for a non-singular threefold, we have

$$\chi(X, K_X) = -\frac{1}{24}\int_X c_1c_2 \tag{3.1}$$

Therefore, if we can express $\chi(X, K_X)$ in terms of numerical characteristics of $\overline{X}$, then we can do the same for the Chern number $c_1c_2$. This is what we are going to do in the following. For a line bundle $\mathbf{F}$ on $Y$, we denote by $\mathcal{O}_Y(\mathbf{F})$ the sheaf of local holomorphic cross-sections of $\mathbf{F}$ over $Y$. Furthermore, we define the following sheaves:

$$\begin{aligned}
\mathcal{O}_Y(\mathbf{F}-\overline{D}) &:= \mathcal{O}_Y(\mathbf{F})\otimes_{\mathcal{O}_Y}\mathcal{I}_{\overline{D}},\\
\mathcal{O}_Y(\mathbf{F}-\overline{X}) &:= \mathcal{O}_Y(\mathbf{F})\otimes_{\mathcal{O}_Y}\mathcal{I}_{\overline{X}},\\
\mathcal{O}_Y(\mathbf{F})_{\overline{D}} &:= \mathcal{O}_Y(\mathbf{F})/\mathcal{O}_Y(\mathbf{F}-\overline{D}), \quad \text{and}\\
\mathcal{O}_Y(\mathbf{F}-\overline{D})_{\overline{X}} &:= \mathcal{O}_Y(\mathbf{F}-\overline{D})/\mathcal{O}_Y(\mathbf{F}-\overline{X}),
\end{aligned}$$

where $\mathcal{I}_{\overline{D}}$ and $\mathcal{I}_{\overline{X}}$ denote the ideal sheaves of $\overline{D}$ and $\overline{X}$ in $\mathcal{O}_Y$, respectively

***Lemma 3.1***. — *There exist the exact sequences of sheaves*

$$0 \longrightarrow \mathcal{O}_Y(\mathbf{F}-\overline{X}) \longrightarrow \mathcal{O}_Y(\mathbf{F}-\overline{D}) \longrightarrow \mathcal{O}_Y(\mathbf{F}-\overline{D})_{\overline{X}} \longrightarrow 0, \tag{3.2}$$

$$0 \longrightarrow \mathcal{O}_Y(\mathbf{F}-\overline{D}) \longrightarrow \mathcal{O}_Y(\mathbf{F}) \longrightarrow \mathcal{O}_Y(\mathbf{F})_{\overline{D}} \longrightarrow 0, \tag{3.3}$$

*over $Y$ and the isomorphism of sheaves*

$$f_*(\mathcal{O}_X(f^*\mathbf{F}-D)) \simeq \mathcal{O}_Y(\mathbf{F}-\overline{D})_{\overline{X}} \tag{3.4}$$

*Proof*. — The exactness of the sequences in (3.2) and (3.3) follows from the definitions of the sheaves $\mathcal{O}_Y(\mathbf{F}-\overline{D})_{\overline{X}}$ and $\mathcal{O}_Y(\mathbf{F})_{\overline{D}}$. In what follows we shall prove the existence of the isomorphism in (3.4). Let $p$ be a point of $\overline{X}$ and $(x, y, z, w)$ a local coordinate with center $p$ in $Y$ such that $\overline{X}$ is defined by one of the equations in (0.1) in the introduction in an open neighborhood of $p$ in $Y$. An element of $\mathcal{O}_Y(\mathbf{F}-\overline{D})_{\overline{X},p}$, the stalk of the sheaf $\mathcal{O}_Y(\mathbf{F}-\overline{D})_{\overline{X}}$ at $p$, can be represented by a local holomorphic function, say $\varphi(x, y, z, w)$, defined in an open neighborhood of $p$, which vanishes on $\overline{D}$. The map which assigns $f^*\varphi \in \mathcal{O}_X(f^*\mathbf{F}-D)_{f^{-1}(p)}$, the pull-back of $\varphi$ by $f$, to $\varphi \in \mathcal{O}_Y(\mathbf{F}-\overline{D})_{\overline{X},p}$ defines the homomorphism of sheaves in (3.4). We are going to show that it is an isomorphism in the cases where $p$ is a quadruple point, or stationary one. We can prove similarly in other cases.

(i) In the case where $p$ is an ordinary quadruple point: In this case $f^{-1}(p)$ of $X$, which we denote by $\{q_1, q_2, q_3, q_4\}$. In a neighborhood of each $q_i$ $(1 \leqslant i \leqslant 4)$, the map $f : X \to Y$ is described as

$$\begin{aligned}
f_1 &: (u_1, v_1, t_1) \to (0, u_1, v_1, t_1) = (x, y, z, w),\\
f_2 &: (u_2, v_2, t_2) \to (u_2, 0, v_2, t_2) = (x, y, z, w),\\
f_3 &: (u_3, v_3, t_3) \to (u_3, v_3, 0, t_3) = (x, y, z, w),\\
f_4 &: (u_4, v_4, t_4) \to (u_4, v_4, t_4, 0) = (x, y, z, w)
\end{aligned}$$

where $(u_i, v_i, t_i)$ $(1 \leqslant i \leqslant 4)$ is a complex analytic local coordinate with center $q_i$. Since $\mathcal{I}_{\overline{D},p}$ is generated by $xyz, xyw, xzw, yzw$, any element of $\mathcal{O}_Y(\mathbf{F} - \overline{D})_{\overline{X},p}$ is represented by a local holomorphic function $\varphi$ at $p$, which has the form

$$\varphi = yzw\,\varphi_1 + xzw\,\varphi_2 + xyw\,\varphi_3 + xyz\,\varphi_4$$

where $\varphi_i{}'s$ $(1 \leqslant i \leqslant 4)$ are local holomorphic functions at $p$. Hence the pull-back of $\varphi$ by $f$, which consists of those of $\varphi$ by $f_i{}'s$ $(1 \leqslant i \leqslant 4)$, are given by

$$\begin{aligned}
(f_1^*\varphi)(u_1, v_1, t_1) &= u_1 v_1 t_1\,\varphi_1(0, u_1, v_1, t_1),\\
(f_2^*\varphi)(u_2, v_2, t_2) &= u_2 v_2 t_2\,\varphi_2(u_2, 0, v_2, t_2),\\
(f_3^*\varphi)(u_3, v_3, t_3) &= u_3 v_3 t_3\,\varphi_3(u_3, v_3, 0, t_3),\\
(f_4^*\varphi)(u_4, v_4, t_4) &= u_4 v_4 t_4\,\varphi_4(u_4, v_4, t_4, 0).
\end{aligned}$$

Therefore, if $f^*\varphi = 0$ in $f_*(\mathcal{O}_X(f^*\mathbf{F} - D))_p = \mathcal{O}_X(f^*\mathbf{F} - D)_{f^{-1}(p)}$, $\varphi_i$ $(1 \leqslant i \leqslant 4)$ must have the forms $\varphi_1 = x\psi_1$, $\varphi_2 = y\psi_2$, $\varphi_3 = z\psi_3$ and $\varphi_4 = w\psi_4$ with $\psi_i$ $(1 \leqslant i \leqslant 4)$ local holomorphic functions at $p$. From this it follows that if $f^*\varphi = 0$ in $f_*(\mathcal{O}_X(f^*\mathbf{F} - D))_p$, then $\varphi = 0$ in $\mathcal{O}_Y(\mathbf{F} - \overline{D})_{\overline{X},p}$. Hence the homomorphism

$$\mathcal{O}_Y(\mathbf{F} - \overline{D})_{\overline{X},p} \longrightarrow f_*(\mathcal{O}_X(f^*\mathbf{F} - D))_p$$

is injective. Next we show that the homomorphism is surjective. Any element $\xi$ of $f_*(\mathcal{O}_X(f^*\mathbf{F} - D))_p$ is represented by a quadruplet of local holomorphic functions $\xi_i(u_i, v_i.t_i)$ $(1 \leqslant i \leqslant 4)$ defined at $q_i$ of the forms

$$\xi_i = u_i v_i t_i\,\eta_i \qquad (1 \leqslant i \leqslant 4)$$

where each $\eta_i$ $(1 \leqslant i \leqslant 4)$ is a local holomorphic function at $q_i$. Therefore, if we put

$$\varphi = yzw\,\eta_1(y, z, w) + xzw\,\eta_2(x, z, w) + xyw\,\eta_3(x, y, w) + xyz\,\eta_4(x, y, z),$$

then we have $\xi = f^*\varphi$ where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$, that is, the homomorphism

$$\mathcal{O}_Y(\mathbf{F} - \overline{D})_{\overline{X},p} \longrightarrow f_*(\mathcal{O}_X(f^*\mathbf{F} - D))_p$$

is surjective. Therefore, the homomorphism

$$\mathcal{O}_Y(\mathbf{F} - \overline{D})_{\overline{X},p} \longrightarrow f_*(\mathcal{O}_X(f^*\mathbf{F} - D))_p$$

is an isomorphism.

(ii) In the case where $p$ is a stationary point: In this case, $f^{-1}(p)$ is two points of $X$, which we denote by $\{q_1, q_2\}$. In a neighborhood of each $q_i$ $(1 \leqslant i \leqslant 2)$,the map $f : X \to Y$ is described as

$$\begin{aligned}
f_1 &: (u_1, v_1, t_1) \to (u_1^2, v_1, u_1 v_1, t_1) = (x, y, z, w),\\
f_2 &: (u_2, v_2, t_2) \to (u_2, v_2, t_2, 0) \qquad = (x, y, z, w),
\end{aligned}$$

where $(u_i, v_i, t_i)$ $(1 \leqslant i \leqslant 2)$ is a complex analytic local coordinate with center $q_i$. Since $\mathcal{I}_{\overline{D},p}$ is generated by $wy, zw, xy^2 - z^2$, any element of $\mathcal{O}_X(f^*\mathbf{F} - \overline{D})_{\overline{X},p}$ is represented by a local holomorphic function $\varphi$ at $p$, which has the form

$$\varphi = wy\,\varphi_1 + zw\,\varphi_2 + (xy^2 - z^2)\,\varphi_3,$$

where $\varphi_i$'s $(1 \leqslant i \leqslant 3)$ are local holomorphic functions at $p$. Hence the pull-backs of $\varphi$ by $f_i$'s $(1 \leqslant i \leqslant 2)$, are given by

$$\begin{aligned}(f_1^*\varphi)(u_1,v_1,t_1) &= t_1\{v_1\,\varphi_1(u_1^2,v_1,u_1v_1,t_1)+u_1v_1\,\varphi_2(u_1^2,v_1,u_1v_1,t_1)\}\\ &= t_1 f_1^*(y\varphi_1+z\varphi_2)(u_1^2,v_1,u_1v_1,t_1),\\ (f_2^*\varphi)(u_2,v_2,t_2) &= (u_2v_2^2-t_2^2)\varphi_3(u_2,v_2,t_2,0).\end{aligned}$$

Therefore, if $f^*\varphi = 0$ in $f_*\mathcal{O}_X(f^*\mathbf{F}-D))_p = \mathcal{O}_X(f^*\mathbf{F}-D)_{f^{-1}(p)}$, $y\varphi_1+z\varphi_2$ and $\varphi_3$ must have the forms

$$y\varphi_1+z\varphi_2 = (xy^2-z^2)\psi_1 \quad \text{and} \quad \varphi_3 = w\psi_2,$$

where $\psi_1,\psi_2$ are local holomorphic functions at $p$. From this the injectivity of the homomorphism $\mathcal{O}_Y(\mathbf{F}-D)_{\overline{X},p} \to f_*(\mathcal{O}_X(f^*\mathbf{F}-D))_p$ follows. Next we show the surjectivity of this homomorphism. First, we should note that $\mathcal{I}_{D,q_1}$ and $\mathcal{I}_{D,q_2}$ are generated by $v_1t_1$ and $u_2v_2^2-t_2^2$, respectively. Hence any element $\xi$ of $f_*(\mathcal{O}_X(f^*\mathbf{F}-D))_p$ is represented by a couple of local holomorphic functions $\xi_i(u_i,v_i,t_i)$ $(1 \leqslant i \leqslant 2)$ each of which is defined at $q_i$ and has the form as follows:

$$\begin{aligned}\xi_1 &= v_1t_1\,\eta_1,\\ \xi_2 &= (u_2v_2^2-t_2^2)\,\eta_2\end{aligned}$$

where each $\eta_i$ $(1 \leqslant i \leqslant 2)$ is a local holomorphic function at $q_i$. We represent $\eta_1$ as

$$\eta_1(u_1,v_1,t_1) = \eta_{11}(u_1^2,v_1,t_1)+u_1\,\eta_{12}(u_1^2,v_1,t_1)$$

where $\eta_{11}$ and $\eta_{12}$ are local holomorphic functions at $q_1$. Then, if we put

$$\varphi = yw\,\eta_{11}(x,y,w)+zw\,\eta_{12}(x,y,w)+(xy^2-z^2)\,\eta_2(x,y,z)$$

we have $f_1^*\varphi = \xi_1$ and $f_2^*\varphi = \xi_2$. Therefore, we conclude that the homomorphism $\mathcal{O}_Y(\mathbf{F}-\overline{D})_{\overline{X},p} \to f_*(\mathcal{O}_X(f^*\mathbf{F}-D))_p$ is surjective. □

We apply Lemma 3.1 to $\mathbf{F} = [(n-5)H]$, the line bundle determined by the divisor $(n-5)H$. Then, by (3.4) we have

$$\chi(X,\mathcal{O}_X(f^*[(n-5)H]-D)) \simeq \chi(\overline{X},\mathcal{O}_Y([(n-5)H]-\overline{D})_{\overline{X}}) \tag{3.5}$$

By (3.2),

(3.6)
$$\begin{aligned}\chi(\overline{X},\mathcal{O}_Y([(n-5)H]-\overline{D})_{\overline{X}}) &= \chi(Y,\mathcal{O}_Y([(n-5)H]-\overline{D}))\\ &\qquad -\chi(Y,\mathcal{O}_Y([(n-5)H]-\overline{X}))\\ &= \chi(Y,\mathcal{O}_Y([(n-5)H]-\overline{D}))-\chi(Y,\mathcal{O}_Y([(-5)H]))\\ &= \chi(Y,\mathcal{O}_Y([(n-5)H]-\overline{D}))-1 \quad (Y=\mathbb{P}^4(\mathbb{C}))\end{aligned}$$

Here the second equality follows from the fact that $\overline{X}$ is linearly equivalent to $nH$ and the third one from the fact $[(-5)H] = [K_Y]$, the canonical divisor class of $Y$.

By (3.3),

$$\chi(Y, \mathcal{O}_Y([(n-5)H] - \overline{D}) = \chi(Y, \mathcal{O}_Y([(n-5)H])) - \chi(\overline{D}, \mathcal{O}_Y([(n-5)H])_{\overline{D}}))$$

$$= \frac{1}{24}(n-4)(n-3)(n-2)(n-1) - \chi(\overline{D}, \mathcal{O}_{\overline{D}}(n-5)) \tag{3.7}$$

where $\mathcal{O}_{\overline{D}}(n-5)$ denotes $\mathcal{O}_{\overline{D}} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y([(n-5)H])$. Consequently, by (3.1), (2.1), (3.5), (3.6) and (3.7), we have the following:

***Theorem 3.2***

$$\begin{aligned}\int_X c_1c_2 &= -24\chi(X, K_X)\\ &= -24\,\chi(Y, \mathcal{O}_Y([(n-5)H] - \overline{D})) + 24\\ &= -(n-4)(n-3)(n-2)(n-1) + 24\,\chi(\overline{D}, \mathcal{O}_{\overline{D}}(n-5)) + 24.\end{aligned}$$

Note that, for a given $\overline{D}$, $\chi(\overline{D}, \mathcal{O}_{\overline{D}}(n-5))$ can be calculated by computing the Hilbert polynomial of the module $\mathcal{O}_{\overline{D}}(n-5)$ on $\mathbb{P}^5(\mathbb{C})$.

## 4. An application

The Todd class $td(X)$ and the Chern character $ch(T_X)$ of the tangent bundle $T_X$ of $X$ are given by

$$\begin{aligned}td(X) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2,\\ ch(T_X) &= 3 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3).\end{aligned}$$

By the Riemann-Roch theorem,

$$\chi(X, \mathcal{T}_X) = \int_X ch(T_X)\cdot td(X).$$

Therefore, by Theorem 1.15, Theorem 2.1 and Theorem 3.2, we have a numerical formula which gives the Euler-Poincaré characteristic $\chi(X, \mathcal{T}_X) = \Sigma_{i=0}^3(-1)^i \dim H^i(X, \mathcal{T}_X)$ with coefficients in the sheaf $\mathcal{T}_X = \mathcal{O}_X(T_X)$ of holomorphic vector fields on $X$.

***Theorem 4.1***

$$\begin{aligned}\chi(X, \mathcal{T}_X) &= \frac{1}{2}\int_X c_1^3 - \frac{19}{24}\int_X c_1c_2 + \frac{1}{2}\int_X c_3\\ &= -5\binom{n}{4} + 5\binom{n}{3} - 20\binom{n}{2} + 15\binom{n}{1}\\ &\quad +(4n^2 - 30n - 2m + 85)m - 2(2n-15)t + (4n-15)\gamma\\ &\quad -\#\Sigma\overline{s} + 4\#\Sigma\overline{q} - \frac{1}{2}\deg k_{\overline{C}} - 19\chi(\overline{D}, \mathcal{O}_{\overline{D}}(n-5)).\end{aligned}$$

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