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ALGORITHMS AND MODULI SPACES FOR DIFFERENTIAL EQUATIONS

by

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Abstract. — This article discusses second and third order differential operators. We will define standard operators, and prove that every differential operator with finite differential Galois group is a so-called pullback of some standard operator. We will also give an algorithm concerning certain field extensions, associated with algebraic solutions of a Riccati equation.

***Résumé* (Algorithmes et espaces modulaires pour les équations différentielles)**

Cet article s'intéresse aux opérateurs différentiels de deuxième et troisième ordre. Nous introduisons une notion d'opérateur standard, et montrons que tout opérateur différentiel de groupe de Galois différentiel fini est image inverse d'un opérateur standard. Nous donnons aussi un algorithme concernant certaines extensions de corps, associées à des solutions algébriques d'une équation de Riccati.

1. Field extensions for Riccati solutions

In this section we consider second order linear differential equations of the form $L: y'' = ry,\ r \in k(x)$. Here $k(x)$ is a *differential field* of characteristic zero, with derivation $\frac{d}{dx}$. The field of constants k is not supposed to be algebraically closed. We will denote its algebraic closure by $\bar{k}$. The differential Galois theory gives us an extension $\bar{k}(x) \subset K$, with K the so called *Picard-Vessiot extension*, which is the minimal differential field extension of $\overline{k}(x)$ which contains a basis $\{y_1, y_2\}$ (over $\bar{k}$) of solutions of L. The solution space $\bar{k}\langle y_1, y_2\rangle := \overline{k}y_1 + \overline{k}y_2 \subset K$ will be denoted V.

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The automorphisms of $K/\bar{k}(x)$ which commute with the differentiation constitute the *differential Galois group* G.

An interesting class of solutions are the so called *Liouvillian solutions*. These are solutions which lie in a Liouvillian extension of $\bar{k}(x)$, which roughly means they can be written down quite explicitly. For a precise definition of a (generalized) Liouvillian extension, see [**Kap76**, p. 39]. Related to this is the *Riccati equation*, denoted R_L, which is an equation depending on L with as solutions elements of the form $u = \frac{y'}{y}$, with y a solution of L. In our case it is the equation $u^2 + u' = r$. We have the following facts (see [**vdPS03**, p. 35,104]).

Fact 1.1. — *$u \in K$ is a solution of $R_L \iff u = \frac{y'}{y}$, for some $y \in V$.*

Fact 1.2. — *$u = \frac{y'}{y}$ is a solution of R_L, algebraic of degree m over $\bar{k}(x) \iff$ The stabilisor in G of the line $\bar{k} \cdot y$ is a subgroup of index m.*

The next fact is concerned with Liouvillian solutions of L.

Fact 1.3. — *L has a Liouvillian solution $\iff R_L$ has an algebraic solution.*

Let u be an algebraic solution of R_L of minimal degree over $\overline{k}(x)$. We define the field k' to be the minimal field in $\overline{k}$ such that the coefficients of the minimal polynomial of u over $\bar{k}(x)$ are elements of $k'(x)$. We want to determine k' as explicit as possible. In [**HvdP95**] bounds on the degree $[k' : k]$ are given, depending on the differential Galois group G of L. We consider G as a subgroup of $\mathrm{GL}_2(\bar{k})$ by its action on y_1, y_2. It is known that G is an algebraic subgroup of $\mathrm{GL}_2(\bar{k})$. Note that changing the basis $\{y_1, y_2\}$ changes G by conjugation. Because in our equation L there is no first order term, we actually have that G lies in $\mathrm{SL}_2(\bar{k})$, see [**Kap76**, p. 41]. We have the following lemma, which is essentially Theorem 5.4 of [**HvdP95**].

Lemma 1.4. — *There are only three cases, with respect to G, for which k' can be different from k. These are (on an appropriate basis):*

(1) $G \subset \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \bar{k}^* \right\}, \#G > 2$, *a subgroup of a torus.*

(2) $G = D_2^{\mathrm{SL}_2}$, *a group of order* 8, *with generators* $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(3) $G = A_4^{\mathrm{SL}_2}$, *a group of order* 24.

We remark that in [**HvdP95**], the group $D_2^{\mathrm{SL}_2}$ is mistakenly denoted by D_4. We have $D_4 \neq D_2^{\mathrm{SL}_2}$, and in fact $D_2^{\mathrm{SL}_2} \cong Q_8$, where Q_8 denotes the quaternion subgroup $\{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}^*$. The notations $D_2^{\mathrm{SL}_2}$ and $A_4^{\mathrm{SL}_2}$ can be explained as follows. Using the natural homomorphism $\mathrm{SL}_2 \to \mathrm{PSL}_2$, these groups are the inverse image of $D_2 \subset \mathrm{PSL}_2$ and $A_4 \subset \mathrm{PSL}_2$ respectively. We will treat these three cases separately.

1.1. Subgroups of a torus. — In this section we consider case (1) of Lemma 1.4. There are exactly two G-invariant lines in V. These correspond to the two solutions of R_L in $\bar{k}(x)$. Such solutions are called *rational*.

For the next lemma we need to introduce the second *symmetric power* of a given differential equation. This is the differential equation with as solutions, all products of two solutions of the given equation. For example take $L : y'' = ry$, with as basis of solutions $\{y_1, y_2\}$. Then the second symmetric power of L, denoted $\mathrm{Sym}(L, 2)$ is the equation $y''' - 4ry' - 2r'y = 0$. It has $\{y_1^2, y_1y_2, y_2^2\}$ as a basis of solutions. Indeed, $\{y_1^2, y_1y_2, y_2^2\}$ are linearly independent over $\overline{k}$ (compare [**SU93**, Lemma 3.5]). In a similar way one defines higher order symmetric powers $\mathrm{Sym}(L, n)$ (see [**vdPS03**, Definition 2.24]), which we will use later on. We note that $\mathrm{Sym}(L, n)$ can have order smaller than $n + 1$. In the proof of the next lemma, we will also use that there is an action of $\mathrm{Gal}(\bar{k}/k)$ on K, which induces an action on V. It acts in the standard way on $\bar{k}(x)$. For details see [**HvdP95**].

Lemma 1.5. — *Assume we are in case* (1) *of Lemma 1.4. Then* $\mathrm{Sym}(L, 2)$ *has (up to constants) a unique non-zero solution* $H \in k(x)$*. If one of the two rational solutions of* R *does not lie in* $k(x)$*, then the rational solutions of* R *are* $\frac{H'}{2H} \pm cH^{-1}$*, for some* $c \in \overline{k} \setminus k, c^2 \in k$*.*

Proof. — For the basis $\{y_1, y_2\}$ for which the representation of G in SL_2 is as in 1. we have that y_1y_2 is G invariant, so $y_1y_2 \in \bar{k}(x)$. It is easily seen that up to constants, this is the only G-invariant solution of $\mathrm{Sym}(L, 2)$. For $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have that $\sigma(y_1y_2)$ is another rational solution of the symmetric square, so it must be a multiple of y_1y_2. Therefore we have a $\mathrm{Gal}(\bar{k}/k)$-invariant line, and thus by Hilbert theorem 90 an invariant point on this line. After multiplying y_1 by a constant, we may suppose $H := y_1y_2 \in k(x)$. Then $\frac{H'}{H} = \frac{y_1'}{y_1} + \frac{y_2'}{y_2}$. The rational solutions of R are $\frac{y_1'}{y_1}$ and $\frac{y_2'}{y_2}$, and since $\mathrm{Gal}(\bar{k}/k)$ acts on the set of solutions of R, each one is fixed by a subgroup of $\mathrm{Gal}(\bar{k}/k)$ of index ≤ 2. Now assume this index is 2, then we can write $\frac{y_1'}{y_1} =: u =: u_0 + du_1,\ u_0, u_1 \in k(x), d^2 \in k, d \notin k$, and then $\frac{y_2'}{y_2} = u_0 - du_1$, so $\frac{H'}{H} = 2u_0$. From $u' + u^2 = r \in k(x)$ one deduces that $2u_0 = -\frac{u_1'}{u_1}$, so u_1 must be $\lambda H^{-1}, \lambda \in k^*$. Therefore we can take $c = d\lambda$, and clearly $\frac{y_2'}{y_2} = \frac{H'}{2H} - cH^{-1}$. □

We note that this gives a way to find in case (1) the rational solutions of the Riccati equation. Indeed H can be found (for example using Maple), and c can be calculated by substituting $\frac{H'}{2H} + cH^{-1}$ into the Riccati equation.

1.2. Klein's theorem. — In the remaining two cases of Lemma 1.4, the differential Galois groups are finite. This implies that the differential Galois group equals the ordinary Galois group. An important tool in studying these cases is *Klein's Theorem*. We present a version of it suggested by F. Beukers. For a different approach we refer to [**BD79**].

It will be convenient to use *differential operators*. These are elements of the skew polynomial ring $\bar{k}(x)[\partial_x]$. The multiplication is defined by $\partial_x x = x\partial_x + 1$. We will identify the linear differential equation $\Sigma_i a_i y^{(i)} = 0$ with the differential operator $\Sigma_i a_i \partial_x^i$.

We recall from [**HvdP95**] the following easy lemma.

Lemma 1.6. — *The $\bar{k}$-algebra homomorphisms $\phi : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$ are given by $\phi(t) = a$ and $\phi(\partial_t) = \frac{1}{a'}\partial_x + b$ with $a \in \bar{k}(x) \setminus \bar{k}$; $a' := \frac{d}{dx}a$ and $b \in \bar{k}(x)$.*

Notation 1.7

– For $F \in \bar{k}(x)\backslash\bar{k}$ we define the $\bar{k}$-homomorphism $\phi_F : \bar{k}(t) \to \bar{k}(x)$, by $\phi_F(t) = F$.
– Let ϕ be an injective homomorphism $\phi : \bar{k}(t) \to \bar{k}(x)$. Then we also write ϕ for the extension of ϕ to the homomorphism of differential operators $\phi : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$, defined by $\phi(\partial_t) = \frac{1}{\phi(t)'}\partial_x$.
– For $F \in \overline{k}(x) \setminus \overline{k}, b \in \bar{k}(x)$, we define $\phi_{F,b} : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$ by $\phi_{F,b}(t) = F$, $\phi_{F,b}(\partial_t) = \frac{1}{F'}(\partial_x + b)$.
– We will call an automorphism of $\overline{k}(t)[\partial_t]$, given by $t \mapsto t, \partial_t \mapsto \partial_t + b$ a *shift*.
– For a differential operator L we define $\mathrm{Aut}(L)$ to be the group $\{\psi \in \mathrm{Aut}_{\bar{k}}\, \bar{k}(t) \mid \mathrm{Norm}(\psi(L)) = L\}$.

First we will discuss the process of normalization. A second order differential operator $L := a_2\partial^2 + a_1\partial + a_0$ is said to be in *normal form* if $a_2 = 1$ and $a_1 = 0$. We can put L into normal form, $\mathrm{Norm}(L)$, by dividing L by a_2, and then applying the shift $\partial \mapsto \partial - \frac{a_1}{2a_2}$. Note that normalization transforms the old solution space V to $f \cdot V$, with $f' = \frac{a_1}{2a_2}f$. The operator remains defined over $k(x)$, but the associated Picard-Vessiot extension K changes if $f \notin K$.

Klein's theorem is concerned with differential operators $L := \partial_x^2 - r$ with finite non-cyclic differential Galois group $G \subset \mathrm{SL}_2(\bar{k})$. If we again use the notation H^{SL_2} for the inverse image in SL_2 of a group $H \subset \mathrm{PSL}_2$, the possibilities for such G are (up to conjugation): $\{D_n^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. In [**BD79**] we find for each such group G a *standard operator*, denoted St_G, which is in normal form, and has differential Galois group G. These are:

$$St_{D_n^{\mathrm{SL}_2}} = \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{3}{16}\frac{1}{(t-1)^2} - \frac{n^2+2}{8n^2}\frac{1}{t(t-1)},$$
$$St_{A_4^{\mathrm{SL}_2}} = \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{3}{16}\frac{1}{t(t-1)},$$
$$St_{S_4^{\mathrm{SL}_2}} = \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{101}{576}\frac{1}{t(t-1)},$$
$$St_{A_5^{\mathrm{SL}_2}} = \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{611}{3600}\frac{1}{t(t-1)}.$$

The so-called *local exponents* of these standard equations are given by the following table.

	0	1	∞
$St_{D_n^{\mathrm{SL}_2}}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{4},\frac{3}{4}$	$-\frac{n+1}{2n},-\frac{n-1}{2n}$
$St_{A_4^{\mathrm{SL}_2}}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{3},\frac{2}{3}$	$-\frac{1}{3},-\frac{2}{3}$
$St_{S_4^{\mathrm{SL}_2}}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{3},\frac{2}{3}$	$-\frac{3}{8},-\frac{5}{8}$
$St_{A_5^{\mathrm{SL}_2}}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{3},\frac{2}{3}$	$-\frac{2}{5},-\frac{3}{5}$

In the proof of Klein's Theorem we will need the following lemma.

Lemma 1.8. — *Let L be a monic second order differential operator over $k(x)$, with finite differential Galois group G, and Picard-Vessiot extension K. Let $\{y_1, y_2\}$ be a basis of solutions of L, and write $s := \frac{y_1}{y_2}$.*

(1) *Normalizing L does not change the field $K^p := \overline{k}(x)(s) \subset K$.*
(2) *Let $L_1 \in \overline{k}(x)[\partial_x]$ be a monic differential operator, which also has a basis of solutions in K of the form $\{sy, y\}$. Then L_1 can be obtained from L by the shift $\partial_x \mapsto \partial_x - (\frac{y}{y_1})'/(\frac{y}{y_1})$.*

If moreover G is non-cyclic and $G \subset \mathrm{SL}_2(\overline{k})$, then also the following statements hold.

(3) *$K^p = K^{\pm I}$, the fixed field of $-I$ in K.*
(4) *$K = K^p(\sqrt{s'})$.*
(5) *$\overline{k}(s)$ is G-invariant and $\exists\ t \in \overline{k}(x)$ such that $\overline{k}(s)^G = \overline{k}(t)$.*

Proof

(1) This follows immediately from the fact that the normalization of L has a basis of solutions $\{fy_1, fy_2\}$ (for some f with $\frac{f'}{f} \in \overline{k}(x)$).

(2) The monic differential operator $\phi_{x,-(\frac{y}{y_1})'/(\frac{y}{y_1})}$ clearly has $\{sy, y\}$ as a basis of solutions, and therefore is equal to L_1.

(3) Since $\bar{k}(x) \subset \overline{k}(x)(y_1, y_2)$ is a finite extension, we have $y_1', y_2' \in \overline{k}(x)(y_1, y_2)$, so $K = \overline{k}(x)(y_1, y_2)$. Because K^p is algebraic over $\bar{k}(x)$ the derivation on K induces a derivation on K^p. So $(\frac{y_1}{y_2})' = \frac{d}{y_2^2} \in \bar{k}(x)(\frac{y_1}{y_2})$, where $d = y_1'y_2 - y_2'y_1$. It is easily seen that $d' = 0$, and $d \neq 0$, so $d \in \overline{k}^*$. We find that $y_2^2 \in K^p$ and for a similar reason also $y_1^2 \in K^p$. So the only elements in G that fix $\bar{k}(x)(\frac{y_1}{y_2})$ are $\pm I$. By Galois correspondence K^p is the fixed field of $\{\pm I\}$.

(4) We have $K = K^p(y_2)$, and $y_2^2 = \frac{d}{s'}$, so $K = K^p(\sqrt{s'})$.

(5) From the G-action on $\overline{k}\langle y_1, y_2\rangle$ one immediately finds that $\overline{k}(s)$ is G-invariant. Since $\bar{k}(s)$ is a purely transcendental extension of $\bar{k}$ we get by Lüroth's theorem that the fixed field of G is also purely transcendental. So we can write $\overline{k}(s)^G = \overline{k}(t)$, and because $t \in K$ is invariant under G, we get $t \in \overline{k}(x)$. □

Theorem 1.9 (Klein). — *Let L be a second order differential operator over $k(x)$ in normal form, with differential Galois group $G \in \{D_n^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. There exists an element $F \in \bar{k}(x)$ such that $\mathrm{Norm}(\phi_F(St_G)) = L$. Moreover $\phi_F : \bar{k}(t) \to \bar{k}(x)$ is unique up to composition with an automorphism $\psi \in \mathrm{Aut}(St_G)$.*

Proof. — We will use the notation of the above lemma. Write $G^p := G/\{\pm I\}$ for $\mathrm{Gal}(K^p/\overline{k}(x)) = \mathrm{Gal}(\overline{k}(s)/\overline{k}(t))$. The field extension $\bar{k}(t) \subset \bar{k}(s)$ corresponds to a covering of $\mathbb{P}^1_t$ by $\mathbb{P}^1_s$, with Galois group G^p. It is known that for the groups $G^p \subset \mathrm{PGL}(2)$ considered here, the map $\mathbb{P}^1_s \to \mathbb{P}^1_t$ is ramified above three points. If necessary replacing t by the image of t under a Möbius-transformation, these three points are $0, 1, \infty$. The list of ramification indices is (up to permutations of $0, 1, \infty$):

G^p	e_0	e_1	e_∞
D_n	2	2	n
A_4	2	3	3
S_4	2	3	4
A_5	2	3	5

We choose t such that we get precisely the above ramification indices for $0, 1, \infty$.

We now want to construct a differential operator in $\overline{k}(t)[\partial_t]$, with differential Galois group G, and with Picard-Vessiot extension some field K_1, such that $K_1^p = \overline{k}(s)$. As suggested by F. Beukers one takes $K_1 := \bar{k}(s, \sqrt{s'})$, where $'$ denotes the unique extension of the derivation $\frac{d}{dt}$ on $\overline{k}(t)$. We write V for the solution space of L in K, and we define $V_1 := \overline{k}\langle \frac{s}{\sqrt{s'}}, \frac{1}{\sqrt{s'}} \rangle \subset K_1$.

Lemma 1.10

(1) *The field K_1 does not depend on the choice of t.*
(2) *K_1 is a Galois extension of $\overline{k}(t)$, and we can identify $\mathrm{Gal}(K_1/\overline{k}(t))$ with G. The vector space V_1 is G-invariant, and isomorphic to V as a G-module.*
(3) *V_1 does not depend on the choice of s.*

Proof

(1) For $t_1 = \frac{at+b}{ct+d}$, $ad - bc = 1$, we have $\frac{ds}{dt} = \frac{ds}{dt_1}\frac{dt_1}{dt} = \frac{ds}{dt_1}\frac{1}{(ct+d)^2}$, so

$$\overline{k}\Big(s, \sqrt{\frac{ds}{dt}}\Big) = \overline{k}\Big(s, \frac{1}{ct+d}\sqrt{\frac{ds}{dt_1}}\Big) = \overline{k}\Big(s, \sqrt{\frac{ds}{dt_1}}\Big)$$

(2) We will show that K_1 is the splitting field over $\bar{k}(t)$ of P_1P_2, where P_1 is the minimal polynomial of s over $\overline{k}(t)$, and P_2 is the minimal polynomial of $\sqrt{s'}$ over $\overline{k}(t)$. By construction the extension $\bar{k}(t) \subset \bar{k}(s)$ is Galois, so all zeroes of P_1 lie in $\bar{k}(s)$. The only thing that remains to be shown is that all roots of P_2 lie in K_1. This minimal polynomial is a factor of $\prod_{\sigma \in G^p}(T^2 - \sigma(s'))$, and $\sigma(s') = \sigma(s)' = \frac{s'}{(cs+d)^2}$, for $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. So all zeroes of the minimal polynomial of $\sqrt{s'}$ are of the form $\frac{\pm\sqrt{s'}}{cs+d}$, and therefore lie in K_1.

We can define an isomorphism $V \to V_1$, by $y_1 \mapsto \frac{s}{\sqrt{s'}}, y_2 \mapsto \frac{1}{\sqrt{s'}}$. This induces a G-action on V_1. A direct computation shows that this action extends to a G-action on K_1, extending the existing G-action on $\bar{k}(s)$. The invariant field in K_1 under this action is $\overline{k}(t)$, as can be seen from the inclusions

$$\overline{k}(t) \overset{G^p}{\subset} \bar{k}(s) \subset K_1.$$

We also conclude from this that $G = \mathrm{Gal}(K_1/\overline{k}(t))$.

(3) We have $(\frac{as+b}{cs+d})' = s' \frac{ad-bc}{(cs+d)^2}$, and it immediately follows that V_1 does not change if we replace s by $\frac{as+b}{cs+d}$, $ad - bc = 1$. Note that changing t in general does change V_1. □

We continue the proof of Klein's Theorem. Since the 2-dimensional vector space V_1 is invariant under the Galois group of K_1 over $\overline{k}(t)$, it is the solution space of some monic second order differential operator M_G over $\bar{k}(t)$. Clearly K_1 is the corresponding Picard-Vessiot extension. Further $s = \big(\frac{s}{\sqrt{s'}}\big)/\big(\frac{1}{\sqrt{s'}}\big)$, so $\bar{k}(s)$ is the corresponding subfield.

Claim: $M_G = St_G$.

We note that a monic second order differential operator with three fixed singular points is completely determined by its *local exponents* (see [**vdPU00**, Chapter 5]). The singular points of the differential operator M_G are $\{0, 1, \infty\}$. So to prove the claim, it suffices to show that the local exponents of M_G and St_G coincide for every singular point. We can calculate the local exponents of M_G. We give the calculation for $t = 0$. After applying a Möbius-transformation to s (which is allowed), we can suppose that s is a local parameter of a point above $0 \in \mathbb{P}^1_t$. So we get an embedding of complete local rings $\bar{k}[[t]] \subset \bar{k}[[s]]$, and we have $t = s^{e_0} + *s^{e_0+1} + \cdots$, where again e_0 is the ramification index of the embedding $\bar{k}(t) \subset \bar{k}(s)$ at $t = 0$. We find $s = t^{\frac{1}{e_0}} + \cdots$, so the power series expansion of the basis of solutions of M_G looks like $\frac{1}{\sqrt{s'}} = t^{\frac{1}{2}-\frac{1}{2e_0}} + \cdots$, and $\frac{s}{\sqrt{s'}} = t^{\frac{1}{2}+\frac{1}{2e_0}} + \cdots$. Therefore the local exponents at $t = 0$ are $\frac{1}{2} \pm \frac{1}{2e_0}$. In the same way we find the local exponents at $t = 1, \infty$ to be $\frac{1}{2} \pm \frac{1}{2e_1}$ and $-\frac{1}{2} \pm \frac{1}{2e_\infty}$ respectively. These are precisely the local exponents of the standard operator, which proves our claim.

Since $t \in \overline{k}(x)$, we can write $t = F \in \overline{k}(x)$. We have that $\phi_F(St_G)$ is a differential operator with corresponding intermediate field $\bar{k}(x)(s)$. By Lemma 1.8 the differential operator $\mathrm{Norm}(\phi_F(St_G))$ also has $\bar{k}(x)(s)$ as corresponding intermediate field, and L can be obtained from $\mathrm{Norm}(\phi_F(St_G))$ by a shift. Since both operators are in normal form, we must have $L = \mathrm{Norm}(\phi_F(St_G))$. This proves the existence of F.

We now consider the unicity of F. First of all, note that the choice of ramification indices over $\{0, 1, \infty\}$ of the covering $\mathbb{P}^1_s \to \mathbb{P}^1_t$ still leaves us some choice for t. To be precise,

- if $G^p = D_2$ we can replace t by its image under an automorphism of the $\mathbb{P}^1_t$ which permutes $\{0, 1, \infty\}$.

– if $G^p = D_n,\ n \neq 2$ we can replace t by $1-t$.
– if $G^p = A_4$ we can replace t by $\frac{t}{t-1}$.

Lemma 1.11. — *Let $\psi \in \mathrm{Aut}_{\bar{k}}\,\overline{k}(t)$ be an automorphism of $\mathbb{P}^1_t$ respecting the ramification data of the covering $\mathbb{P}^1_s \to \mathbb{P}^1_t$. Then $\psi \in \mathrm{Aut}(St_G)$.*

Proof. — Suppose we can replace t by $z, t = \frac{az+b}{cz+d},\ ad-bc=1$, without changing the ramification indices at $\{0,1,\infty\}$ of the covering induced by the field extension $\overline{k}(t) \subset \overline{k}(s)$. The resulting vector space $\widetilde{V_1}$ can be written as $\widetilde{V_1} = (cz+d)V_1$. Let $\widetilde{M_G}$ be the monic differential operator in $\overline{k}(z)[\partial_z]$, with solution space $\widetilde{V_1}$. We find that $\widetilde{M_G} = \frac{1}{(cz+d)^4}\phi_{\frac{az+b}{cz+d},\frac{c}{cz+d}}(M_G)$. Indeed $\phi_{\frac{az+b}{cz+d}}(M_G)$ is a differential operator over $\overline{k}(z)$ with solution space V_1, and multiplying all solutions by $cz+d$ corresponds to the shift $\partial_z \mapsto \partial_z - \frac{c}{cz+d}$. Because $\widetilde{M_G}$ is constructed in the same way as M_G, we have that $\phi_t(\widetilde{M_G}) = St_G$, $\phi_t : \bar{k}(z)[\partial_z] \to \overline{k}(t)[\partial_t]$. We find that $\frac{1}{(ct+d)^4}\phi_{\frac{at+b}{ct+d},\frac{c}{ct+d}}(St_G) = St_G$, so $\frac{at+b}{ct+d} \in \mathrm{Aut}(St_G)$. □

We will now show that ϕ_F is unique up to composition with an element in $\mathrm{Aut}(St_G)$. Our constructions give rise to the following diagram,

$$\begin{array}{ccccc} \bar{k}(x) & \subset & \bar{k}(x)(s) & \subset & \bar{k}(x)(y_1,y_2) \\ \cup & & \cup & & \\ \bar{k}(t) & \subset & \bar{k}(s) & \subset & \bar{k}(s,\sqrt{s'}) \end{array}$$

Now suppose we can write $L = \mathrm{Norm}(\phi_P(St_G))$ for some $P \in \bar{k}(x)$. Then we can make a diagram as above, where the image of t in $\overline{k}(x)$ is now P. As we proved above, t is almost unique up to composition with some $\psi \in \mathrm{Aut}(St_G)$. Therefore we must have $\phi_P = \phi_F \circ \psi$, for some $\psi \in \mathrm{Aut}(St_G)$. □

Remark 1.12. — In this remark we want to explain the following phenomenon. Let $\mathbb{C}(x) \subset K_G$ be a Picard-Vessiot extension for $St_G,\ G \in \{S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. For each G, we find two normalized differential operators in [**vdPU00**] with Picard-Vessiot extension equal to K_G (and satisfying certain nice properties). They correspond to the two irreducible two-dimensional representations of G. One of these two operators is St_G. Write L_G for the other operator. By Klein's theorem, we have that L_G is a pullback of St_G. On the other hand we will show that St_G is not a pullback of L_G, so L_G cannot be used as "standard operator" in Klein's theorem.

We will now explain this phenomenon in detail. First we consider the case $G = S_4^{\mathrm{SL}_2}$. The two operators of interest are

$$St_{S_4^{\mathrm{SL}_2}} = \partial_x^2 + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} - \frac{101}{576}\frac{1}{x(x-1)},$$
$$L_{S_4^{\mathrm{SL}_2}} = \partial_x^2 + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} - \frac{173}{576}\frac{1}{x(x-1)}.$$

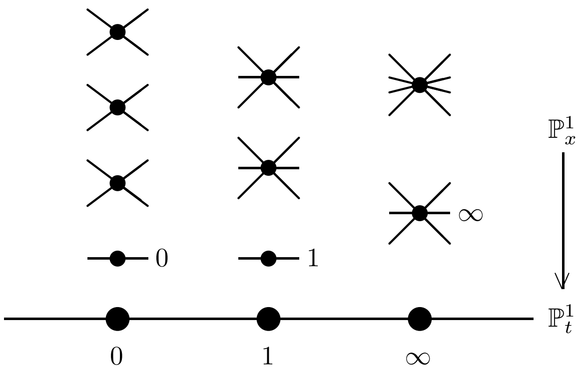
The local exponents of $St_{S_4^{\mathrm{SL}_2}}$ and $L_{S_4^{\mathrm{SL}_2}}$ are given by the following table.

	0	1	∞
$St_{S_4^{\mathrm{SL}_2}}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}$	$-\frac{3}{8}, -\frac{5}{8}$
$L_{S_4^{\mathrm{SL}_2}}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}$	$-\frac{1}{8}, -\frac{7}{8}$

Using the pullback formula of Theorem 2.7 we find that

$$L_{S_4^{\mathrm{SL}_2}} = \phi_{F,b}(St_{S_4^{\mathrm{SL}_2}}),\ F = \frac{(x-1)(144x^2-232x+81)^3}{(28x-27)^4} + 1,\ b = \frac{F''}{2F'}.$$

As we will see in Lemma 1.18, the difference of the local exponents of $L_{S_4^{\mathrm{SL}_2}}$ in a point a is equal to the ramification index of F at a times the difference of the local exponents of $St_{S_4^{\mathrm{SL}_2}}$ in $F(a)$. This is in accordance with the fact that the difference of the local exponents of $L_{S_4^{\mathrm{SL}_2}}$ at ∞ is $\frac{3}{4}$. Indeed, F has ramification index 3 at ∞, and the difference of the local exponents of $St_{S_4^{\mathrm{SL}_2}}$ at ∞ is $\frac{1}{4}$ (and $F(\infty) = \infty$). It also follows that $St_{S_4^{\mathrm{SL}_2}}$ cannot be written as a pullback of $L_{S_4^{\mathrm{SL}_2}}$. The complete ramification data of F are given by the following figure.



We note that the local exponents of $\phi_F(St_{S_4^{\mathrm{SL}_2}})$ at the ramified points ($\neq \infty$) above $0, \infty$ lie in $\frac{1}{2}\mathbb{Z}$ (see the proof of Lemma 1.18), but after applying the shift over $\frac{F''}{2F'}$, the local exponents become $\{0, 1\}$ at these points.

We will now explain how the representation of $S_4^{\mathrm{SL}_2}$ on the solution space changes by applying the pullback $\phi_{F,b}$. As in the proof of Klein's theorem (using the variables x, u instead of t, s), we can write $K = \mathbb{C}(u, \sqrt{u'})$, $' = \frac{d}{dx}$ for the Picard-Vessiot extension of $St_{S_4^{\mathrm{SL}_2}}$. The solution space of $St_{S_4^{\mathrm{SL}_2}}$ is $V := \langle \frac{u}{\sqrt{u'}}, \frac{1}{\sqrt{u'}} \rangle$, and $K^p := K^{\pm I} = \mathbb{C}(u)$. We can assume that the ramification data of $\mathbb{C}(x) \subset \mathbb{C}(u)$ is as in the proof of Klein's theorem. Let $W := \langle w_1, w_2 \rangle$ be the solution space of $L_{S_4^{\mathrm{SL}_2}}$, and define $s := \frac{w_1}{w_2}$. Then the group S_4 acts on $\mathbb{C}(s)$, and we define $\mathbb{C}(t) := \mathbb{C}(s)^{S_4}$, with the appropriate ramification data. These constructions give rise to the following diagram.

$$\begin{array}{ccccc} \mathbb{C}(x) & \subset & \mathbb{C}(x)(s)=\mathbb{C}(u) & \subset & \mathbb{C}(u,\sqrt{u'}) \\ \cup & & \cup & & \\ \mathbb{C}(t) & \subset & \mathbb{C}(s) & & \end{array}$$

We have $t = F \in \mathbb{C}(x)$, and s is some rational expression of degree 7 in u, say $s = g(u)$. We will now calculate g.

The extension $\mathbb{C}(x) \subset \mathbb{C}(u)$ has degree 24, and using [**BD79**], we find that we can write $x = h(u)$, where

$$h = -\frac{(u^8 + 14u^4 + 1)^3}{108u^4(u^4 - 1)^4} + 1.$$

We can also take $t = h(s)$, so $t = F(x) = F(h(u))$ and $t = h(s) = h(g(u))$. Therefore g satisfies $h(g(u)) = F(h(u))$. Using the ramification data of F and h, we can calculate the ramification data for g. Using these ramification data, together with some heuristics, we find

$$g = -\frac{u^3(u^4 + 7)}{7u^4 + 1}.$$

We can now express W in terms of u and $\sqrt{u'}$. We have $\frac{ds}{dx} = \frac{w_1'w_2 - w_1w_2'}{w_2^2}$, and since the operator $L_{S_4^{\mathrm{SL}_2}}$ is in normal form $w_1'w_2 - w_1w_2' \in \mathbb{C}$. So we find that $W = \langle \frac{s}{\sqrt{s'}}, \frac{1}{\sqrt{s'}} \rangle$, $' = \frac{d}{dx}$. Clearly $\frac{ds}{dx} = \frac{dg(u)}{du} \cdot \frac{du}{dx}$, and $\frac{dg(u)}{du} = -21\big(\frac{u(u^4-1)}{7u^4+1}\big)^2$. So we find a basis for W in terms of u and $\sqrt{u'}$, namely

$$\Big\{ \frac{u^2(u^4 + 7)}{(u^4 - 1)\sqrt{u'}}, \frac{7u^4 + 1}{u(u^4 - 1)\sqrt{u'}} \Big\}.$$

We will now examine the group $S_4^{\mathrm{SL}_2}$ in detail, and we will see how we can distinguish between the two irreducible representations ρ_1, ρ_2 of $S_4^{\mathrm{SL}_2}$ in $\mathrm{GL}_2(\mathbb{C})$. The abstract group $S_4^{\mathrm{SL}_2}$ is generated by two elements α, β, with image $(1234), (12)$ in S_4 respectively. For ρ_1 we take the representation $S_4^{\mathrm{SL}_2} \to \mathrm{GL}_2(\mathbb{C}), \alpha \mapsto \left(\begin{smallmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{smallmatrix}\right)$, $\beta \mapsto \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, $\zeta_8 = e^{\frac{2\pi i}{8}}$ (see [**Kov86**, p. 30]). Then for ρ_2 we can take the representation obtained by composition of ρ_1 with the automorphism of $\mathbb{Q}(\zeta_8)$ given by $\zeta_8 \mapsto \zeta_8^3$. We remark that the induced representations of S_4 in $\mathrm{PGL}(2, \mathbb{C})$ are conjugate. We can distinguish ρ_1 from ρ_2 by the eigenvalues of $\rho_i(\alpha)$. For ρ_1 these are $\{\zeta_8, \zeta_8^{-1}\}$ and for ρ_2 they are $\{\zeta_8^3, \zeta_8^{-3}\}$.

We fix an identification of $\mathrm{Gal}(K/\mathbb{C}(x))$ with $S_4^{\mathrm{SL}_2}$. We remark that since the group $\mathrm{Out}(S_4^{\mathrm{SL}_2})$ has two elements, there are essentially two ways to do this. We may assume that $S_4^{\mathrm{SL}_2}$ acts on V via the representation ρ_1. So $\alpha\big(\frac{u}{\sqrt{u'}}\big) = \zeta_8 \frac{u}{\sqrt{u'}}$ and $\alpha\big(\frac{1}{\sqrt{u'}}\big) = \zeta_8^{-1}\frac{1}{\sqrt{u'}}$. We will now calculate the action of α on W. We have $\alpha(u) = \zeta_8^2 u$, so $\alpha\big(\frac{u^2(u^4+7)}{(u^4-1)\sqrt{u'}}\big) = \zeta_8^3 \frac{u^2(u^4+7)}{(u^4-1)\sqrt{u'}}$ and $\alpha\big(\frac{7u^4+1}{u(u^4-1)\sqrt{u'}}\big) = \zeta_8^{-3}\frac{7u^4+1}{u(u^4-1)\sqrt{u'}}$. It immediately follows that the representation of $S_4^{\mathrm{SL}_2}$ in W is conjugate to ρ_2, which is what we wanted to show.

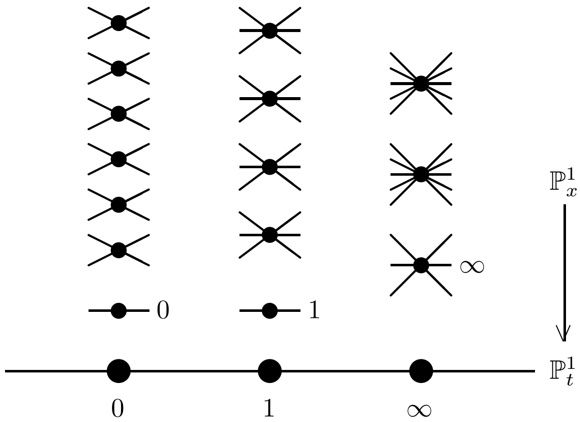
Now consider the case $G = A_5^{\mathrm{SL}_2}$. We will use the same terminology as in the $S_4^{\mathrm{SL}_2}$-case. The equations of interest are

$$St_{A_5^{\mathrm{SL}_2}} := \partial_x^2 + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} - \frac{611}{3600}\frac{1}{x(x-1)},$$
$$L_{A_5^{\mathrm{SL}_2}} := \partial_x^2 + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} - \frac{899}{3600}\frac{1}{x(x-1)}.$$

We have that $L_{A_5^{\mathrm{SL}_2}}$ is a pullback of $St_{A_5^{\mathrm{SL}_2}}$, with pullback function

$$F = \frac{(1-x)(147456x^4 - 403456x^3 + 379296x^2 - 57591x - 59049)^3}{(1664x^2 - 2457x + 729)^5} + 1.$$

The ramification of F is given by the following diagram.



As in the $S_4^{\mathrm{SL}_2}$ case we have the following diagram.

$$\begin{array}{ccccc} \mathbb{C}(x) & \subset & \mathbb{C}(u) & \subset & \mathbb{C}(u,\sqrt{u'}) \\ \cup & & \cup & & \\ \mathbb{C}(t) & \subset & \mathbb{C}(s) & & \end{array}$$

Again write $x = h(u)$ and $s = g(u)$. In the same way as in the $S_4^{\mathrm{SL}_2}$-case, we find

$$h = \frac{(u^{20} - 228u^{15} + 494u^{10} + 228u^5 + 1)^3}{1728u^5(u^{10} + 11u^5 - 1)^5} + 1,$$
$$g = -\frac{u^3(u^{10} - 39u^5 - 26)}{26u^{10} - 39u^5 - 1}.$$

We have $W = \langle \frac{s}{\sqrt{s'}}, \frac{1}{\sqrt{s'}} \rangle$, $' = \frac{d}{dx}$, and $\frac{ds}{dx} = \frac{dg(u)}{du} \cdot \frac{du}{dx}$. Using the fact that $\frac{dg(u)}{du} = -78\big(\frac{u(u^{10}+11u^5-1)}{26u^{10}-39u^5-1}\big)^2$, we obtain the following basis for W

$$\Big\{\frac{u^2(u^{10}-39u^5-26)}{(u^{10}+11u^5-1)\sqrt{u'}}, \frac{26u^{10}-39u^5-1}{u(u^{10}+11u^5-1)\sqrt{u'}}\Big\}.$$

The group $A_5^{\mathrm{SL}_2}$ has two irreducible representations ρ_1, ρ_2 in $\mathrm{GL}_2(\mathbb{C})$. We have that $A_5^{\mathrm{SL}_2}$ is generated by two elements α, β, with image (12345) and (12)(34) in A_5 respectively. We fix ρ_1 to be the representation of $A_5^{\mathrm{SL}_2}$ in $\mathrm{GL}_2(\mathbb{C})$ given by $\alpha \mapsto \begin{pmatrix} \zeta_{10} & 0 \\ 0 & \zeta_{10}^{-1} \end{pmatrix}$, $\beta \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, $\zeta_{10} = e^{\frac{2\pi i}{10}}$, $a = \frac{1}{5}(3\zeta_{10}^3 - \zeta_{10}^2 + 4\zeta_{10} - 2)$, $b = \frac{1}{5}(\zeta_{10}^3 + 3\zeta_{10}^2 - 2\zeta_{10} + 1)$. This explicit formulas come from [**Kov86**, p. 30], note that we can also write $a = i\sqrt{\frac{1}{2} + \frac{1}{10}\sqrt{5}}$, $b = \frac{\sqrt{5}-1}{2}a$. Then ρ_2 is the representation obtained by composition of ρ_1 with the automorphism of $\mathbb{Q}(\zeta_{10})$ given by $\zeta_{10} \mapsto \zeta_{10}^3$. In contrast to the $S_4^{\mathrm{SL}_2}$-case, the induced representations of A_5 in $\mathrm{PGL}(2, \mathbb{C})$ are not isomorphic. As in the $S_4^{\mathrm{SL}_2}$-case, we can distinguish ρ_1 from ρ_2 by the eigenvalues of $\rho_i(\alpha)$. For ρ_1 these are $\{\zeta_{10}, \zeta_{10}^{-1}\}$ and for ρ_2 they are $\{\zeta_{10}^3, \zeta_{10}^{-3}\}$.

Fix an identification of $\mathrm{Gal}(K/\mathbb{C}(x))$ with $A_5^{\mathrm{SL}_2}$. Again there are essentially two ways to do this. We may assume that $A_5^{\mathrm{SL}_2}$ acts on V via the representation ρ_1. So $\alpha\big(\frac{u}{\sqrt{u'}}\big) = \zeta_{10}\frac{u}{\sqrt{u'}}$ and $\alpha\big(\frac{1}{\sqrt{u'}}\big) = \zeta_{10}^{-1}\frac{1}{\sqrt{u'}}$. Again we calculate the action of α on W. We have $\alpha(u) = \zeta_{10}^2 u$, so $\alpha\big(\frac{u^2(u^{10}-39u^5-26)}{(u^{10}+11u^5-1)\sqrt{u'}}\big) = \zeta_{10}^3\frac{u^2(u^{10}-39u^5-26)}{(u^{10}+11u^5-1)\sqrt{u'}}$ and $\alpha\big(\frac{26u^{10}-39u^5-1}{u(u^{10}+11u^5-1)\sqrt{u'}}\big) = \zeta_{10}^{-3}\frac{26u^{10}-39u^5-1}{u(u^{10}+11u^5-1)\sqrt{u'}}$. It follows that the representation of $A_5^{\mathrm{SL}_2}$ in W is conjugate to ρ_2.

Only for some specific $F \in \bar{k}(x)$ the differential operator $\mathrm{Norm}(\phi_F(St_G))$ lies in $k(x)[\partial_x]$. The next corollary makes this precise.

Corollary 1.13

(1) $\mathrm{Norm}(\phi_F(St_G))$ *is defined over* $k \Longleftrightarrow \forall\ \sigma \in \mathrm{Gal}(\bar{k}/k)\ \exists\ S(\sigma) \in \bar{k}(t)$ *such that* $\phi_{S(\sigma)} \in \mathrm{Aut}(St_G)$ *and* $\phi_{\sigma(F)} = \phi_F \circ \phi_{S(\sigma)}$.

(2) *Furthermore,* ϕ_F *satisfies the equivalent properties of* (1) *if and only if* $\phi_F = \phi_f \circ \phi_h$, *with* $f \in k(x)$, *and* ϕ_h *an automorphism of* $\bar{k}(t)$ *satisfying the equivalent properties of* (1).

Proof

(1) '$\Longleftarrow$' $\forall\ \sigma \in \mathrm{Gal}(\bar{k}/k)$ we have $\sigma(\mathrm{Norm}(\phi_F(St_G))) = \mathrm{Norm}(\sigma(\phi_F(St_G))) = \mathrm{Norm}(\phi_{\sigma(F)}(St_G)) = \mathrm{Norm}(\phi_F \circ \phi_{S(\sigma)}(St_G)) = \mathrm{Norm}(\phi_F(St_G))$, so the operator is $\mathrm{Gal}(\bar{k}/k)$ invariant, hence has coefficients in $k(x)$. '$\Longrightarrow$' Because $\mathrm{Norm}(\phi_F(St_G))$ is $\mathrm{Gal}(\bar{k}/k)$ invariant we get $\mathrm{Norm}(\phi_F(St_G)) = \sigma(\mathrm{Norm}(\phi_F(St_G))) = \mathrm{Norm}(\phi_{\sigma(F)}(St_G))$ $\forall\ \sigma \in \mathrm{Gal}(\bar{k}/k)$. Hence Klein's theorem gives $\phi_{\sigma(F)} = \phi_F \circ \phi_{S(\sigma)}$, with $\phi_{S(\sigma)} \in \mathrm{Aut}(St_G)$. This proves (1).

(2) The if-part follows immediately from $\phi_{\sigma(F)} = \phi_{\sigma(f)} \circ \phi_{\sigma(h)} = \phi_f \circ \phi_{\sigma(h)}$. For the other implication write $\phi_{\sigma(F)} = \phi_F \circ \phi_{S(\sigma)}$, with $\phi_{S(\sigma)} \in \mathrm{Aut}(St_G)$ an automorphism of $\bar{k}(t)$ that permutes $0, 1, \infty$. Then there is also an automorphism ϕ_h of $\bar{k}(t)$, with $\phi_{\sigma(h)} = \phi_h \circ \phi_{S(\sigma)}\ \forall\ \sigma \in \mathrm{Gal}(\bar{k}/k)$. Namely, take $h = \frac{a_1 - a_\infty}{a_1 - a_0} \frac{t - a_0}{t - a_\infty}$, where $a_0, a_1, a_\infty \in \bar{k}$ are elements which are permuted in the same way by every $\sigma \in \mathrm{Gal}(\bar{k}/k)$ as $0, 1, \infty$ by $\phi_{S(\sigma)}$. These elements are proven to exist in the lemma below. Note that for such a_0, a_1, a_∞ the extension $k(a_0, a_1, a_\infty)/k$ has degree at most 6. Define $f \in \bar{k}(x)$ by $\phi_f := \phi_F \circ \phi_h^{-1}$. Then $\phi_F = \phi_f \circ \phi_h$, and we only need to show that f is $\mathrm{Gal}(\bar{k}/k)$ invariant. But we have that $\phi_{\sigma(f)} = \phi_{\sigma(F)} \circ \phi_{\sigma(h)}^{-1} = \phi_F \circ \phi_{S(\sigma)} \circ (\phi_h \circ \phi_{S(\sigma)})^{-1} = \phi_f$ and therefore $f \in k(x)$. □

Remark 1.14. — The above corollary states that every differential operator $\partial_x^2 - r$, with $r \in k(x)$ is the pullback of a differential operator over $k(x)$ with three singularities, and with the same local exponents as the corresponding standard operator (use $\mathrm{Norm}(\phi_h(St_G))$). So we can see this corollary as a "rational version" of Klein's theorem.

In the proof above we used the following lemma. Its content is well known, and we prove it only for the sake of completeness.

Lemma 1.15. — *Given an action of $G := Gal(\bar{k}/k)$ on the set $\{1, 2, 3\}$, there exists a Galois extension $k \subset k(a_1, a_2, a_3) \subset \overline{k}$, such that G permutes the set $\{a_1, a_2, a_3\}$ in the corresponding manner.*

Proof. — We first assume G acts as S_3. Let H be the subgroup of G which fixes $\{1, 2, 3\}$. Then $F := \bar{k}^H$ is a Galois extension of k of degree 6. We have an action of $G/H \cong S_3$ on F. For some element σ of order two in S_3, write $k(a_1) = F^\sigma$. Then $k \subset k(a_1)$ is an extension of degree 3, which is not a Galois extension. Writing a_2, a_3 for the conjugates of a_1 in F, we have $F = k(a_1, a_2, a_3)$. Furthermore G acts as S_3 on the set $\{a_1, a_2, a_3\}$. We can rename the a_i, in such a way that G permutes the set $\{a_1, a_2, a_3\}$ in the desired manner. The remaining cases, where G acts as $1, C_2$ or C_3 are easy. □

Notation 1.16

- Let $L \in k(x)[\partial_x]$ be an arbitrary second order differential operator, with differential Galois group $G \subset \mathrm{GL}_2(\overline{k})$. We write G^p for the image of G in $\mathrm{PGL}(2)$, and call G^p the *projective differential Galois group* of L. This definition of G^p is consistent with the definition of G^p in the proof of Klein's theorem.
- For L as above, and $a \in \mathbb{P}^1(\overline{k})$, we have a set of local exponents $\{l_1, l_2\}$ at a. We will call $|l_1 - l_2|$ the *local exponent difference* at a.

Again let $L \in k(x)[\partial_x]$ be a second order differential operator, with projective differential Galois group $G^p \in \{D_n, A_4, S_4, A_5\}$. We have that $\mathrm{Norm}(L)$ has the same projective differential Galois group. Indeed L and $\mathrm{Norm}(L)$ define the same field extension $\overline{k}(x) \subset \overline{k}(x)(s)$ (notation from the proof of Klein's theorem), and we can identify G^p with $\mathrm{Gal}(\overline{k}(x)(s)/\overline{k}(x))$, where $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G^p$ acts on s by $\sigma(s) = \frac{as+b}{cs+d}$. Consequently, the differential Galois group of $\mathrm{Norm}(L)$ is an element of $\{D_n^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. Using Klein's theorem we find that there exist elements $a, F, b \in \overline{k}(x)$, such that $L = a \cdot \phi_{F,b}(St_G)$.

1.3. Differential Galois group $D_2^{\mathrm{SL}_2}$. — For generality, we formulate the following theorem for differential operators with projective differential Galois group D_2. This of course includes differential operators in normal form with differential Galois group $D_2^{\mathrm{SL}_2}$.

Theorem 1.17. — *Let $L \in k(x)[\partial_x]$ be a second order differential operator, with projective differential Galois group $G^p = D_2$. There exists a point $a \in \mathbb{P}^1(\bar{k})$ for which L has local exponent difference in $\frac{1}{2} + \mathbb{Z}$. For any such a there is an algebraic solution of minimal degree of the corresponding Riccati equation, with minimal polynomial in $k(a)[x]$.*

Proof. — We will first show that we can assume L to be in normal form. We can write $\mathrm{Norm}(L) = a \cdot \phi_{x,b}(L)$, for some $a, b \in k(x)$. If u is a solution of the Riccati equation $R_{\mathrm{Norm}(L)}$, then $u + b$ is the corresponding solution of R_L. Writing f_u for the minimal polynomial of u over $\overline{k}(x)$, we clearly have $f_u \in k'(x)[T] \iff f_{u+b} \in k'(x)[T]$. Furthermore normalization does not affect the local exponent difference at a point.

Klein's theorem gives an $F \in \bar{k}(x)$ such that $L = \mathrm{Norm}(\phi_F(St_G))$, where $G := D_2^{\mathrm{SL}_2}$. We will use notations as in the proof of Klein's theorem. We have that $\left\{\frac{s}{\sqrt{s'}}, \frac{1}{\sqrt{s'}}\right\}$ is a basis of solutions of St_G. Then $\left\{\frac{s}{\sqrt{F's'}}, \frac{1}{\sqrt{F's'}}\right\}$ is a basis of solutions of L, where $'$ now denotes $\frac{d}{dx}$. We find that the solutions of R_L are precisely the elements $F'u - \frac{1}{2}\frac{F''}{F'}$, with u a solution of R_{St_G}. From the explicit description of $D_2^{\mathrm{SL}_2}$ in Lemma 1.4, we know that there are six solutions of R_{St_G} of degree two over $\overline{k}(t)$, which correspond to three minimal polynomials $\{P_1, P_2, P_3\}$. By [**HvdP95**, 6.5.3] we know that $P_i \in k(t)[T], i = 1, 2, 3$. Let u be one of the six solutions of R_{St_G} of degree 2 over $\overline{k}(t)$. Write $\widetilde{u} := F'u - \frac{1}{2}\frac{F''}{F'}$ for the corresponding solution of R_L. If P_j is the minimal polynomial of u, then $F(P_j) := (F')^2 P_j\left(\frac{T}{F'} + \frac{1}{2}\frac{F''}{(F')^2}\right) \in \bar{k}(x)[T]$ is the minimal polynomial of $\widetilde{u}$. Let $k \subset \widetilde{k}$ be a minimal extension, such that $F \in \widetilde{k}(x)$. Then $F(P_j) \in \widetilde{k}(x)[T]$, so we can take $k' \subset \widetilde{k}$, where k' is the field defined in the beginning of this section. Because $L \in k(x)[\partial_x]$, we have that F satisfies the properties stated in Corollary 1.13. Using notation as in the proof of this corollary, we see that we can take $\widetilde{k}$ to be the extension of k generated by the coefficients of h, so $\widetilde{k} = k(a_0, a_1, a_\infty)$. This is a field extension of k of degree at most 6.

Claim: for any $j \in \{0, 1, \infty\}$ there is a solution of R_L with minimal polynomial in $k(a_j)[T]$.

The Galois group $\mathrm{Gal}(\overline{k}/k)$ acts as a group of permutations on the set $\{F(P_1), F(P_2), F(P_3)\}$. In fact $\sigma(F(P_i)) = \sigma(F)(P_i)$ for $\sigma \in \mathrm{Gal}(\overline{k}/k)$. By Corollary 1.13 we have $\phi_{\sigma(F)} = \phi_F \circ \phi_{S(\sigma)}$ with $\phi_{S(\sigma)} \in \mathrm{Aut}(St_G)$. We know the polynomials P_i explicitly, see Example 1.20. A calculation shows that all non-trivial automorphisms $\phi_S, S \in \left\{\frac{1}{t}, 1-t, \frac{1}{1-t}, \frac{t}{t-1}, \frac{t-1}{t}\right\}$ act non-trivially on the P_i. Using this we see that there exists a $\mathrm{Gal}(\overline{k}/k)$-equivariant bijection between $\{a_0, a_1, a_\infty\}$ and $\{F(P_1), F(P_2), F(P_3)\}$. This immediately proves the claim.

Let f be as in Corollary 1.13 (2). If $f(a) = a_i$, then $k(a_i) \subset k(a)$. So the only thing left to prove is that there exist points a with local exponent difference in $\frac{1}{2} + \mathbb{Z}$, and that any such point satisfies $f(a) \in \{a_0, a_1, a_\infty\}$. For this we need the following lemma.

Lemma 1.18. — *With the above notation the following holds.*

The extension $\bar{k}(t) \subset \bar{k}(x)$ corresponds to a covering $\mathbb{P}^1_x \to \mathbb{P}^1_t$. Suppose that this covering is ramified with index e in a point $a \in \mathbb{P}^1_x(\bar{k})$ lying above some $b \in \mathbb{P}^1_t(\bar{k})$. The local exponent difference of $L = \mathrm{Norm}(\phi_F(St_G))$ at a is $|e(l_1 - l_2)|$, where $\{l_1, l_2\}$ are the local exponents of St_G at b.

Proof. — By a calculation as in the proof of Klein's theorem, we find that the local exponents of $\phi_F(St_G)$ at a are $\{el_1, el_2\}$. The lemma now follows from the fact that normalization does not change the local exponent difference at a point. □

We continue the proof of Theorem 1.17. Using the above lemma, we see that if a point $a \in \mathbb{P}^1_x(\bar{k})$ does not lie above one of the points $0, 1, \infty$, then the local exponent difference of L at a lies in $\mathbb{Z}$. If a does lie above $b \in \{0, 1, \infty\}$, then the local exponents of St_G at b are $\{l_1, l_2\} = \pm\left\{\frac{1}{4}, \frac{3}{4}\right\}$, so the local exponent difference of L at a is in $\frac{1}{2} + \mathbb{Z}$ if e is odd, and in $\mathbb{Z}$ if e is even.

The only thing left to prove is that there exist points $a \in \mathbb{P}^1_x(\bar{k})$, such that L has local exponent difference in $\frac{1}{2} + \mathbb{Z}$ at a. By [**vdPS03**, Theorem 5.8], the differential Galois group of L is equal to the monodromy group, so there is a local monodromy matrix which has order 2 in $\mathrm{PGL}_2(\overline{k})$. It follows that the local exponents at the corresponding singular point have local exponent difference in $\frac{1}{2} + \mathbb{Z}$. □

1.4. Differential Galois group $A_4^{\mathrm{SL}_2}$

Theorem 1.19. — *Let $L \in k(x)[\partial_x]$ be a second order differential operator, with projective differential Galois group $G^p = A_4$. There exists a point $a \in \mathbb{P}^1(\bar{k})$ for which L has local exponent difference in $\frac{1}{3}\mathbb{Z}\backslash\mathbb{Z}$. For any such a there is an algebraic solution of minimal degree of the corresponding Riccati equation, with minimal polynomial in $k(a)[x]$.*

Proof. — This case can be treated similarly to the D_2-case above, now taking $G := A_4^{\mathrm{SL}_2}$. Again we will use notation of Corollary 1.13. We will only give the differences with the proof of Theorem 1.17.

The Riccati equation R_{St_G} has eight solutions of degree 4 over $\overline{k}(t)$, corresponding to two minimal polynomials $P_1, P_2, \in k(t)[T]$ (see Example 1.21, [**HvdP95**, 6.5.4], or [**Kov86**, 5.2]). The group $\mathrm{Aut}(St_G)$ consists of two elements, namely $\{\phi_t, \phi_{\frac{t}{t-1}}\}$. Therefore an automorphism of St_G can only permute the singular points $\{1, \infty\}$. This implies $a_0 \in k$, $k(a_1) = k(a_\infty) = \widetilde{k}$, and $[\widetilde{k} : k] \leq 2$. So $F(P_i) := (F')^4 P_i\big(\frac{T}{F'} + \frac{1}{2}\frac{F''}{(F')^2}\big) \in k(a_1)[T] = k(a_\infty)[T]$. The only thing left to prove is that all points a with local exponent difference in $\frac{1}{3}\mathbb{Z}\backslash\mathbb{Z}$ satisfy $f(a) \in \{a_1, a_\infty\}$, and that there exists such a point.

The local exponents of St_G at the point 0 are $\left\{\frac{1}{4}, \frac{3}{4}\right\}$. At the point 1 the local exponents are $\left\{\frac{1}{3}, \frac{2}{3}\right\}$, and at the point ∞ they are $\left\{-\frac{1}{3}, -\frac{2}{3}\right\}$. Now Lemma 1.18 gives the following. The points with local exponent difference in $\frac{1}{3}\mathbb{Z}\backslash\mathbb{Z}$ are precisely the points $a \in \mathbb{P}^1_x(\bar{k})$ lying above $1, \infty$ with ramification index not divisible by 3. To prove that indeed there are such points a, we again use that the differential Galois group is equal to the monodromy group. We may assume that L is of the form $L = \phi_F(St_G)$. It follows that if the local exponent difference at a point lies in $\mathbb{Z}$, then the local exponents lie in $\frac{1}{2}\mathbb{Z}$. If all local exponents lie in $\frac{1}{2}\mathbb{Z}$, then the monodromy group is generated by elements of order ≤ 2. This contradicts the assumption that $G = A_4^{\mathrm{SL}_2}$, because $A_4^{\mathrm{SL}_2}$ is not generated by elements of order 2. □

1.5. Examples. — In the following examples we will give explicitly the minimal polynomials of solutions of R_{St_G} of minimal degree over $\overline{\mathbb{Q}}(t)$, for $G \in \{D_2^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}\}$. We will also calculate these minimal polynomials corresponding to pullbacks of standard equations.

Example 1.20. — In the proof of Theorem 1.17 we showed that the Riccati equation R_{St_G}, $G := D_2^{\mathrm{SL}_2}$ has six algebraic solutions of degree two over $\overline{\mathbb{Q}}(t)$. Let $\{y_1, y_2\}$ be a basis of solutions of $St_{D_2^{\mathrm{SL}_2}}$, on which the differential Galois group G has the explicit form of Lemma 1.4. Then these six solutions of the Riccati equation are $\frac{y'}{y}, y \in \{y_1, y_2, y_1 + y_2, y_1 - y_2, y_1 + iy_2, y_1 - iy_2\}$, which are the solutions of the three polynomials

$$P_1 := T^2 - \Big(\frac{1}{2}\frac{1}{t} + \frac{1}{t-1}\Big)T + \frac{1}{16}\frac{9t^2 - 7t + 1}{t^2(t-1)^2},$$

$$P_2 := T^2 - \Big(\frac{1}{2}\frac{1}{t} + \frac{1}{2}\frac{1}{t-1}\Big)T + \frac{1}{16}\frac{3t^2 - 3t + 1}{t^2(t-1)^2},$$

$$P_3 := T^2 - \Big(\frac{1}{t} + \frac{1}{2}\frac{1}{t-1}\Big)T + \frac{1}{16}\frac{9t^2 - 11t + 3}{t^2(t-1)^2}.$$

Now consider the function $F := \frac{2x}{x-\sqrt{2}}$, mapping $0, -\sqrt{2}, \sqrt{2}$ to $0, 1, \infty$ respectively. We have that ϕ_F satisfies the properties of Corollary 1.13 (1), so $L := \mathrm{Norm}(\phi_F(St_G))$

is defined over $\mathbb{Q}$. A calculation gives

$$L = \partial_x^2 + \frac{3}{8}\frac{3x^2+2}{x^2(x^2-1)^2}.$$

Using the formula in the proof of Theorem 1.17, we find that the six solutions of R_L of degree two over $\overline{\mathbb{Q}}(x)$ are the solutions of the polynomials

$$T^2 - \frac{1}{2}\frac{4x^2-\sqrt{2}x-2}{x(x^2-2)}T + \frac{1}{8}\frac{8x^4-4\sqrt{2}x^3-9x^2+4\sqrt{2}x+2}{x^2(x^2-1)^2},$$

$$T^2 - \frac{1}{2}\frac{4x^2+\sqrt{2}x-2}{x(x^2-2)}T + \frac{1}{8}\frac{8x^4+4\sqrt{2}x^3-9x^2-4\sqrt{2}x+2}{x^2(x^2-1)^2},$$

$$T^2 - \frac{2(x^2-1)}{x(x^2-2)}T + \frac{1}{8}\frac{8x^4-15x^2+6}{x^2(x^2-1)^2}.$$

We remark that the local exponent difference is $\frac{1}{2}$ for each singular point of L. This is in accordance with Theorem 1.17. In [**HvdP95**] it is stated that $[k' : k] \in \{1,3\}$ for $G = D_2^{\mathrm{SL}_2}$. This does not contradict the fact that we find $k' = \mathbb{Q}(\sqrt{2})$ for some of the solutions of R_L, because in [**HvdP95**] only fields k' of minimal degree over k are considered. ☐

Example 1.21. — We consider the standard equation St_G, $G := A_4^{\mathrm{SL}_2}$. In [**Kov86**, 5.2] one of the two minimal polynomials for solutions of R_{St_G} of degree 4 over $\overline{\mathbb{Q}}(t)$ is computed. It is the polynomial

$$P_1 := T^4 - \frac{7t-3}{3t(t-1)}T^3 + \frac{48t^2-41t+9}{24t^2(t-1)^2}T^2 - \frac{320t^3-409t^2+180t-27}{432t^3(t-1)^3}T + \frac{2048t^4-3484t^3-2313t^2-702t+81}{20736t^4(t-1)^4}.$$

The other minimal polynomial is $P_2 := S(P_1)$, $S = \frac{t}{t-1}$, where we use notation of the proof of Theorem 1.17. A calculation gives

$$P_2 = T^4 - \frac{8t-3}{3t(t-1)}T^3 + \frac{64t^2-49t+9}{24t^2(t-1)^2}T^2 - \frac{512t^3-598t^2+225t-27}{432t^3(t-1)^3}T + \frac{-530t^4+2788t^3-909t^2-918t+81}{20736t^4(t-1)^4}.$$

Let $a \in \mathbb{Q}$, and define $F := \frac{2x}{x-\sqrt{a}}$ which maps $0, -\sqrt{a}, \sqrt{a}$ to $0, 1, \infty$ respectively. Then $L := \mathrm{Norm}(\phi_F(St_G))$ is

$$\partial_x^2 + \frac{3}{16}\frac{1}{x^2} - \frac{3}{16}\frac{1}{x^2-a} + \frac{8}{9}\frac{a}{(x^2-a)^2}.$$

The local exponents at $0, -\sqrt{a}, \sqrt{a}$ are $\{\frac{1}{4}, \frac{3}{4}\}, \{\frac{1}{3}, \frac{2}{3}\}, \{\frac{1}{3}, \frac{2}{3}\}$ respectively. So theorem 1.19 states that there is a solution of R_L of degree 4 over $\overline{\mathbb{Q}}(x)$, such that the corresponding field k' lies in $\mathbb{Q}(\sqrt{a})$. A calculations shows that in fact for each solution of R_L of degree 4 over $\overline{\mathbb{Q}}(x)$, the corresponding field k' equals $\mathbb{Q}(\sqrt{a})$. ☐

1.6. Algorithm. — We will now give an algorithm to compute a field k' (as defined in the beginning of this chapter). We will also give some examples.

Let $L \in k(x)[\partial_x]$ be a second order differential operator in normal form with known differential Galois group G in $\{D_2^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}\}$. We can use theorems 1.17 and 1.19 to find a field k'. Write $L = \partial_x^2 - \frac{T}{N}, T, N \in k[x]$, where $\gcd(T,N) = 1$, and N is monic. Because G is finite, all singularities of L are regular singular (see [**vdPS03**, Definition 3.9]). Therefore, the zeros of N can at most have order two. So we can write $N = N_1 \cdot N_2^2$, such that N_1, N_2 have only zeros of order one, and are monic. We can make a decomposition $\frac{T}{N} = \frac{A}{N_2^2} + \frac{B}{N_1}$. Now the local exponents at some point $p \in \bar{k}$ are the solutions of the equation $\lambda(\lambda-1) = \frac{A\cdot(x-p)^2}{N_2^2}|_{x=p}$. So the local exponents λ satisfy $\lambda(\lambda-1) = \frac{A(p)}{N_2'(p)^2}$.

For the D_2-case we search for points with local exponent difference in $\frac{1}{2}+\mathbb{Z}$. Because L is in normal form, the local exponents of L at such a point are $\left\{\frac{2n+1}{4}, \frac{3-2n}{4}\right\}$, for some $n \in \mathbb{Z}$. Therefore we get the system of equations:

$$D_2\text{-case}: \begin{cases} (3+4n-4n^2)N_2'(p)^2 + 16A(p) = 0 \\ N_2(p) = 0. \end{cases}$$

To solve this system we can calculate the resultant of $(3+4n-4n^2)N_2'(x)^2 + 16A(x)$ and $N_2(x)$ with respect to x. This gives a polynomial in n, for which it is easy to determine if it has integer solutions. If this resultant is zero for some n_0, then we can substitute $n = n_0$ into the system of equations. Then solutions of the system are given by $\gcd((3+4n_0-4n_0^2)(N_2')^2 + 16A, N_2) = 0$.

For the A_4-case we search for points with local exponent difference in $\frac{1}{3}\mathbb{Z}\backslash\mathbb{Z}$. At such a point the local exponents are $\left\{\frac{3+n}{6}, \frac{3-n}{6}\right\}$, for some $n \in \mathbb{Z}$. We find the system of equations:

$$A_4\text{-case}: \begin{cases} (9-n^2)N_2'(p)^2 + 36A(p) = 0 \\ N_2(p) = 0, \end{cases}$$

We search solutions, with $n \not\equiv 0 \bmod 3$. This system can be solved in the same way as in the D_2-case. We conclude that for a differential operator L satisfying our assumptions, we can find a corresponding field k'.

Example 1.22. — We will demonstrate the algorithm for

$$L := \partial_x^2 + \frac{16x^{18} - 288x^{15} + 2160x^{12} - 8947x^9 + 20745x^6 - 25056x^3 + 13456}{8(x^9 - 9x^6 + 27x^3 - 29)^2 x^2}.$$

This is the operator obtained as the pullback of $St_{D_2^{\mathrm{SL}_2}}$ with $F = h \circ (x^3 - 3)$, where h is some automorphism of $\bar{k}(t)$ that sends the roots of $x^3 - 2$ to $\{0, 1, \infty\}$. With the notation of the algorithm we have $T = \frac{1}{8}(16x^{18} - 288x^{15} + 2160x^{12} - 8947x^9 + 20745x^6 - 25056x^3 + 13456)$, $N = (x^9 - 9x^6 + 27x^3 - 29)^2 x^2$.

We calculate N_2 by $N_2 = \gcd(N, N')$ which obviously is $(x^9 - 9x^6 + 27x^3 - 29)x$ and furthermore $A = T$. Using for example Maple we find that the resultant of $(3 + 4n - 4n^2)(N_2')^2 + 16A$ and N_2 over x is

$$(457668486144n^3 - 1373005458432n^4 + 1373005458432n^5 - 457668486144n^6)^3 (29435 + 3364n - 3364n^2).$$

This expression has as integer solutions $n = 0$ and $n = 1$, which both correspond to the same set of local exponents. Substituting $n = 0$ we get $\gcd(3N_2' + 16A, N_2) = x^9 - 9x^6 + 27x^3 - 29$, a polynomial in x^3, with as a solution $a := (3 + 2^{\frac{1}{3}})^{\frac{1}{3}}$. So there is a field $k' \subset \mathbb{Q}(a)$. We know that $[k' : k] \leq 3$, so $k' = k$ or $[k' : k] = 3$. We can calculate all subfields of $\mathbb{Q}(a)$ of degree 3 over $\mathbb{Q}$ in Maple 7, with the command

```
evala(Subfields(x^9-9x^6+27x^3-29,3));
```

It turns out that the only such subfield is $\mathbb{Q}(\sqrt[3]{2})$. It follows that there is a field $k' \subset \mathbb{Q}(\sqrt[3]{2})$. □

Example 1.23. — Let h to be the automorphism of $\mathbb{P}^1$ sending $1, -\sqrt{2}, \sqrt{2}$ to $0, 1, \infty$ respectively. Let $f = x^2 - 3$, and $F = h \circ f$, then $\mathrm{Norm}(\phi_F(St_{A_4}))$ is

$$\partial_x^2 - \frac{27x^{12} - 540x^{10} + 4145x^8 - 16366x^6 + 37160x^4 - 46872x^2 + 21168}{(6x(x-2)(x+2)(x^4 - 6x^2 + 7)^2}.$$

As before we write this as $\partial_x^2 - \left(\frac{A}{N_2^2} + \frac{B}{N_1}\right)$. The resultant of $(9-n^2)(N_2')^2 + 36A$ and N_2 is $-2^{58}3^{26}7^6(n-6)(n+6)(2n-3)^2(2n+3)^2(n-1)^4(n+1)^4$. The integer solutions for n are $n \in \{-6, -1, 1, 6\}$. So only $n = 1$ (which gives the same as $n = -1$) is of interest, for $-6, 6 \equiv 0 \bmod 3$. We now substitute $n = 1$ into $(9-n^2)(N_2')^2 + 36A$, and calculate the greatest common divisor with N_2. This gives the polynomial $x^4 - 6x^2 + 7 = (x-3)^2 - 2$. A zero of this polynomial is $a = \sqrt{\sqrt{2} + 3}$, so there is a field k' of degree ≤ 2 over $\mathbb{Q}$ in $\mathbb{Q}\big(\sqrt{\sqrt{2} + 3}\big)$. By a calculation in Maple 7 we find that the only field extension of $\mathbb{Q}$ of order 2 in $\mathbb{Q}\big(\sqrt{\sqrt{2} + 3}\big)$ is $\mathbb{Q}(\sqrt{2})$. Therefore there is a field $k' \subset \mathbb{Q}(\sqrt{2})$. Note that in this example we can explicitly calculate k' from knowing only the operator and the differential Galois group. □

2. Algorithms for finding the pullback function

The material in this section is joint work with Mark van Hoeij and Jacques-Arthur Weil. A short preliminary version is published as [**Wei**].

Let $L = \partial_x^2 + a_1\partial_x + a_0 \in k(x)[\partial_x]$ be a monic order 2 differential operator. We suppose the differential Galois group G over $\bar{k}(x)$ is known and is a finite subgroup of $\mathrm{GL}(2, \bar{k})$. We will write G^p for the image of G in the $\mathrm{PGL}_2(\bar{k})$. The normalization $\mathrm{Norm}(L)$ of L is obtained by a shift $\partial_x \mapsto \partial_x - \frac{a_1}{2}$, and we write G^n for the differential Galois group of $\mathrm{Norm}(L)$. We assume G^n is non-cyclic, which implies $G^n \in \big\{A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}, D_n^{\mathrm{SL}_2}\big\}$. By Klein's theorem we have $\mathrm{Norm}(L) =$

$\mathrm{Norm}(\phi_F(St_{G^n}))$, for some $F \in \overline{k}(x)$. Therefore $\exists\, b \in \overline{k}(x)$, such that for $\phi := \phi_{F,b}$ we have $L = (\phi(t)')^2 \phi(St_{G^n})$.

In this section we will concentrate on finding $\phi(t)$, which we will do case by case with respect to G^p. We will define new standard equations St_{G^p}, with projective Galois group G^p, for $G^p \in \{A_4, S_4, A_5, D_n\}$. For this standard equations Klein's theorem still holds, and we are able to give an explicit formula for $\phi(t)$.

Notation 2.1

- Let $\phi : \overline{k}(t)[\partial_t] \to \overline{k}(x)[\partial_x], \phi(\partial_t) = \frac{1}{\phi(t)'}(\partial_x + b)$ be a homomorphism. Then we call $\phi(t)$ the *pullback function* corresponding to ϕ.
- Let $L_1 \in \overline{k}(t)[\partial_t], L_2 \in \overline{k}(x)[\partial_x]$, be differential operators, such that we can write $L_2 = a\phi_{F,b}(L_1), a, F, b \in \overline{k}(x)$. If $b = 0$, we call L_2 a *pullback* of L_1. If $b \neq 0$, we call L_2 a *weak pullback* of L_1.

2.1. Projective Galois group A_4. — We define the following new standard equation:

$$St_{A_4} := \partial_t^2 + \frac{8t+3}{6t(t+1)}\partial_t + \frac{s}{t(t+1)^2}, \quad s = \frac{1}{48}.$$

This differential operator is obtained from $St_{A_4^{\mathrm{SL}_2}}$ by first making the shift $\partial_t \mapsto \partial_t + \frac{1}{4t} + \frac{1}{3(t-1)}$, and then applying the coordinate transformation $t \mapsto \frac{t}{t+1}$. So $St_{A_4} = \phi_{\frac{t}{t+1}, \frac{1}{4t} - \frac{7}{12(t+1)}}(St_{A_4^{\mathrm{SL}_2}})$. We will now motivate this new choice of a standard operator.š

From the fact that the projective differential Galois group of $St_{A_4^{\mathrm{SL}_2}}$ is A_4, it follows, using some representation theory, that this operator has solutions $y_1, \cdots, y_4$ such that $\frac{(y_1\cdots y_4)'}{y_1\cdots y_4} \in \overline{k}(t)$. This translates into the existence of a degree one right-hand factor of $\mathrm{Sym}(St_{A_4^{\mathrm{SL}_2}}, 4)$. In fact there are precisely two such right-hand factors. By a direct computation, we find that these right-hand factors are $\partial_t - \frac{1}{t} - \frac{4}{3(t-1)}$ and $\partial_t - \frac{1}{t} - \frac{5}{3(t-1)}$. We constructed St_{A_4} such that $\mathrm{Sym}(St_{A_4}, 4)$ has a right-hand factor ∂_t. To see this, note that for any differential operator L, we have $\mathrm{Sym}(\phi_{f,b}(L), n) = \phi_{f,nb}(\mathrm{Sym}(L, n))$. So applying the shift $\partial_t \mapsto \partial_t + \frac{1}{4t} + \frac{1}{3(t-1)}$ to $St_{A_4^{\mathrm{SL}_2}}$ gives a differential operator $\widetilde{St} = \partial_t^2 + \frac{7t-3}{6t(t-1)}\partial_t - \frac{1}{48t(t-1)}$ with the property that $\mathrm{Sym}(\widetilde{St}, 4)$ has a right-hand factor ∂_t. The coordinate transformation $t \mapsto \frac{t}{t+1}$ does not changes this property, and will make the pullback formula in Theorem 2.3 somewhat nicer. Note that applying the shift $\partial_t \mapsto \partial_t + \frac{1}{4t} + \frac{5}{12(t-1)}$ to $St_{A_4^{\mathrm{SL}_2}}$ also results in a differential operator such that its fourth symmetric power has a right-hand factor ∂_t. This differential operator is different from $\widetilde{St}$. There is a non-trivial automorphism of $\overline{k}(t)[\partial_t]$ mapping St_{A_4} to a multiple of itself, namely $\phi_{\frac{-t}{t+1}, \frac{1}{12(t+1)}}$. It follows immediately from the proof of Klein's theorem that this is the unique non-trivial automorphism of St_{A_4}.

Proposition 2.2. — *The differential Galois group G of St_{A_4} is a central extension of A_4 by the cyclic group C_4.*

Proof. — Let F be a fundamental matrix for St_{A_4}, i.e., a matrix $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$, where $\{y_1, y_2\}$ is a basis of solutions of St_{A_4}. The determinant $\mathrm{Det}(F)$ of F, satisfies the differential operator $\partial_t + \frac{8t+3}{6t(t+1)} = \bigwedge^2 St_{A_4}$. For $g \in G$, we have $g(\mathrm{Det}(F)) = \mathrm{Det}(g) \cdot \mathrm{Det}(F)$, so the differential Galois group of $\partial_t + \frac{8t+3}{6t(t+1)}$ is precisely the image of G under the determinant map. The differential Galois group of $\partial_t + \frac{8t+3}{6t(t+1)}$ is easily seen to be the group μ_6 consisting of the sixth roots of unity. So $G \subset H := \{M \in \mathrm{GL}_2(\overline{\mathbb{Q}}) | \, \mathrm{Det}(M)^6 = 1, M^p \in A_4\}$, where M^p denotes the image of M in $\mathrm{PGL}_2(\overline{\mathbb{Q}})$.

By a calculation in Maple, we find a basis $\{y_1, y_2\}$ of solutions for St_{A_4}, with $y_1 y_2 = \sqrt{\frac{a+1}{a}}, y_1^4 = \frac{\sqrt{3}(a-1)+2\sqrt{a^2-a+1}}{a}, a^3 + t + 1 = 0$. From this we see that the Picard-Vessiot extension $\overline{\mathbb{Q}}(t)(y_1, y_2)$ lies in the degree 48 extension

$$K := \overline{\mathbb{Q}}(t) \left(a, \sqrt{a^2 - a + 1}, \sqrt{\frac{a+1}{a}}, \sqrt[4]{\frac{\sqrt{3}(a-1) + 2\sqrt{a^2 - a + 1}}{a}} \right)$$

of $\overline{\mathbb{Q}}(t)$, where $a^3 + t + 1 = 0$. In order to determine G precisely, we will make use of the local exponents of St_{A_4}. Let E_p denote the set of local exponents at the point p. Then we have $E_0 = \{0, \frac{1}{2}\}, E_{-1} = \{\frac{1}{4}, \frac{-1}{12}\}$ and $E_\infty = \{0, \frac{1}{3}\}$. Now Proposition 5.1 in [**vdPU00**], provides us with an element $g_{-1} \in G$, which is conjugated to $e^{2\pi i D}$, where D is the diagonal matrix with $\frac{1}{4}, \frac{-1}{12}$ on the diagonal. So the eigenvalues of g_{-1} are $\{e^{\frac{\pi i}{2}}, e^{\frac{-\pi i}{6}}\}$, and therefore $\mathrm{Det}(g_{-1}) = e^{\frac{\pi i}{3}}$. We have $g_{-1}^3 = -i \cdot Id$. So the kernel of the natural map $G \to G^p$ has at least order 4, and we find that G has at least order 48. We already found that G had maximally order 48, so G is a central extension of A_4 by C_4 of order 48, and K is a Picard-Vessiot extension for St_{A_4}. □

We will now give the pullback function for a second order differential operator $L \in k(x)[\partial_x]$ with projective Galois group A_4. After applying a shift, we can suppose (as in the case of $St_{A_4^{\mathrm{SL}_2}}$) that $\mathrm{Sym}(L, 4)$ has a right-hand factor ∂_x. This shift does not change the pullback function. This shift can be found in the following way. By representation theory it follows that the operator $\mathrm{Sym}(L, 4)$ has two degree one right-hand factors, say $(\partial_x + b_1)$ and $(\partial_x + b_2)$. The b_i are rational solutions of the Riccati equation corresponding to $\mathrm{Sym}(L, 4)$, and therefore the b_i can be computed. The group $\mathrm{Gal}(\overline{k}/k)$ acts on $\{b_1, b_2\}$, so we find that $b_1, b_2 \in k'(x)$ for some minimal field $k' \subset \overline{k}$ of degree ≤ 2 over k. In fact this field k' is the field defined the beginning of this chapter. To see this, let $u = \frac{y'}{y}$ be an algebraic solution of the Riccati equation R_L of degree 4 over $\overline{k}(x)$. Then the sum b of the conjugates of u under the differential Galois group of L is a rational solution of the Riccati equation corresponding to $\mathrm{Sym}(L, 4)$, and we see that $b \in k'(x)$ if and only if the minimal polynomial of u is defined over $k'(x)$.

Theorem 2.3. — *Let $L = \partial_x^2 + a_1 \partial_x + a_0$, with $a_0, a_1 \in k(x)$, be a differential operator with projective Galois group A_4 such that* $\mathrm{Sym}(L, 4)$ *has a right-hand factor ∂_x. Then*

L is the pullback of St_{A_4}, with pullback function $\phi(t) := \frac{9s}{a_0}(\frac{a_0'}{a_0} + 2a_1)^2, s = \frac{1}{48}$. The only other (weak) pullback is obtained by composition with the unique non-trivial automorphism of St_{A_4}.

Proof. — We will first show, that for the suitable choice of $\phi(t)$ no shift is needed. By Klein's theorem, there exists $\phi : \overline{k}(t)[\partial_t] \to \overline{k}(x)[\partial_x]$, with $L = (\phi(t)')^2\phi(St_{A_4})$. The $\mathrm{Sym}(St_{A_4}, 4)$ has a right-hand factor ∂_t, and because $\phi(\mathrm{Sym}(St_{A_4}, 4)) = \mathrm{Sym}(\phi(St_{A_4}), 4)$, the $\mathrm{Sym}(L, 4)$ has a right-hand factor $\phi(\partial_t)$. The $\mathrm{Sym}(L, 4)$ has two right-hand factors of degree one, ∂_x and $\partial_x - u$ for some $u \in \overline{k}(x)$, so $\phi(\partial_t) \in \{\frac{1}{\phi(t)'}\partial_x, \frac{1}{\phi(t)'}(\partial_x - u)\}$. If $\phi(\partial_t) = \frac{1}{\phi(t)'}\partial_x$ we are done, otherwise consider the automorphism $\psi := \phi_{\frac{-t}{t+1}, \frac{1}{12(t+1)}}$ of St_{A_4}. Then $\phi \circ \psi$ is the other possible pullback, with a different image for ∂_t, so this image must be $\frac{1}{\phi(t)'}\partial_x$. Therefore we may suppose that ϕ has no shift. We will now calculate $\phi(t)$.

The formula for $\phi(t)$ can be obtained using the following trick. Write St_{A_4} as $\partial_t^2 + s_1\partial_t + s_0$, so $s_0 = \frac{s}{t(t+1)^2}$ and $s_1 = \frac{8t+3}{6t(t+1)}$. In the following we will use $\phi(f)' = \phi(t)'\phi(f')$, where $f \in \overline{k}(t)$, and $'$ denotes $\frac{d}{dx}$ or $\frac{d}{dt}$. Applying ϕ to $t = \frac{s}{s_0}\big(\frac{1}{t+1}\big)^2$, we get $\phi(t) = \frac{s}{\phi(s_0)}\big(\frac{1}{\phi(t+1)}\big)^2 = \frac{s}{(\phi(t)')^2\phi(s_0)}\big(\frac{\phi(t)'}{\phi(t+1)}\big)^2$. Furthermore $a_0 = (\phi(t)')^2\phi(s_0)$, so $\phi(t) = \frac{s}{a_0}\big(\frac{\phi(t)'}{\phi(t+1)}\big)^2$. We are done if we can prove $\frac{\phi(t)'}{\phi(t+1)} = 3\frac{a_0'}{a_0} + 6a_1$. Using $a_1 = \phi(t)'\phi(s_1) - \frac{\phi(t)''}{\phi(t)'}$, we can write $3\frac{a_0'}{a_0} + 6a_1$ as $3\big(2\frac{\phi(t)''}{\phi(t)'} + \frac{\phi(s_0)'}{\phi(s_0)}\big) + 6\big(\phi(t)'\phi(s_1) - \frac{\phi(t)''}{\phi(t)'}\big) = 3\phi(t)'\phi(\frac{s_0'}{s_0}) + 6\phi(t)'\phi(s_1) = \phi(t)'\phi\big(-3\big(\frac{1}{t} + \frac{2}{t+1}\big) + \frac{8t+3}{t(t+1)}\big) = \frac{\phi(t)'}{\phi(t+1)}$, which finishes the proof. □

Remark 2.4. — This pullback formula was found using *semi-invariants*. The representation of A_4 in the $\mathrm{PGL}(\overline{k}x_1 + \overline{k}x_2)$ induces an action of A_4 on $\overline{k}[x_1, x_2]$. A polynomial P in this ring is a semi-invariant if $\forall\ \sigma \in A_4\ \exists\ c_\sigma \in \overline{k}^*$ such that $\sigma(P) = c_\sigma P$. There are two semi-invariants $H_1(x_1, x_2), H_2(x_1, x_2)$ of degree 4, such that for a basis of solutions $\{y_1, y_2\}$ of St_{A_4} we have $\frac{H_1(y_1,y_2)^3}{H_2(y_1,y_2)^3} = t + 1$. Let $\{v_1, v_2\}$ be a basis of solutions of L. Then we find $\frac{H_1(v_1,v_2)^3}{H_2(v_1,v_2)^3} = \phi(t + 1)$. The expressions $H_1(v_1, v_2), H_2(v_1, v_2)$ are so-called *exponential solutions* of $\mathrm{Sym}(L, 4)$, i.e., $\frac{H_i(v_1,v_2)'}{H_i(v_1,v_2)} \in \overline{k}(x)$. These exponential solutions can be found (up to constants). We can also give a formula for one of these exponential solutions in terms of the other and the coefficients of L. So if we suppose $H_1(v_1, v_2) = 1$, we find a formula for the pullback function in terms of the coefficients of L.

Corollary 2.5. — *Let $L = \partial_x^2 + a_1\partial_x + a_0$ be a differential operator, with projective Galois group A_4. There are two differential operators $L_i, i = 1, 2$ obtained from L by a shift $\partial_x \mapsto \partial_x + b_i$, such that $\mathrm{Sym}(L_i, 4)$ has a right-hand factor ∂_x. Let F_i be the pullback function of L_i as in Theorem 2.3. Then $F_2 = \frac{-F_1}{F_1+1}$ and $b_2 = b_1 - \frac{F_1'}{12(F_1+1)}$.*

Proof. — We recall that the unique non-trivial automorphism of St_{A_4} is $\phi_{\frac{-t}{t+1},\frac{1}{12(t+1)}}$. We have $F_1^2\phi_{F_1}(St_{A_4}) = L_1 = \phi_{t,b_1-b_2}(L_2) = F_2^2\phi_{F_2,b_1-b_2}(St_{A_4})$. Because $b_2 \neq b_1$ we must have $\phi_{F_2,b_1-b_2} = \phi_{F_1} \circ \phi_{\frac{-t}{t+1},\frac{1}{12(t+1)}}$, so $F_2 = \frac{-F_1}{F_1+1}$ and $b_2 = b_1 - \frac{F_1'}{12(F_1+1)}$. □

2.2. Projective Galois group S_4 or A_5. — These two cases can be treated in almost the same way as the A_4-case. We will only give the differences. The new standard equations we will use are:

$$St_{G^p} := \partial_t^2 + \frac{8t+3}{6t(t+1)}\partial_t + \frac{s}{t(t+1)^2}$$

with $s = \frac{5}{576}$ for $G^p = S_4$, and $s = \frac{11}{3600}$ for $G^p = A_5$. In both cases there are no automorphisms (i.e., no automorphisms of $\overline{\mathbb{Q}}(t)[\partial_t]$ mapping St_{G^p} to a multiple of itself). Using representation theory we find that S_4 and A_5 have a unique semi-invariant of degree $m = 6, 12$, respectively. The new standard equations are chosen in such a way that $\mathrm{Sym}(St_{G^p}, m)$ has a right-hand factor ∂_t.

Proposition 2.6. — *The Galois group of St_{G^p}, $G^p \in \{S_4, A_5\}$ is a central extension of G^p by the cyclic group C_6.*

Proof. — We start by calculating G_1, the Galois group of St_{S_4}. The local exponents of St_{S_4} are given by $E_0 = \{0, \frac{1}{2}\}, E_{-1} = \{\frac{5}{24}, -\frac{1}{24}\}$ and $E_\infty = \{0, \frac{1}{3}\}$. As in the A_4 case, we conclude that there is an element $g_{-1} \in G_1$ of order 24 with eigenvalues $\{e^{-\frac{1}{12}\pi i}, e^{\frac{5}{12}\pi i}\}$, so with $\mathrm{Det}(g_{-1}) = e^{\frac{1}{3}\pi i}$. We have that $g_{-1}^4 = e^{-\frac{1}{3}\pi i}\cdot Id$ is an element in the kernel of the map $G_1 \mapsto G_1^p = S_4$. We find that this kernel has at least order 6, so G_1 has order ≥ 144. Reasoning as in the A_4-case we find that G_1 has order 144, and it is a central extension of S_4 by C_6.

We will now calculate the Galois group G_2 of St_{A_5}. The local exponents of St_{A_5} are given by $E_0 = \{0, \frac{1}{2}\}, E_{-1} = \{\frac{11}{60}, -\frac{1}{60}\}$ and $E_\infty = \{0, \frac{1}{3}\}$. So there is an element $h_{-1} \in G_2$ of order 60, with eigenvalues $e^{-\frac{1}{30}\pi i}, e^{\frac{11}{30}\pi i}$. We have $\mathrm{Det}(h_{-1}) = e^{\pi i \frac{1}{3}}$, so h_{-1}^5 is an element in the kernel of the map $G_2 \mapsto G_2^p = A_5$. Again reasoning as in the A_4-case we get that G_2 has order 360, and it is a central extension of A_5 by C_6. □

Let a differential operator L with projective Galois group $G^p \in \{S_4, A_5\}$ be given. Then after applying a shift we can assume that $\mathrm{Sym}(L, m)$ has a right-hand factor ∂_x, where $m = 6$ if $G^p = S_4$ and $m = 12$ if $G^p = A_5$. The shift we have to apply is $\partial_x \mapsto \partial_x + \frac{b}{m}$, with b a rational solution of the Riccati equation corresponding to $\mathrm{Sym}(L, m)$. From the uniqueness of b it also follows that the field k' as defined in the beginning of this chapter is equal to k (compare the A_4-case).

It is now clear that we get the following generalization of Theorem 2.3.

Theorem 2.7. — *Let $L = \partial_x^2 + a_1\partial_x + a_0$, with $a_0, a_1 \in k(x)$, be a differential operator, with projective Galois group $G^p \in \{A_4, S_4, A_5\}$. Set $m = 4, s = \frac{1}{48}$ if $G^p = A_4$, set $m = 6, s = \frac{5}{576}$ if $G^p = S_4$, and set $m = 12, s = \frac{11}{3600}$ if $G^p = A_5$. If* $\mathrm{Sym}(L, m)$

has a right-hand factor ∂_x, then L is the pullback of St_{G^p}, with pullback function $\phi(t) := \frac{9s}{a_0}\left(\frac{a_0'}{a_0} + 2a_1\right)^2$.

2.3. Projective Galois group D_n, $n \geq 2$. — Let $\widetilde{L} \in k(x)[\partial_x]$ be a second order differential operator with $G^p = D_n$. Then for $n \geq 3$, we have that $\mathrm{Sym}(\widetilde{L}, 2)$ has precisely one right-hand factor of degree one over $\overline{k}(x)$, say $\partial_x + a$. The shift $\partial_x \mapsto \partial_x - \frac{a}{2}$ transforms $\widetilde{L}$ into a differential operator L, such that $\mathrm{Sym}(L, 2)$ has a right-hand factor ∂_x. In the case $G^p = D_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the operator $\mathrm{Sym}(\widetilde{L}, 2)$ has three degree one right-hand factors. So there are three possible shifts transforming $\widetilde{L}$ into a differential operator L such that $\mathrm{Sym}(L, 2)$ has a right-hand factor ∂_x. We note that from the above we can conclude that the field k' defined in the beginning of this chapter satisfies $k' = k$ for $G^p = D_n, n > 2$ and $[k' : k] \leq 3$ for $G^p = D_2$ (see the A_4-case for details).

A calculation shows that if $L = \partial_x^2 + a_1\partial_x + a_0$ satisfies $\mathrm{Sym}(L, 2) = * \cdot \partial_x$, then $\frac{a_0'}{a_0} = -2a_1$ and a basis of solutions is given by $\{y, \frac{1}{y}\}, y = e^{\int \sqrt{-a_0}dx}$. We will now calculate the possibilities for the differential Galois group G of an operator L with these properties, and moreover with $G^p = D_n, n \geq 2$. We have that the extension $\overline{k}(t) \subset K^p = \overline{k}(t)(y^2)$ is Galois with Galois group D_n. So $\overline{k}(t)(y^2)$ is a differential field, and consequently $K = \overline{k}(t)(y)$ is a differential field, too. Therefore it is a Picard-Vessiot extension for L. The extension $\overline{k}(t)(y^2) \subset \overline{k}(t)(y)$ has degree one or two.

A small calculation shows that on the basis $\{y, \frac{1}{y}\}$, the differential Galois group G lies in $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \cup \left(\begin{smallmatrix} 0 & b \\ b^{-1} & 0 \end{smallmatrix}\right)$. The image of G in PGL_2 must be

$$G^p = D_n = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix} \right\rangle$$

where ζ_{2n} is a $2n$-th root of unity. We have $|G|/|G^p| \leq 2$ and we find that $|G| = |G^p|$ can only occur when n is odd. Then $G^p \cong G = \left\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{smallmatrix}\right) \right\rangle$ or $G = \left\langle \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{smallmatrix}\right) \right\rangle$, where we can take $\zeta_n = -\zeta_{2n}$. In case $|G| = 2|G^p|$ we have $D_{2n} \cong G = \left\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{smallmatrix}\right) \right\rangle$.

We will use the following new standard equations:

$$St_{D_n} := \partial_t^2 + \frac{t}{t^2 - 1}\partial_t - \frac{1}{4n^2(t^2 - 1)}.$$

For $n > 2$ we have one non-trivial automorphism of St_{D_n}, namely $\psi = \phi_{-t}$. The group of automorphisms of St_{D_2} is isomorphic to S_3, and has generators ψ, ψ_1, with ψ as above, and $\psi_1 = \phi_{\frac{t+3}{t-1}, -\frac{1}{4(t-1)}}$. The operator $\mathrm{Sym}(St_{D_n}, 2)$ has a right-hand factor ∂_t. The differential Galois group of St_{D_n} is D_{2n}, which follows from the fact that $\{y, \frac{1}{y}\}$, $y = (t + \sqrt{t^2 - 1})^{\frac{1}{2n}}$ is a basis of solutions. Indeed, y satisfies the irreducible polynomial $T^{4n} - 2tT^{2n} + 1$.

Lemma 2.8. — $\mathrm{Sym}(St_{D_n}, 2n)$ *has a basis of rational solutions* $\{1, t\}$. *Furthermore* t *is up to constants the unique rational solution of the right-hand factor* $\partial_t^2 + \frac{t}{t^2-1}\partial_t - \frac{1}{(t^2-1)}$ *of* $\mathrm{Sym}(St_{D_n}, 2n)$.

Proof. — Write $St_{D_n} = \partial_t^2 + s_1\partial_t + s_0$. We have that St_{D_n} has a basis of solutions $\{y, \frac{1}{y}\}$. A direct calculation (or an examination of the explicit form of the solutions presented above) shows that for any non-zero integer k, the operator $\partial_t^2 + s_1\partial_t + k^2 s_0$ has as basis of solutions $\{y^k, y^{-k}\}$. Therefore $\partial_t^2 + s_1\partial_t + k^2 s_0$ is a right-hand factor of $\mathrm{Sym}(St_{D_n}, k)$. In particular $\partial_t^2 + \frac{t}{t^2-1}\partial_t - \frac{1}{(t^2-1)}$ is a right-hand factor of $\mathrm{Sym}(St_{D_n}, 2n)$. Now observe that t is a solution of $\partial_t^2 + \frac{t}{t^2-1}\partial_t - \frac{1}{(t^2-1)}$. We note that $1 = y^n \cdot y^{-n}$ is also a solution of $\mathrm{Sym}(St_{D_n}, 2n)$. It is easily seen that the space of rational solutions of $\mathrm{Sym}(St_{D_n}, 2n)$ is 2-dimensional. Indeed the differential Galois group D_{2n} of St_{D_n} is generated by σ, τ with $\sigma(y) = \zeta_{2n} y$, with ζ_{2n} a primitive $2n$-th root of unity, and $\tau(y) = y^{-1}$. A basis of solutions of $\mathrm{Sym}(St_{D_n}, 2n)$ is $\{y^{2n}, y^{2n-2}, \cdots, y^{-2n}$, and it immediately follows that $\{1, y^{2n} + y^{-2n}\}$ is a basis of the rational solutions of $\mathrm{Sym}(St_{D_n}, 2n)$. □

Theorem 2.9. — *Let* $L = \partial_x^2 + a_1\partial_x + a_0$ *be a differential operator, with projective differential Galois group* $D_n, n \geq 3$, *such that* $\mathrm{Sym}(L, 2)$ *has a right-hand factor* ∂_x. *The right-hand factor* $\partial_x^2 + a_1\partial_x + 4n^2 a_0$ *of* $\mathrm{Sym}(L, 2n)$ *has, up to constants, a unique rational solution, say* a. *Write* $b := \frac{a'}{a}$, *then* b *is independent of the choice of* a. *Now* L *is the pullback of* St_{D_n}, *with pullback function* $\phi(t) := \left(1 + \frac{b^2}{4n^2 a_0}\right)^{-\frac{1}{2}}$. *The only other pullback function is* $-\phi(t)$.

Proof. — The proof is somewhat similar to the proof of Theorem 2.3. The fact that no shift is needed follows from the fact that $\mathrm{Sym}(L, 2)$ has a unique degree one right-hand factor. An argument as in the proof of Lemma 2.8 shows the existence and unicity of the rational solution a. As before, write $St_{D_n} = \partial_t^2 + s_1\partial_t + s_0$. From the expression $s_0 = \frac{-1}{4n^2(t^2-1)}$ it follows that $t = \left(1 + \frac{1}{4n^2 s_0 t^2}\right)^{-\frac{1}{2}}$. The pullback map transforms this expression into $\phi(t) = \left(1 + \frac{\phi(t)'^2}{4n^2 a_0 \phi(t)^2}\right)^{-\frac{1}{2}}$, since $a_0 = \phi(t)'^2 \phi(s_0)$. By the previous lemma, t is a rational solution of $\partial_t^2 + s_1\partial_t + 4n^2 s_0$. Therefore $\phi(t)$ is a rational solution of $\phi\left(\partial_t^2 + s_1\partial_t + 4n^2 s_0\right) = \left(\frac{1}{\phi(t)'}\right)^2 (\partial_x^2 + a_1\partial_x + 4n^2 a_0)$. Consequently $b = \frac{\phi(t)'}{\phi(t)}$, and it follows that $\phi(t) = \left(1 + \frac{b^2}{4n^2 a_0}\right)^{-\frac{1}{2}}$. We see that a different choice for the square root changes $\phi(t)$ into $-\phi(t)$. It also follows from Klein's Theorem (1.9) that $-\phi(t)$ is the only other possible pullback function. □

Remark 2.10. — In the above proof, we see that $\phi(t) = c \cdot a$ for some constant $c \in \overline{k}$, and a as in Theorem 2.9.

For the D_2 case, we get the following variant of Theorem 2.9.

Theorem 2.11. — *Let* $L = \partial_x^2 + a_1\partial_x + a_0$ *be a differential operator, with projective differential Galois group* D_2*, such that* $\mathrm{Sym}(L,2)$ *has a right-hand factor* ∂_x*. Then* $\mathrm{Sym}(L,2)$ *has three right-hand factors of degree one, say* $\partial_x, \partial_x+b_1, \partial_x+b_2$*. Write* $b := \frac{4b_1b_2}{b_1+b_2}$*. Now* L *is the pullback of* St_{D_2}*, with pullback function* $\phi(t) := (1+\frac{b^2}{4n^2a_0})^{-\frac{1}{2}}$*. The other (weak) pullbacks are obtained by composition with automorphisms of* St_{D_2}*.*

Proof. — By the argument in the proof of Theorem 2.9, we only have to show that $b = \frac{\phi(t)'}{\phi(t)}$. We have that $\mathrm{Sym}(St_{D_2}, 2)$ has three right-hand factors of degree one $\partial_t, \partial_t + \frac{1}{2}\frac{1}{t+1}, \partial_t + \frac{1}{2}\frac{1}{t-1}$. So the three degree one right-hand factors of L are $\partial_x, \partial_x + \frac{1}{2}\frac{\phi(t)'}{\phi(t)+1}, \partial_x + \frac{1}{2}\frac{\phi(t)'}{\phi(t)-1}$. Therefore we can write $b_1 = \frac{1}{2}\frac{\phi(t)'}{\phi(t)+1}, b_2 = \frac{1}{2}\frac{\phi(t)'}{\phi(t)-1}$. It follows that $b = \frac{\phi(t)'}{\phi(t)}$. □

***Algorithm 2.12* (Determining n)**. — In the above, we assumed that the projective differential Galois group was known. For a second order differential operator $L \in k(x)[\partial_x]$ it is not hard to determine whether or not the projective differential Galois group is a group $D_n, n \in \mathbb{N}_{\geq 2} \cup \infty$. For completeness, we give an algorithm to determine n in case k is a number field, and L is a second order differential operator with dihedral differential Galois group. We note that this is a known algorithm (see [**BD79**, Section 6]).

As above we may assume that $\mathrm{Sym}(L,2)$ has a right-hand factor ∂_x. So L has a solution $y = e^{\int \sqrt{-a_0}dx}$. Let $K = \overline{k}(x)(y,y')$ be a Picard-Vessiot extension for L. Consider the tower of fields $\overline{k}(x) \subset \overline{k}(x,\sqrt{-a_0}) \subset K$, where $\sqrt{-a_0} = \frac{y'}{y}$. Since we assume the projective Galois group to be dihedral, it follows that a_0 is not a square in $\overline{k}(x)$. The field extension $\overline{k}(x,\sqrt{-a_0}) \subset K = \overline{k}(x)(y)$ is infinite in case $n = \infty$, and otherwise cyclic of order n or $2n$. Consider the differential $\omega := 2\sqrt{-a_0}dx$ on the hyperelliptic curve H with function field $\overline{k}(H) := \overline{k}(x,\sqrt{-a_0})$. We want to find the degree over $\overline{k}(x,\sqrt{-a_0})$ of the solution y^2 of the equation $\omega = \frac{dy^2}{y^2}$.

Suppose y is algebraic over $\overline{k}(x)$. Then by the action of the differential Galois group we find that $y^{2n} \in \overline{k}(H)$. Let $D := \mathrm{Div}(y^{2n})$ be the divisor of y^{2n}. If $D = \sum a_i[p_i]$, then the residue of $n\omega = \frac{dy^{2n}}{y^{2n}}$ in p_i is a_i. We also find that ω has only poles of order 1 and no zeroes. So a necessary condition for y to be algebraic is $\mathrm{ord}_h(\omega) \in \{-1,0\}, \mathrm{res}_h(\omega) \in \mathbb{Q}\ \forall\ h \in H$. In the following we will assume ω to satisfy these easily verifiable conditions.

Let m_1 be the least common multiple of the denominators of all nonzero residues of ω. Then $D_1 := \sum \mathrm{res}_h(m_1\omega)[h]$ is a divisor on H, and we want to find the smallest integer m_2 such that m_2D_1 is a principal divisor. If such an integer m_2 exists, then $n = m_1m_2$ and otherwise $n = \infty$. Indeed if $m_2D_1 = \mathrm{Div}(f), f \in \overline{k}(H)$, then $\frac{df}{f} = m_1m_2\omega$ and we can take $y^2 = f^{\frac{1}{m_1m_2}}$. Because $m_1m_2\omega$ is defined over k, one finds using Hilbert theorem 90 that we may suppose f to be defined over k (compare the argument in the proof of Lemma 1.5).

We want to find the order m_2 of the element $D_1 \in \mathrm{Jac}(H)(k)$. We will use the following known result.

Lemma 2.13. — *Let k be a number field, and A/k an abelian variety. Let $\mathfrak{p}$ be a prime ideal in the ring of integers $\mathcal{O}_k$ of k, extending the prime number p. Suppose:*

1. *A has good reduction at $\mathfrak{p}$,*
2. *the ramification index $e_{\mathfrak{p}}$ is smaller than $p-1$.*

Then reduction modulo $\mathfrak{p}$ yields an injective homomorphism

$$A(k)_{tors} \longrightarrow A \bmod \mathfrak{p}(\mathcal{O}_k/\mathfrak{p}).$$

Proof. — Let $a \in A(k)_{\mathrm{tors}}$ be a point of prime order ℓ. The subgroup $\langle a \rangle \subset A$ defines a constant group scheme of order ℓ over k. Since $e_{\mathfrak{p}} < p-1$, by Theorem 4.5.1 in [**Tat97**] this group scheme extends uniquely to the finite flat group scheme $\underline{\mathbb{Z}/\ell\mathbb{Z}}_{\mathcal{O}_{\mathfrak{p}}}$, where $\mathcal{O}_{\mathfrak{p}}$ denotes the completion of $\mathcal{O}$ at $\mathfrak{p}$. This shows that a reduces modulo $\mathfrak{p}$ to a point of again order ℓ. So the kernel of the reduction map $A(k)_{\mathrm{tors}} \to A \bmod \mathfrak{p}(\mathcal{O}_k/\mathfrak{p})$ contains no points of prime order, and therefore the map is injective. □

Now let $\mathfrak{p}$ be a prime ideal in the ring of integers of k, such that H has good reduction modulo $\mathfrak{p}$ and $e_{\mathfrak{p}} < p-1$. Then we can apply the above lemma to the abelian variety $\mathrm{Jac}(H)$, and the prime ideal $\mathfrak{p}$. It follows that if $\mathrm{ord}(D_1) < \infty$ then this order equals the order of D_1 in $\mathrm{Jac}(H) \bmod \mathfrak{p}$. We can calculate this order $\widetilde{m_2}$ using the algorithm in [**GH00**, 3.2]. Write $\widetilde{n} = \widetilde{m_2}m_1$. If $\partial_x^2 + a_1\partial_x + 4\widetilde{n}^2 a_0$ has a rational solution then $n = \widetilde{n}$, otherwise $n = \infty$.

Note that it is not strictly necessary to calculate the order of D_1 in a reduction of H. The Hasse-Weil bound gives an upper bound for this order. This produces a number N which is an upper bound for n in case n is finite. Now n is the smallest integer such that $\partial_x^2 + a_1\partial_x + 4n^2 a_0$ has a rational solution. If there is no such solution for $n < N$, then $n = \infty$.

We remark that in case $k = \mathbb{Q}$ the order of D_1 can be calculated using the computer algebra package MAGMA.

3. A generalization of Klein's theorem

In this section we will give a variant of Klein's theorem for third order operators. We will define a notion of standard operator, such that each differential operator L with finite irreducible differential Galois group $G \subset \mathrm{SL}_3$ is a weak pullback of a standard operator for G. We start by giving an alternative construction of standard operators of order 2, more in line with our construction of order 3 standard operators, which we will give subsequently. In this section we will work over an algebraically closed field of characteristic zero, denoted by C.

3.1. Standard operators of order 2 revisited. — Let V be a 2-dimensional vector space over C, and let $G \subset \mathrm{SL}(V)$ be an irreducible finite group.

Notation 3.1

- $Z(G)$ denotes the center of G. We have $G^p \cong G/Z(G)$ (with G^p the image of G in $\mathrm{PGL}(V)$).
- $\mathbb{P}(V) := \mathrm{Proj}\ C[V]$, where $C[V]$ is the *symmetric algebra* of V.
- $K^p := C(\mathbb{P}(V))$, the function field of $\mathbb{P}(V)$. Note that $K^p = C[V]_{((0))}$, i.e., K^p consists of quotients of homogeneous elements of $C[V]$ of the same degree.

There is an action of G^p on K^p, and by Lüroth's theorem we can write $(K^p)^{G^p}$ as $C(t)$, where t is unique up to a Möbius-transformation. We will construct a Galois extension $K^p \subset K$, such that $\mathrm{Gal}(K/C(t)) \cong G$, and a G-invariant C-vector space $W \subset K$ that is G-isomorphic to V. The corresponding monic differential operator with solution space W will be called a standard operator for G.

***Construction 3.2* (Second order standard operators).** — For $0 \neq \ell \in V$, we can see $\frac{V}{\ell}$ as a set of functions on $\mathbb{P}(V)$. This gives an injection $\frac{V}{\ell} \hookrightarrow K^p$. For $\sigma \in G$ we have $\sigma\big(\frac{V}{\ell}\big) = \frac{\ell}{\sigma(\ell)}\frac{V}{\ell}$. The set $\frac{V}{\ell}$ is not G-invariant, for $\sigma(\frac{V}{\ell}) = \frac{V}{\ell}\ \forall\ \sigma \in G$ would imply $\frac{\ell}{\sigma(\ell)} \in C\ \forall\ \sigma \in G$, but there are no G-invariant lines in V. Roughly spoken, we want to construct some f in an extension of K^p such that $f\frac{V}{\ell}$ is a G-invariant vector space.

The map $c : G \to (K^p)^*,\ \sigma \mapsto \frac{\ell}{\sigma(\ell)}$ is a 1-cocyle in $H^1(G,(K^p)^*)$. We want to use Hilbert theorem 90 to construct a G-invariant space, but the problem is that G is not the Galois group of $K^p/C(t)$, which is G^p. We can avoid this problem by considering the map $d : G^p \to (K^p)^*, \tau \mapsto c(\sigma)^2$, where $\sigma \in G$ is some lift of $\tau \in G^p$. The value $c(\sigma)^2$ is independent of the chosen lift. So d is an element of $H^1(G^p,(K^p)^*)$, and therefore Hilbert theorem 90 implies that there exists an $f \in (K^p)^*$, with $d(\tau) = \frac{f}{\tau(f)}\ \forall\ \tau \in G^p$. In other words, $\frac{f}{\bar\sigma(f)} = \frac{\ell^2}{\sigma(\ell)^2}\ \forall\ \sigma \in G$, where $\bar\sigma \in G^p$ denotes the image of σ. This f is unique up to multiplication by an element in $C(t)^*$. We define $K := K^p(f_2)$, by $f_2^2 = f$. We have $f_2 \notin K^p$, for otherwise $\widetilde{V} := f_2\frac{V}{\ell}$ would have a G^p-action, which is impossible because $G \twoheadrightarrow G^p$ has no section. The field extension $C(t) \subset K$ is a Galois extension. This follows from the fact that for $\tau \in G^p$ with lift $\sigma \in G$, we have $\tau(f) = c(\sigma)^2 f$, so the square roots of the conjugates of f are present. Note that the choice of $\ell \in V$ is irrelevant, because for an other choice ℓ', we can take $f_2' = f_2\frac{\ell'}{\ell}$, which leaves $\widetilde{V}$ unchanged.

Lemma 3.3. — *Using the above notations, there is a natural isomorphism* $\mathrm{Gal}(K/C(t)) \cong G$, *and* $\widetilde{V} := f_2\frac{V}{\ell}$ *is G-invariant and G-isomorphic to V.*

Proof. — We will extend the G^p-action on K^p to a G-action on K. Write $K = K^p + K^p f_2$. We define a G-action on K by $\sigma(\alpha + \beta f_2) = \bar\sigma(\alpha) + \bar\sigma(\beta)\frac{\sigma(\ell)}{\ell} f_2$. Using

$\frac{f}{\sigma(f)} = \frac{\ell^2}{\sigma(\ell)^2}$ it is clear that G acts by automorphisms. A counting argument shows $G = \mathrm{Gal}(K/C(t))$, and clearly $\widetilde{V}$ is G-isomorphic to V. □

To $\widetilde{V}$ corresponds a differential operator $\widetilde{L} = \partial_t^2 + a\partial_t + b \in C(t)[\partial_t]$ with solution space $\widetilde{V}$. We will now show that the normalization L of $\widetilde{L}$ corresponds to a different choice for f. Let $\{y_1, y_2\}$ be a basis of solutions of $\widetilde{L}$. Then $q := \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ lies in $C(t)$ because the differential Galois group is unimodular. We have $q' = -aq$. We can normalize $\widetilde{L}$ by making the shift $\partial_t \mapsto \partial_t - \frac{1}{2}a$ which changes the solution space $\widetilde{V}$ into $q_2^{-1}\widetilde{V}$, with $q_2^2 = q$. This corresponds to replacing the f above by $q^{-1}f$. This is allowed, because f was defined up to multiplication by elements of $C(t)$. We have that K^p is a purely transcendental extension of C. The Galois extension $C(t) \subset K^p$, with Galois group G^p is ramified in three points, which we can suppose to be $\{0, 1, \infty\}$ by making an appropriate choice for t. For any such t we call the constructed operator L a standard operator for G.

With the appropriate choice for t, the constructed differential operator L is equal to the differential operator St_G defined in Section 1.1. This follows from Theorem 3.9, which we prove for third order operators, but which is also valid for second order operators.

3.2. Standard operators of order 3. — Now let V be a 3-dimensional vector space over C, and again let $G \subset \mathrm{SL}(V)$ be an irreducible finite group. We will now give a construction of third order standard operators, with projective differential Galois group isomorphic to G^p. This construction is to some extend a copy of the construction in the previous section.

Definition 3.4. — Let $Z \subset \mathbb{P}(V)$ be a G^p-invariant irreducible curve, such that $Z/G^p \cong \mathbb{P}^1_C$. Note that by Remark 3.8 such a curve always exists. We write $C(t) := C(Z/G^p)$. We define a standard operator corresponding to Z and G^p, to be a differential operator $L_Z \in C(t)[\partial_t]$ given by the construction below.

***Construction 3.5* (Standard operator corresponding to Z)**. — For the construction of standard operators, we must consider two different cases for G^p. Let $\pi : \mathrm{SL}(V) \to \mathrm{PGL}(V)$ be the canonical map. We have that the center of G is trivial, or a cyclic group of order three. We will write C_3 for a cyclic group of order three. The cases we have to consider are:

1. the natural map $\pi^{-1}(G^p) \to G^p$ has no section, so $G = \pi^{-1}(G^p)$,
2. the natural map $\pi^{-1}(G^p) \to G^p$ has a section, so $\pi^{-1}(G^p) \cong C_3 \times G^p$. In this case $G = \pi^{-1}(G^p)$ or $G \cong G^p$.

We will now give the construction of standard operators case by case.

Case 1. For $0 \neq \ell \in V$, we regard $\frac{V}{\ell}$ as a set of functions on $\mathbb{P}(V)$, which induce functions on Z. This gives a map $\frac{V}{\ell} \to C(Z)$, which is an injection, for otherwise Z

would be a line in $\mathbb{P}(V)$. This is impossible because G is irreducible, and therefore has no G-invariant planes.

As in the construction of second order standard operators, we consider the cocycle $d : G^p \to C(Z)^*, \sigma \mapsto \left(\frac{\ell}{\sigma(\ell)}\right)^3$. Then $d \in H^1(G^p, C(Z)^*)$, and therefore Hilbert theorem 90 implies that there exists an $f \in C(Z)^*$, with $d(\tau) = \frac{f}{\tau(f)}\ \forall\ \tau \in G^p$. Now take f_3, with $f_3^3 = f$. We have $f_3 \notin C(Z)$, for otherwise $f_3\frac{V}{\ell}$ would be G^p-invariant, which is impossible because $G \twoheadrightarrow G^p$ has no section. So we consider the degree 3 extension $C(Z) \subset C(Z)(f_3)$. We have that $C(t) \subset C(Z)(f_3)$ is a Galois extension. As in the second order case, we have $\mathrm{Gal}(C(Z)(f_3)/C(t)) = G$, and $\widetilde{V} := f_3\frac{V}{\ell}$ is G-invariant.

To $\widetilde{V}$ corresponds a unique monic differential operator $\widetilde{L}$. As in the second order case, normalizing $\widetilde{L}$ corresponds to making a different choice for f. This normalization L_Z of $\widetilde{L}$ is now uniquely determined and will be called the standard differential operator corresponding to Z. Note that the standard operator depends on the choice of t. By construction, the differential Galois group of L_Z is G.

Case 2. Let H be a lift of G^p in $\mathrm{SL}(V)$ that is isomorphic to G^p. Now Hilbert theorem 90, applied to $H^1(H, C(Z)^*)$, implies the existence of an $f \in C(Z)^*$ with $\frac{\ell}{h(\ell)} = \frac{f}{h(f)}\ \forall\ h \in H$. So $\widetilde{V} := f\frac{V}{\ell}$ is H-invariant. This defines an operator $\widetilde{L}$ which in general is not in normal form. We call the normalization L_Z of $\widetilde{L}$ a standard operator for Z. The projective differential Galois group of L_Z is G^p, but the differential Galois group can be different from G!

From the construction above, we get the following properties for a standard operator L_Z with solution space V_Z and Picard-Vessiot extension K_Z.

1. L_Z is uniquely defined, up to a Möbius-transformation of t.
2. The projective differential Galois group of L_Z is isomorphic to G^p, and $\mathbb{P}(V_Z)$ is G^p-isomorphic to $\mathbb{P}(V)$.
3. There is a G^p-equivariant isomorphism $K_Z^{Z(G)} \cong C(Z)$.

3.3. A Klein-like theorem for order 3 operators. — Let $G \subset \mathrm{SL}_3$ be a finite irreducible group. Let L be a monic third order differential operator over $C(x)$ with Picard-Vessiot extension K, and solution space $V \subset K$. We assume that the representation of the differential Galois group in V is isomorphic to G.

Remark 3.6. — Let $L_1 = \partial_x^3 + a\partial_x^2 + \cdots$ be a differential operator with finite differential Galois group in GL_3. Then $a = -\frac{q'}{q}$, $q = \det(F)$, where F is a "fundamental matrix" as in 1.3.1. In particular q is algebraic. Applying the shift $\partial_x \mapsto \partial_x - \frac{1}{3}a$ to L_1 produces a differential operator L with differential Galois group in SL_3. Writing V_1 for the solution space of L_1, the solution space of L is $q_3^{-1}V_1,\ q_3^3 = q$. So the solutions of L are also algebraic, and therefore the differential Galois group of L is finite. We will prove that L is the pullback of some standard equation, and therefore L_1 is a pullback of this standard equation, too. So the restriction to the case $G \subset \mathrm{SL}(3)$ is no real restriction.

We will start by constructing an irreducible curve $Z \subset \mathbb{P}(V)$ corresponding to L. This Z will be G^p invariant, and satisfy $Z/G^p \cong \mathbb{P}^1_C$.

Construction 3.7. — The map $V \to K$ extends to a map $\varphi : C[V] \to K$. This map is G-equivariant. Now take some $v_0 \in V \setminus \{0\}$. For $f := \prod_{\sigma \in G} \sigma(v_0)$, we can consider the ring $(C[V][\frac{1}{f}])_0$ of homogeneous elements of degree zero in $C[V]_f$. We can extend φ to a map $\psi : \big(C[V]\big[\frac{1}{f}\big]\big)_0 \to K$. Write $I := \ker(\psi)$. We have that $\big(C[V]\big[\frac{1}{f}\big]\big)_0 = \mathcal{O}(\mathbb{P}(V)\backslash Z(f))$, where $Z(f)$ is the variety given by $f = 0$. Now I defines a subset $Z_1 \subset \mathbb{P}(V) \setminus Z(f)$, and we write Z for its closure in $\mathbb{P}(V)$. Note that Z is independent of the choice of v_0. The function field of Z is $C(Z) = \mathrm{frac}\big(\big(C[V]\big[\frac{1}{f}\big]\big)_0/I\big)$. Furthermore ψ induces a G^p-equivariant injection of $C(Z)$ in $K^p := K^{Z(G)}$. The fixed field $C(Z)^{G^p}$ is a subfield of $C(x)$ of transcendence degree 1 over C, so it can be written as $C(t)$, for some $t \in C(Z)$, where t is unique up to a Möbius transformation. We conclude that L defines a G^p-invariant irreducible curve $Z \subset \mathbb{P}(V)$, with $Z/G^p \cong \mathbb{P}^1_C$.

Remark 3.8. — We can use the above construction to show that for every finite group $G \subset \mathrm{SL}(3)$ there exists a curve Z as in Definition 3.4. Indeed, let $G \subset \mathrm{SL}(3)$ be a finite group. We can make a Galois extension $C(z) \subset K$ with Galois group G, by realizing G as a quotient of the fundamental group of $\mathbb{P}^1_C$ minus a finite number of points. As in [**vdPU00**], we can construct a third order differential operator over $C(z)$, with Picard-Vessiot extension K, such that the Galois action on the solution space equals $G \subset \mathrm{SL}(3)$. Now the construction above gives the desired curve Z.

We can now state an equivalent of Klein's theorem, for third order operators.

Theorem 3.9. — *Let L and G be as above. These data define a G^p-invariant projective curve $Z \subset \mathbb{P}(V)$ with $Z/G^p \cong \mathbb{P}^1_C$. If L_Z is a corresponding standard differential operator, then L is a weak pullback of L_Z.*

Proof. — From the construction of Z above, we get the following diagram:

$$\begin{array}{ccccc} C(x) & \subset & K^p & \subset & K \\ \cup & & \cup & & \\ C(t) & \subset & C(Z) & & \end{array}$$

We have that the G^p-action on $C(Z)$ corresponds with the G^p-action on K^p. Let K_Z be the Picard-Vessiot extension of L_Z. By the definition of a standard operator we can write $K_Z^{Z(G)} = C(Z)$. Let $V_Z \subset K_Z$ be the solution space of L_Z. In the compositum

$\mathcal{K}$ of K and K_Z over $C(Z)$, we have the identity $V_Z = \frac{f}{\ell}V$, for the appropriate f, ℓ as defined in the construction of L_Z. We will now use Notation 1.7. If F is the image of t in $C(x)$, then the pullback $\phi_F(L_Z)$ again has solution space V_Z. The derivation $\frac{d}{dt}$ extends uniquely to $\mathcal{K}$. Also $\frac{d}{dx}$ extends uniquely to $\mathcal{K}$, and $\frac{d}{dx}(a) := \frac{dF}{dx}\frac{d}{dt}(a),\ a \in \mathcal{K}$. So we can define $b := \frac{d}{dx}(\frac{f}{\ell})/(\frac{f}{\ell})$. Applying the shift $\partial_x \mapsto \partial_x + b$ to $\phi_F(L_Z)$ defines a differential operator with solution space V. So $(F')^2\phi_{F,b}(L_Z) = L$, and therefore L is a weak pullback of L_Z. □

3.4. Examples with Galois group A_5 or G_{168}. — In this subsection we will give, for the cases $G = A_5$ and $G = G_{168}$, all possible non-singular curves Z, as in Definition 3.4. We will also give an example of a standard operator with projective differential Galois group A_5.

Consider $A_5 \subset \mathrm{SL}(3, \mathbb{C})$, with generators

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_5 & 0 \\ 0 & 0 & \zeta_5^{-1} \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 & 2 \\ 1 & \zeta_5^2+\zeta_5^{-2} & \zeta_5+\zeta_5^{-1} \\ 1 & \zeta_5+\zeta_5^{-1} & \zeta_5^2+\zeta_5^{-2} \end{pmatrix}$$

Here ζ_5 is a primitive 5-th root of unity.

We have an action of A_5 on the polynomial ring $\mathbb{C}[x, y, z]$. It acts on linear terms by $g(ax + by + cz) = \left\langle g\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} cx \\ y \\ z \end{smallmatrix}\right)\right\rangle$, and this action is extended to all of $\mathbb{C}[x, y, z]$. In [**Ben96**] we find the basic invariants:

$$f_n := c_n 5^{\frac{n}{2}-1}\left(x^n + \sum_{i=0}^{4}\left(\frac{x + \zeta_5^i y + \zeta_5^{-i} z}{\sqrt{5}}\right)^n\right), n \in \{2, 6, 10\}$$

We take as constants $c_2 = \frac{1}{2}, c_6 = 1, c_{10} = 3$.

There is one more basic invariant f_{15} which is the determinant of the Jacobian matrix of (f_2, f_6, f_{10}). So we have $\mathbb{C}[x, y, z]^{A_5} = \mathbb{C}[f_2, f_6, f_{10}, f_{15}]$, where

$$\begin{aligned}
f_2 = {} & x^2 + yz, \\
f_6 = {} & 2\,(13x^6 + 3xy^5 + 15x^4yz + 45x^2y^2z^2 + 10y^3z^3 + 3xz^5, \\
f_{10} = {} & 3\big(626x^{10} + y^{10} + 90x^8yz + 1260x^6y^2z^2 + 4200x^4y^3z^3 + 3150x^2y^4z^4 \\
& +252y^5z^5 + z^{10} + (252x^5(y^5+z^5) + 840x^3yz + 360xy^2z^2)(y^5+z^5)\big), \\
f_{15} = {} & 180\big(2x + (\zeta_5+\zeta_5^4)(y+z)\big) \\
& \big(2x + (\zeta_5^2+\zeta_5^3)(y+z)\big)\big(2x + (1+\zeta_5)y + (1+\zeta_5^4)z\big) \\
& \big(2x + (1+\zeta_5^2)y + (1+\zeta_5^3)z\big)\big(2x + (1+\zeta_5^3)y + (1+\zeta_5^2)z\big) \\
& \big(2x + (1+\zeta_5^4)y + (1+\zeta_5)z\big)\big(2x + (\zeta_5+\zeta_5^2)y + (\zeta_5^3+\zeta_5^4)z\big) \\
& \big(2x + (\zeta_5^2+\zeta_5^4)y + (\zeta_5+\zeta_5^3)z\big)\big(2x + (\zeta_5^3+\zeta_5^4)y + (\zeta_5+\zeta_5^2)z\big) \\
& \big(2x + (\zeta_5+\zeta_5^3)y + (\zeta_5^2+\zeta_5^4)z\big)(y-z)(y-\zeta_5 z)(y-\zeta_5^2 z)(y-\zeta_5^3 z)(y-\zeta_5^4 z).
\end{aligned}$$

We have the relation

$$\frac{1}{400}f_{15}^2 = 3f_{10}^3 - 1590f_{10}^2f_6f_2^2 + 25014f_{10}^2f_2^5 - 90f_{10}f_6^3f_2 + 285840f_{10}f_6^2f_2^4 - 8928000f_{10}f_6f_2^7 + 70060500f_{10}f_2^{10} + 18f_6^5 + 14860f_6^4f_2^3 - 17651900f_6^3f_2^6 + 810582000f_6^2f_2^9 - 12634745000f_6f_2^{12} + 65956225000f_2^{15}.$$

We want to find all irreducible A_5-invariant plane curves Z, with $Z/A_5 \cong \mathbb{P}^1_{\mathbb{C}}$. Such a curve Z is given by $f = 0$, for some $f \in \mathbb{C}[f_2, f_6, f_{10}, f_{15}]$. We get a Galois covering $Z \to Z/A_5$. The ramification points of this covering are points in $\mathbb{P}^2_{\mathbb{C}}$ which are fixed by a cyclic subgroup of A_5. So to calculate the genus of Z/A_5 and the ramification data, we need information on the points in $\mathbb{P}^2_{\mathbb{C}}$ fixed by a cyclic subgroup of A_5. From [**Web96**] we get the following table. The first column gives the type of cyclic subgroup of A_5. The second column gives the number of subgroups of that type. The third column gives the number of points in $\mathbb{P}^2_{\mathbb{C}}$, which have a stabilizer of the type given by the first column.

H	#	*pts.*
C_2	15	∞
C_3	10	20
C_5	6	12

There are 15 lines in $\mathbb{P}^2_{\mathbb{C}}$, given by $f_{15} = 0$, and the points with stabilizer C_2 are the points that lie on precisely one of these lines. Note that each line as a whole is invariant under a group $C_2 \times C_2 \subset A_5$.

From this data we get the following information. For a branch point of the covering $Z \to Z/A_5$, the ramification index e must be in $\{2, 3, 5\}$. Then the stabilizer of a ramification point above this branch point is C_e. Above a branch point with ramification index 3, there lie 20 points. So there is at most one branch point with ramification index 3. In the same way we see that there is at most one branch point with ramification index 5.

The ramification points with ramification index 2 are intersection points of Z with $Z(f_{15}) := \{p \in \mathbb{P}^2_{\mathbb{C}} | f_{15}(p) = 0\}$. If $Z = Z(f)$, and f has degree $d, f \nmid f_{15}$, then for a fixed line $l \subset Z(f_{15})$, $Z \cap l$ consists of at most d points. Such a line l is fixed by a group D_2, so a point $p \in Z \cap l$ with stabilizer C_2 has a conjugate in $Z \cap l$ different from p. All lines in $Z(f_{15})$ are images under G of l, so all branch points with ramification index 2 are given by the images of $Z \cap l$ in Z/A_5. We see that there are at most $\frac{d}{2}$ such branch points.

Now we can use Hurwitz's formula to calculate all possibilities for non-singular Z. Hurwitz's formula states $2g-2 = 60(2g_0-2)+60\sum_i \frac{e_i-1}{e_i}$. Here i runs over the branch points of the covering $Z \to Z/A_5$, and e_i are the corresponding ramification indices. Further, g denotes the genus of Z, and g_0 denotes the genus of Z/A_5. We write n_i for the number of branch points with ramification index i, and d for the degree of $f, Z = Z(f)$. Then for non-singular Z, we have $g = \frac{(d-1)(d-2)}{2}$. So we can rewrite Hurwitz's formula as $d^2-3d+120 = 120g_0+30n_2+40n_3+48n_5$.

If $g_0 = 0$, the restrictions $n_3, n_5 \leq 1, n_2 \leq \frac{d}{2}$ give the bound $d \leq 15$. A computation in Maple shows that the homogeneous irreducible nonsingular $f \in \mathbb{C}[f_2, f_6, f_{10}, f_{15}]$ of degree ≤ 15 lie in the set $\{f_2, \lambda f_2^3+f_6, \lambda f_2^5+\mu f_2^2 f_6+f_{10}, \lambda f_2^6+\mu f_2^3 f_6+\nu f_6^2+f_2 f_{10} \mid \lambda, \mu, \nu \in \mathbb{C}\}$.

For such a polynomial f of degree d, the values of n_2, n_3, n_5 can be computed using Hurwitz's formula. For $d \in \{10, 12\}$, there is the possibility that $g_0 = 1$, but an explicit calculation of the number of intersection points of f_{10} and $f_6^2+f_2f_{10}$ with the invariant line $y = z$ rules out this possibility. We find the following table for the possibilities for d, n_2, n_3, n_5, for non-singular Z.

d	n_2	n_3	n_5
2	1	1	1
6	3	0	1
10	5	1	0
12	6	0	1

From this table, we can see that the 12 points with stabilizer C_5 lie on $Z(f_2)$ and on $Z(f_6)$. Therefore by Bezout's theorem, they are the points $Z(f_2) \cap Z(f_6)$. So we have a complete list of all non-singular curves Z satisfying definition 3.4, and we see that there are infinitely many such curves Z.

Unfortunately we are not able to give a complete list of all singular curves Z, with $Z/A_5 \cong \mathbb{P}^1_{\mathbb{C}}$. We can give the list up to a certain degree. By the previous, we see that the singular curves of degree 10 are of the form $f_2^5+\lambda f_{10}, \lambda \in \mathbb{C}^*$, and the genus is 36. For degree 12 we find the family $f_2^6+\lambda f_2^3 f_6+\mu f_6^2, \lambda \in \mathbb{C}, \mu \in \mathbb{C}^*$ of genus 19. For degree 16, all irreducible curves in our family are non-singular, so here $g_0 \geq 1$.

The group G_{168}. — We take $G_{168} \subset \mathrm{SL}(3, \mathbb{C})$, with generators:

$$\begin{pmatrix} \zeta_7 & 0 & 0 \\ 0 & \zeta_7^2 & 0 \\ 0 & 0 & \zeta_7^4 \end{pmatrix}, \begin{pmatrix} \zeta_7^5-\zeta_7^2 & \zeta_7^6-\zeta_7 & \zeta_7^3-\zeta_7^4 \\ \zeta_7^6-\zeta_7 & \zeta_7^3-\zeta_7^4 & \zeta_7^5-\zeta_7^2 \\ \zeta_7^3-\zeta_7^4 & \zeta_7^5-\zeta_7^2 & \zeta_7^6+\zeta_7 \end{pmatrix}.$$

Here ζ_7 is a primitive 7-th root of unity.

In [**Ben96**] we find that the ring of invariants for G_{168} is $C[f_4, f_6, f_{14}, f_{21}]$.

$$
\begin{aligned}
f_4 &= 2(xy^3 + yz^3 + zx^3),\\
f_6 &= \tfrac{1}{216}\operatorname{Det}(\operatorname{Hes}(f_4)),\\
f_{14} &= \tfrac{1}{144}\operatorname{Det}\begin{pmatrix}
\frac{\partial^2 f_4}{\partial x^2} & \frac{\partial^2 f_4}{\partial x\partial y} & \frac{\partial^2 f_4}{\partial x\partial z} & \frac{\partial f_6}{\partial x}\\
\frac{\partial^2 f_4}{\partial y\partial x} & \frac{\partial^2 f_4}{\partial y^2} & \frac{\partial^2 f_4}{\partial y\partial z} & \frac{\partial f_6}{\partial y}\\
\frac{\partial^2 f_4}{\partial z\partial x} & \frac{\partial^2 f_4}{\partial z\partial y} & \frac{\partial^2 f_4}{\partial z^2} & \frac{\partial f_6}{\partial z}\\
\frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial y} & \frac{\partial f_6}{\partial z} & 0
\end{pmatrix},\\
f_{21} &= \tfrac{1}{28}\operatorname{Det}(\operatorname{Jac}(f_4, f_6, f_{14})).
\end{aligned}
$$

For completeness, we give f_6, f_{14} explicitly.

$$
\begin{aligned}
f_6 &= 2(5z^2x^2y^2 - z^5x - y^5z - x^5y),\\
f_{14} &= z^{14} + x^{14} + y^{14} + 18y^7x^7 + 18y^7z^7 + 18z^7x^7 - 126z^3x^6y^5 - 250y^4x^9z\\
&\quad -34y^2z^{11}x - 34z^2x^{11}y + 375z^4x^8y^2 - 250z^4xy^9 + 375z^8x^2y^4\\
&\quad -34zx^2y^{11} - 126z^5x^3y^6 - 250z^9x^4y + 375z^2x^4y^8 - 126z^6x^5y^3.
\end{aligned}
$$

f_{21} factors as a product of linear terms over $\mathbb{Q}(\zeta_7)$. In fact f_{21} has as linear factors: $x - y(1 + \zeta_7^5 + \zeta_7^6) + z(\zeta_7^3 + \zeta_7^5)$ and its 5 conjugates, $x - y(\zeta_7 + \zeta_7^4 + \zeta_7^5) + z(\zeta_7 + \zeta_7^6)$ and its 5 conjugates, $x - y(\zeta_7^3 + \zeta_7^4 + \zeta_7^5) - z(\zeta_7^4 + \zeta_7^6)$ and its 5 conjugates, and $x - y(1 + \zeta_7^2 + \zeta_7^5) + z(\zeta_7^2 + \zeta_7^5)$ and its 2 conjugates.

There is one relation between the f_i:

$$
\begin{aligned}
f_{21}^2 &= 4f_{14}^3 - 8f_{14}f_4^7 - 44f_{14}^2f_4^2f_6 - 8f_4^9f_6 + 68f_{14}f_4^4f_6^2\\
&\quad +172f_4^6f_6^3 + 126f_{14}f_4f_6^4 - 938f_4^3f_6^5 + 54f_6^7
\end{aligned}
$$

According to [**Web96**], we have the following table of points in $\mathbb{P}^2_{\mathbb{C}}$, fixed by some subgroup of G_{168} (for details, see the A_5-case):

H	#	*pts.*
C_2	21	∞
C_3	28	56
C_4	21	42
C_7	8	24

There are 21 lines with stabilizer C_2. To be precise, each point which is on precisely one line is fixed by a group C_2, and each line as a whole is invariant under a group $C_2 \times C_2$.

For Z a G_{168}-invariant curve of degree d, the covering $Z \to Z/G_{168}$ can have ramification indices in $\{2, 3, 4, 7\}$. For a non-singular curve Z such that the quotient has genus 0, the Hurwitz formula writes:

$$d^2 - 3d + 336 = 84n_2 + 112n_3 + 126n_4 + 144n_7.$$

By calculating the number of ramification points, we find $n_3, n_4, n_7 \in \{0,1\}$, and $n_2 \leq \frac{d}{4}$. This gives the following possibilities for d, n_2, n_3, n_4, n_7:

d	n_2	n_3	n_4	n_7
4	1	1	0	1
6	1	0	1	1
14	3	1	1	0
18	4	0	1	1
20	5	1	0	1

There are infinitely many G_{168}-invariant non-singular irreducible curves Z in $\mathbb{P}^2_{\mathbb{C}}$, with $Z/G_{168} \cong \mathbb{P}^1_{\mathbb{C}}$. They are given by the irreducible polynomials in the set $\{f_4, f_6, f_{14}, \lambda f_6^3 + f_{14}f_4, \lambda f_4^5 + f_{14}f_6 | \lambda \in \mathbb{C}^*\}$.

Example 3.10. — We will give here explicit standard operators corresponding to some of the G^p-invariant irreducible curves Z (with $Z/G^p \cong \mathbb{P}^1_C$) described above. We will give the calculation for the case $G = G_{168}$, $Z = Z(f_6)$, but the method can be applied to the A_5-case, and to arbitrary Z of the above form. Independent from the author, Mark van Hoeij also calculated such operators. For his method, see `http://www.math.fsu.edu/~hoeij/files/G168/`.

We have that $C(Z) = \mathrm{Quo}(C[x,y,z]/f_6)_{((0))}$, i.e., the quotients of homogeneous elements in $C[x,y,z]/f_6$ of the same degree, where we use the grading on $C[x,y,z]/f_6$ induced by the one on $C[x,y,z]$. We will now calculate $C(Z)^G$. Obviously $\mathrm{Quo}(C[x,y,z]/f_6)^G = \mathrm{Quo}((C[x,y,z]/f_6)^G)$ which equals $\mathrm{Quo}(C[f_4, f_{14}, f_{21}])$, where we consider $C[f_4, f_{14}, f_{21}]$ as a subring of $C[x,y,z]/f_6$. So $C(Z)^G = \mathrm{Quo}(C[f_4, f_{14}, f_{21}])_{((0))}$, and since f_{21}^2 equals some polynomial in f_4, f_{14} modulo f_6, we find $C(Z)^G = \mathrm{Quo}(C[f_4, f_{14}])_{((0))}$. So we can write $C(Z)^G = C(t)$, with $t := \frac{f_4^7}{f_{14}^2}$.

Let $V_1 := \langle x, y, z\rangle_C$, then $V := \frac{f_4^5}{f_{21}} V_1 \subset C(Z)$ is a G-invariant vector space, and $C(Z)$ is generated as algebra by the elements of V. We want to calculate the corresponding differential operator. For this we write $C(Z)$ as $\mathrm{Quo}(C[u,v]/g_6)$, where $g_i := f_i(u,v,1)$. Then $t = \frac{g_4^7}{g_{14}^2}$, and we write $f := \frac{g_4^5}{g_{21}}$. The differential equation corresponding to V is the determinant of the wronskian matrix $W(Y, fu, fv, f)$ (see [**vdPS03**, Definition 1.11]). This equation can be calculated in terms of u, v using $\frac{\partial u}{\partial t} = \frac{\partial h}{\partial v}\big(\frac{\partial t}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial t}{\partial v}\frac{\partial h}{\partial u}\big)^{-1}$, $\frac{\partial v}{\partial t} = \frac{\partial h}{\partial u}\big(\frac{\partial t}{\partial v}\frac{\partial h}{\partial u} - \frac{\partial t}{\partial u}\frac{\partial h}{\partial v}\big)^{-1}$, where $h := g_6$. If we divide by the coefficient of Y''', we get an equation L with coefficients in $C(Z)^G = C(t)$.

From our calculations, we get the coefficients of L as rational expressions in u, v. We can find the corresponding rational function in t as follows. Let $c \in C(Z)^G$ be an expression in u, v. We can write $c = r(t)$ for some rational function r. Let $(u_i)_{i=1}^n$, $u_i \in C$ be some tuple of elements of C, and let $(v_i)_{i=1}^n$, $v_i \in C$ be a tuple with the property that $h(u_i, v_i) = 0$, $i = 1\cdots n$. Then $c(u_i, v_i) = r(t(u_i, v_i))$, so we have an n-tuple $((t(u_i,v_i),c(u_i,v_i))_{i=1}^n$ of points on the graph of r. If we know that r has degree m with $n \geq 2m+1$, then r is determined by these points, and can be easily calculated.

Returning to our example, let c be some coefficient of L. We can make some heuristic guess for the degree of c. Using floating point approximations for the v_i we get a good approximation of the coefficients of r (which are unique if we take the denominator to be monic). Rounding gives a sophisticated guess for r, which can then be rigorously checked by verifying that $r\big(\frac{g_4^7}{g_{14}^2}\big) \equiv c \bmod g_6$.

This method leads to the following standard operator, where we applied some möbius transformation to t and a shift in order to get a nicer formula (note that a shift does not change the projective differential Galois group).

$$\partial^3 + \frac{3(5t-3)}{4t(t-1)}\,\partial^2 + \frac{3(67t-14)}{112t^2(t-1)}\,\partial - \frac{57}{21952t^2(t-1)}$$

In [**vdPU00**, 8.2.1 (2)] we find the operator (up to an automorphism of $\mathbb{Q}(t)[\partial]$)

$$\partial^3 + \frac{13t-7}{4t(t-1)}\,\partial^2 + \frac{137t-14}{112t^2(t-1)}\,\partial - \frac{27}{21952t^2(t-1)}$$

which also corresponds to f_6, so it must be a pullback of the previous operator.

Similar calculations lead to the operator with projective differential Galois group A_5 corresponding to the invariant f_6

$$\partial^3 + \frac{14t^3+17325t^2+6824280t+945465625}{2t(t^3+1575t^2+853035t+189093125)}\,\partial^2 + \frac{32t^3+28720t^2+7040545t+370622525}{4t^2(t^3+1575t^2+853035t+189093125)}\,\partial - \frac{40885(2t-185)}{8t^3(t^3+1575t^2+853035t+189093125)}.$$

This equation does not appear in [**vdPU00**], since it has four singular points.

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