

## SIX RESULTS ON PAINLEVÉ VI

by

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**Abstract.** — After recalling some of the geometry of the sixth Painlevé equation, we describe how the Okamoto symmetries arise naturally from symmetries of Schlesinger’s equations and summarise the classification of the Platonic Painlevé six solutions.

**Résumé (Six résultats sur Painlevé VI).** — Après quelques rappels sur la géométrie de la sixième équation de Painlevé, nous expliquons comment les symétries d’Okamoto résultent de façon naturelle des symétries des équations de Schlesinger et comment elles conduisent à la classification des solutions platoniques de la sixième équation de Painlevé.

### 1. Background

The Painlevé VI equation is a second order nonlinear differential equation which governs the isomonodromic deformations of linear systems of Fuchsian differential equations of the form

$$(1) \quad \frac{d}{dz} - \left( \frac{A_1}{z} + \frac{A_2}{z-t} + \frac{A_3}{z-1} \right), \quad A_i \in \mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$$

as the second pole position  $t$  varies in  $B := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ . (The general case—varying all four pole positions—reduces to this case using automorphisms of  $\mathbb{P}^1$ .)

By ‘isomonodromic deformation’ one means that as  $t$  varies the linear monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}) \rightarrow \mathrm{SL}_2(\mathbb{C})$$

of (1) does not change (up to overall conjugation). Of course, this is not quite well-defined since as  $t$  varies one is taking fundamental groups of different four-punctured

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spheres, and it is crucial to understand this in order to understand the global behaviour (nonlinear monodromy) of Painlevé VI solutions. For small changes of  $t$  there are canonical isomorphisms between the fundamental groups: if  $t_1, t_2$  are in some disk  $\Delta \subset B$  in the three-punctured sphere then one has a canonical isomorphism

$$\pi_1(\mathbb{P}^1 \setminus \{0, t_1, 1, \infty\}) \cong \pi_1(\mathbb{P}^1 \setminus \{0, t_2, 1, \infty\})$$

coming from the homotopy equivalences

$$\mathbb{P}^1 \setminus \{0, t_1, 1, \infty\} \hookrightarrow \{(t, z) \in \Delta \times \mathbb{P}^1 \mid z \neq 0, t, 1, \infty\} \hookrightarrow \mathbb{P}^1 \setminus \{0, t_2, 1, \infty\}.$$

(Here we view the central space as a family of four-punctured spheres parameterised by  $t \in \Delta$  and are simply saying that it contracts onto any of its fibres.)

In turn, by taking the space of such  $\rho$ 's, i.e., the space of conjugacy classes of  $\mathrm{SL}_2(\mathbb{C})$  representations of the above fundamental groups, one obtains canonical isomorphisms:

$$\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, t_1, 1, \infty\}), G)/G \cong \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, t_2, 1, \infty\}), G)/G$$

where  $G = \mathrm{SL}_2(\mathbb{C})$ . Geometrically this says that the spaces of representations

$$\widetilde{M}_t := \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}), G)/G$$

constitute a ‘local system of varieties’ parameterised by  $t \in B$ . In other words, the natural fibration

$$\widetilde{M} := \{(t, \rho) \mid t \in B, \rho \in \widetilde{M}_t\} \longrightarrow B$$

over  $B$  (whose fibre over  $t$  is  $\widetilde{M}_t$ ) has a natural flat (Ehresmann) connection on it. Moreover, this connection is complete: over any disk in  $B$  any two fibres have a canonical identification.

To get from here to Painlevé VI ( $P_{\mathrm{VI}}$ ) one pulls back the connection on the fibre bundle  $\widetilde{M}$  along the Riemann–Hilbert map and writes down the resulting connection in certain coordinates. Consequently we see immediately that the monodromy of  $P_{\mathrm{VI}}$  solutions corresponds (under the Riemann–Hilbert map) to the monodromy of the connection on the fibre bundle  $\widetilde{M}$ . However, since this connection is flat and complete, its monodromy is given by the action of the fundamental group of the base  $\pi_1(B) \cong \mathcal{F}_2$  (the free group on 2 generators) on a fibre  $\widetilde{M}_t \subset \widetilde{M}$ , which can easily be written down explicitly.

Before describing this in more detail let us first restrict to linear representations  $\rho$  having local monodromies in fixed conjugacy classes:

$$M_t := \{\rho \in \widetilde{M}_t \mid \rho(\gamma_i) \in \mathcal{C}_i, i = 1, 2, 3, 4\} \subset \widetilde{M}_t$$

where  $\mathcal{C}_i \subset G$  are four chosen conjugacy classes, and  $\gamma_i$  is a simple positive loop in  $\mathbb{P}^1 \setminus \{0, t, 1, \infty\}$  around  $a_i$ , where  $(a_1, a_2, a_3, a_4) = (0, t, 1, \infty)$  are the four pole positions. (By convention we assume the loop  $\gamma_4 \cdots \gamma_1$  is contractible, and note that  $M_t$  is two-dimensional in general.) The connection on  $\widetilde{M}$  restricts to a (complete flat Ehresmann) connection on the fibration

$$M := \{(t, \rho) \mid t \in B, \rho \in M_t\} \rightarrow B$$

whose fibre over  $t \in B$  is  $M_t$ . The action of  $\mathcal{F}_2 = \pi_1(B)$  on the fibre  $M_t$  (giving the monodromy of the connection on the bundle  $M$  and thus the monodromy of the corresponding  $P_{VI}$  solution) is given explicitly as follows. Let  $w_1, w_2$  denote the generators of  $\mathcal{F}_2$ , thought of as simple positive loops in  $B$  based at  $1/2$  encircling 0 (resp. 1) once. Then,  $w_i$  acts on  $\rho \in M_t$  as the square of  $\omega_i$  where  $\omega_i$  acts by fixing  $M_j$  for  $j \neq i, i+1$ , ( $1 \leq j \leq 4$ ) and

$$(2) \quad \omega_i(M_i, M_{i+1}) = (M_{i+1}, M_{i+1}M_iM_{i+1}^{-1})$$

where  $M_j = \rho(\gamma_j) \in G$  is the  $j$ th monodromy matrix. Indeed,  $\mathcal{F}_2$  can naturally be identified with the pure mapping class group of the four-punctured sphere and this action comes from its natural action (by push-forward of loops) as outer automorphisms of  $\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\})$ , cf. [5]. (The geometric origins of this action in the context of isomonodromy can be traced back at least to Malgrange's work [28] on the global properties of the Schlesinger equations.)

On the other side of the Riemann–Hilbert correspondence we may choose some adjoint orbits  $\mathcal{O}_i \subset \mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$  such that

$$\exp(2\pi\sqrt{-1}\mathcal{O}_i) = \mathcal{C}_i$$

and construct the space of residues:

$$\mathcal{O} := \mathcal{O}_1 \times \cdots \times \mathcal{O}_4 // G = \left\{ (A_1, \dots, A_4) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_4 \mid \sum A_i = 0 \right\} / G$$

where, on the right-hand side,  $G$  is acting by diagonal conjugation:  $g \cdot (A_1, \dots, A_4) = (gA_1g^{-1}, \dots, gA_4g^{-1})$ . This space  $\mathcal{O}$  is also two-dimensional in general. To construct a Fuchsian system (1) out of such a four-tuple of residues one must also choose a value of  $t$ , so the total space of linear connections we are interested in is:

$$\mathcal{M}^* := \mathcal{O} \times B$$

and we think of a point  $(\mathbf{A}, t) \in \mathcal{M}^*$ , where  $\mathbf{A} = (A_1, \dots, A_4)$ , as representing the linear connection

$$\nabla = d - Adz, \quad \text{where } A = \sum_1^3 \frac{A_i}{z - a_i}, \quad (a_1, a_2, a_3, a_4) = (0, t, 1, \infty)$$

or equivalently the Fuchsian system (1).

If we think of  $\mathcal{M}^*$  as being a (trivial) fibre bundle over  $B$  with fibre  $\mathcal{O}$  then, provided the residues are sufficiently generic (e.g., if no eigenvalues differ by positive integers), the Riemann–Hilbert map (taking linear connections to their monodromy representations) gives a bundle map

$$\nu : \mathcal{M}^* \rightarrow M.$$

Written like this the Riemann–Hilbert map  $\nu$  is a holomorphic map (which is in fact injective if the eigenvalues are also nonzero cf. e.g., [25, Proposition 2.5]). We may

then pull-back (restrict) the nonlinear connection on  $M$  to give a nonlinear connection on the bundle  $\mathcal{M}^*$ , which we will refer to as the *isomonodromy connection*.

The remarkable fact is that even though the Riemann–Hilbert map is transcendental, the connection one obtains in this way is algebraic. Indeed Schlesinger [31] showed that locally horizontal sections  $\mathbf{A}(t) : B \rightarrow \mathcal{M}^*$  are given (up to overall conjugation) by solutions to the Schlesinger equations:

$$(3) \quad \frac{dA_1}{dt} = \frac{[A_2, A_1]}{t}, \quad \frac{dA_2}{dt} = \frac{[A_1, A_2]}{t} + \frac{[A_3, A_2]}{t-1}, \quad \frac{dA_3}{dt} = \frac{[A_2, A_3]}{t-1}$$

which are (nonlinear) *algebraic* differential equations.

To get from the Schlesinger equations to  $P_{VI}$  one proceeds as follows (cf. [24, Appendix C]). Label the eigenvalues of  $A_i$  by  $\pm\theta_i/2$  (thus choosing an order of the eigenvalues or equivalently, if the reader prefers, a quasi-parabolic structure at each singularity), and suppose  $A_4$  is diagonalisable. Conjugate the system so that

$$A_4 = -(A_1 + A_2 + A_3) = \text{diag}(\theta_4, -\theta_4)/2$$

and note that Schlesinger’s equations preserve  $A_4$ . Since the top-right matrix entry of  $A_4$  is zero, the top-right matrix entry of

$$(4) \quad z(z-1)(z-t) \sum_1^3 \frac{A_i}{z-a_i}$$

is a degree one polynomial in  $z$ . Define  $y(t)$  to be the position of its unique zero on the complex  $z$  line.

**Theorem -1 (see [24]).** — *If  $\mathbf{A}(t)$  satisfies the Schlesinger equations then  $y(t)$  satisfies  $P_{VI}$ :*

$$\begin{aligned} \frac{d^2 y}{dt^2} = & \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} \\ & + \frac{y(y-1)(y-t)}{2t^2(t-1)^2} \left( (\theta_4-1)^2 - \frac{\theta_1^2 t}{y^2} + \frac{\theta_3^2(t-1)}{(y-1)^2} + \frac{(1-\theta_2^2)t(t-1)}{(y-t)^2} \right). \end{aligned}$$

Phrased differently, for each fixed  $t$ , the prescription above defines a function  $y$  on  $\mathcal{O}$ , which makes up half of a system of (canonical) coordinates, defined on a dense open subset. A conjugate coordinate  $x$  can be explicitly defined and one can write the isomonodromy connection explicitly in the coordinates  $x, y$  on  $\mathcal{O}$  to obtain a coupled system of first-order nonlinear equations for  $x(t), y(t)$  (see [24], where our  $x$  is denoted  $\tilde{z}$ ). Then, eliminating  $x$  yields the second order equation  $P_{VI}$  for  $y$ . (One consequence is that if  $y$  solves  $P_{VI}$  there is a direct relation between  $x$  and the derivative  $y'$ , as in equation (6) below.)

In the remainder of this article the main aims are to:

- 1) Explain how Okamoto’s affine  $F_4$  Weyl group symmetries of  $P_{VI}$  arise from natural symmetries of Schlesinger equations, and

- 2) Describe the classification of the Platonic solutions to  $P_{VI}$  (i.e., those solutions having linear monodromy group equal to the symmetry group of a Platonic solid).

The key step for •1) (which also led us to •2)) is to use a different realisation of  $P_{VI}$ , as controlling isomonodromic deformations of certain  $3 \times 3$  Fuchsian systems. Note that these results have been written down elsewhere, although the explicit formulae of Remarks 6 and 7 are new and constitute a direct verification of the main results about the  $3 \times 3$  Fuchsian realisation. Note also that the construction of the Platonic solutions has evolved rapidly recently (e.g., since the author's talk in Angers and since the first version of [13] appeared). For example, there are now simple explicit formulae for all the Platonic solutions (something that we had not imagined was possible for a long time<sup>(1)</sup>).

**Remark 1.** — Let us briefly mention some other possible directions that will not be discussed further here. Firstly, by describing  $P_{VI}$  in this way the author is trying to emphasise that  $P_{VI}$  is the explicit form of the simplest non-abelian Gauss–Manin connection, in the sense of Simpson [34], thereby putting  $P_{VI}$  in a very general context (propounded further in [9, section 7], especially p. 192). For example, suppose we replace the above family of four-punctured spheres (over  $B$ ) by a family of projective varieties  $X$  over a base  $S$ , and choose a complex reductive group  $G$ . Then (by the same argument as above), one again has a local system of varieties

$$M_B = \text{Hom}(\pi_1(X_s), G)/G$$

over  $S$  and one can pull-back along the Riemann–Hilbert map to obtain a flat connection on the corresponding family  $M_{DR}$  of moduli spaces of connections. Simpson proves this connection is again algebraic, and calls it the non-abelian Gauss–Manin connection, since  $M_B$  and  $M_{DR}$  are two realisations of the first non-abelian cohomology group  $H^1(X_s, G)$ , the Betti and De Rham realisations.

Also, much of the structure found in the regular (-singular) case may be generalised to the irregular case. For example, as Jimbo–Miwa–Ueno [25] showed, one can also consider isomonodromic deformations of (generic) irregular connections on a Riemann surface and obtain explicit deformation equations in the case of  $\mathbb{P}^1$ . This can also be described in terms of nonlinear connections on moduli spaces and there are natural symplectic structures on the moduli spaces which are preserved by the connections [9, 7]. Perhaps most interestingly, one obtains extra deformation parameters in the irregular case (one may vary the ‘irregular type’ of the linear connections as well as the moduli of the punctured curve). These extra deformation parameters turn out to be related to quantum Weyl groups [10].

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<sup>(1)</sup>Mainly because the 18 branch genus one icosahedral solution of [18] took 10 pages to write down and we knew quite early on that the largest icosahedral solution had genus seven and 72 branches.

As another example, in the regular (-singular) case non-abelian Hodge theory [33] gives a third “Dolbeault” realisation of  $H^1(X_s, G)$  as a moduli space of Higgs bundles, closely related to the existence of a hyperKähler structure on the moduli space. The moduli spaces of (generic) irregular connections on curves may also be realised in terms of Higgs bundles and admit hyperKähler metrics [4].

## 2. Affine Weyl group symmetries

If we subtract off  $y'' = \frac{d^2 y}{dt^2}$  from the right-hand side of the  $P_{VI}$  equation and multiply through by  $t^2(t-1)^2 y(y-1)(y-t)$  then we obtain a polynomial

$$P(t, y, y', y'', \theta) \in \mathbb{C}[t, y, y', y'', \theta_1, \theta_2, \theta_3, \theta_4]$$

where  $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$  are the parameters.

Suppose  $\Pi$  is a Riemann surface equipped with a holomorphic map  $t : \Pi \rightarrow U$  onto some open subset  $U \subset B := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ , with non-zero derivative (so  $t$  is always a local isomorphism). (For example, one could take  $\Pi = U$  with  $t$  the inclusion, or take  $\Pi$  to be the upper half-plane, and  $t$  the universal covering map onto  $U = B$ .) Then, a meromorphic function  $y$  on  $\Pi$  will be said to be a solution to  $P_{VI}$  if

$$(5) \quad P(t, y, y', y'', \theta) = 0$$

as functions on  $\Pi$ , for some choice of  $\theta$ , where  $y' = \frac{dy}{dt}$ ,  $y'' = \frac{d^2 y}{dt^2}$  are defined by using  $t$  as a local parameter on  $\Pi$ . (With this  $t$ -dependence understood we will abbreviate (5) as  $P(t, y, \theta) = 0$  below.) By definition, the finite branching solutions to  $P_{VI}$  are those with  $\Pi$  a finite cover of  $B$ , i.e., so that  $t$  is a Belyi map. Such  $\Pi$  admits a natural compactification  $\bar{\Pi}$ , on which  $t$  extends to a rational function. The solution is “algebraic” if  $y$  is a rational function on  $\bar{\Pi}$ . Given an algebraic solution  $(\bar{\Pi}, y, t)$  we will say the curve  $\bar{\Pi}$  is “minimal” or is an “efficient parameterisation” if  $y$  generates the function field of  $\bar{\Pi}$  as an extension of  $\mathbb{C}(t)$ . The “degree” (or number of “branches”) of an algebraic solution is the degree of the map  $t : \bar{\Pi} \rightarrow \mathbb{P}^1$  (for  $\bar{\Pi}$  minimal) and the genus of the solution is the genus of the (minimal) curve  $\bar{\Pi}$ . (The genus can easily be computed in terms of the nonlinear monodromy of the  $P_{VI}$  solution using the Riemann–Hurwitz formula, i.e., in terms of the explicit  $\mathcal{F}_2$  action above on the linear monodromy data.)

Four symmetries of  $P_{VI}$  (which we will label  $R_1, \dots, R_4$ ) are immediate:

$$\begin{aligned} (R_1) \quad & P(t, y, \theta) = P(t, y, -\theta_1, \theta_2, \theta_3, \theta_4) \\ (R_2) \quad & = P(t, y, \theta_1, -\theta_2, \theta_3, \theta_4) \\ (R_3) \quad & = P(t, y, \theta_1, \theta_2, -\theta_3, \theta_4) \\ (R_4) \quad & = P(t, y, \theta_1, \theta_2, \theta_3, 2 - \theta_4) \end{aligned}$$

since  $P$  only depends on the squares of  $\theta_1, \theta_2, \theta_3$  and  $\theta_4 - 1$ .

Okamoto [30] proved there are also much less trivial symmetries:

**Theorem 0.** — If  $P(t, y, \theta) = 0$  then

$$(R_5) \quad P(t, y + \delta/x, \theta_1 - \delta, \theta_2 - \delta, \theta_3 - \delta, \theta_4 - \delta) = 0$$

where  $\delta = \sum_1^4 \theta_i/2$  and

$$(6) \quad 2x = \frac{(t-1)y' - \theta_1}{y} + \frac{y' - 1 - \theta_2}{y-t} - \frac{ty' + \theta_3}{y-1}.$$

**Remark 2.** — This can be verified directly by a symbolic computation in differential algebra. On actual solutions however it is not always well-defined since, for example, one may have  $y = t$  (identically) or find  $x$  is identically zero. It seems one can avoid these problems by assuming  $y$  is not a Riccati solution (cf. [35]). For example, if one finds  $x = 0$  then we see  $y$  solves a first order (Riccati) equation, so was a Riccati solution. Moreover, the Riccati solutions are well understood and correspond to the linear representations  $\rho$  which are either reducible or rigid, so little generality is lost.

**Remark 3.** — In terms of the symmetries  $s_0, \dots, s_4$  of [29],  $R_1, \dots, R_4$  are  $s_4, s_0, s_3, s_1$  respectively and  $R_5$  is conjugate to  $s_2$  via  $R_1 R_2 R_3 R_4$ , where the parameters  $\alpha_4, \alpha_0, \alpha_3, \alpha_1$  of [29] are taken to be  $\theta_1, \theta_2, \theta_3, \theta_4 - 1$  respectively, and  $p = x + \sum_1^3 \theta_i/(y - a_i)$ .

A basic observation (of Okamoto) is that these five symmetries generate a group isomorphic to the affine Weyl group of type  $D_4$ . More precisely let  $\varepsilon_1, \dots, \varepsilon_4$  be an orthonormal basis of a Euclidean vector space  $V_{\mathbb{R}}$  with inner product  $(\cdot, \cdot)$  and complexification  $V$ , and consider the following set of 24 unit vectors

$$D_4^- = \{\pm\varepsilon_i, (\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2\}.$$

This is a root system isomorphic to the standard  $D_4$  root system

$$D_4 = \{\pm\varepsilon_i \pm \varepsilon_j (i < j)\}$$

but with vectors of length 1 rather than  $\sqrt{2}$ . (Our main reference for root systems etc. is [14]. One may identify  $D_4^-$  with the group of units of the Hurwitzian integral quaternions [15], and then identify with  $D_4$  by multiplying by the quaternion  $1 + \mathbf{i}$ .) Each root  $\alpha \in D_4^-$  determines a coroot  $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$  ( $= 2\alpha$  here) as well as a hyperplane  $L_\alpha$  in  $V$ :

$$L_\alpha := \{v \in V \mid (\alpha, v) = 0\}.$$

In turn  $\alpha$  determines an orthogonal reflection  $s_\alpha$ , the reflection in this hyperplane:

$$s_\alpha(v) = v - 2 \frac{(\alpha, v)}{(\alpha, \alpha)} \alpha = v - (\alpha^\vee, v) \alpha.$$

The Weyl group  $W(D_4^-) \subset O(V)$  is the group generated by these reflections:

$$W(D_4^-) = \langle s_\alpha \mid \alpha \in D_4^- \rangle$$

which is of order 192. Similarly the choice of a root  $\alpha \in D_4^-$  and an integer  $k \in \mathbb{Z}$  determines an affine hyperplane  $L_{\alpha,k}$  in  $V$ :

$$L_{\alpha,k} := \{ v \in V \mid (\alpha, v) = k \}$$

and the reflection  $s_{\alpha,k}$  in this hyperplane is an affine Euclidean transformation

$$s_{\alpha,k}(v) = s_\alpha(v) + k\alpha^\vee.$$

The affine Weyl group  $W_a(D_4^-) \subset \text{Aff}(V)$  is the group generated by these reflections:

$$W_a(D_4^-) = \langle s_{\alpha,k} \mid \alpha \in D_4^-, k \in \mathbb{Z} \rangle$$

which is an infinite group isomorphic to the semi-direct product of  $W(D_4^-)$  and the coroot lattice  $Q((D_4^-)^\vee)$  (which is the lattice in  $V$  generated by the coroots  $\alpha^\vee \in (D_4^-)^\vee = D_4^+ = 2D_4^-$ ). By definition the connected components of the complement in  $V_{\mathbb{R}}$  of all the (affine) reflection hyperplanes are the  $D_4^-$  alcoves. The closure  $\overline{\mathcal{A}}$  in  $V_{\mathbb{R}}$  of any alcove  $\mathcal{A}$  is a fundamental domain for the action of the affine Weyl group; every  $W_a(D_4^-)$  orbit in  $V_{\mathbb{R}}$  intersects  $\overline{\mathcal{A}}$  in precisely one point.

Now, if we write a point of  $V$  as  $\sum \theta_i \varepsilon_i$  (i.e., the parameters  $\theta_i$  are being viewed as coordinates on  $V$  with respect to the  $\varepsilon$ -basis) then, on  $V$ , the five symmetries above correspond to the reflections in the five hyperplanes:

$$\theta_1 = 0, \quad \theta_2 = 0, \quad \theta_3 = 0, \quad \theta_4 = 1, \quad \sum \theta_i = 0.$$

The reflections in these hyperplanes generate  $W_a(D_4^-)$  since the region:

$$\theta_1 < 0, \quad \theta_2 < 0, \quad \theta_3 < 0, \quad \theta_4 < 1, \quad \sum \theta_i > 0$$

that they bound in  $V_{\mathbb{R}}$  is an alcove. (With respect to the root ordering given by taking the inner product with the vector  $4\varepsilon_4 - \sum_1^3 \varepsilon_i$ , the roots  $-\varepsilon_1, -\varepsilon_2, -\varepsilon_3, \sum \varepsilon_i/2$  are a basis of positive roots of  $D_4^-$ , and the highest root is  $\varepsilon_4$ , so by [14, p. 175] this is an alcove.)

In fact, as Okamoto showed, the full symmetry group of  $P_{V1}$  is the affine Weyl group of type  $F_4$ . (The  $F_4$  root system is the set of 48 vectors in the union of  $D_4$  and  $D_4^-$ .) This is not surprising if one recalls that  $W_a(F_4)$  is the normaliser of  $W_a(D_4^-)$  in the group of affine transformations;  $W_a(F_4)$  is the extension of  $W_a(D_4^-)$  by the symmetric group on four letters,  $S_4$  thought of as the automorphisms of the affine  $D_4$  Dynkin diagram (a central node with four satellites). This extension breaks into two pieces corresponding to the exact sequence

$$1 \longrightarrow K_4 \longrightarrow S_4 \longrightarrow S_3 \longrightarrow 1$$

where  $K_4 \cong (\mathbb{Z}/2)^2$  is the Klein four-group. On one hand the group of translations is extended by a  $K_4$ ; the lattice  $Q(D_4^+)$  is replaced by  $Q(F_4^\vee) = Q(D_4)$ . (In general [14, p. 176] one replaces  $Q(R^\vee)$  by  $P(R^\vee) = Q(R)^*$ .) On the other hand the Weyl group is extended by an  $S_3$ , thought of as the automorphisms of the usual  $D_4$  Dynkin



diagram;  $W(D_4^-)$  is replaced by the full group of automorphisms  $A(D_4^-)$  of the root system, which in this case is equal to  $W(F_4)$ .

Likewise, the corresponding symmetries of  $P_{VI}$  break into two pieces. First, one has an  $S_3$  permuting  $\theta_i$  ( $i = 1, 2, 3$ ) generated, for example, by the symmetries (denoted  $x^1, x^3$  respectively in [30, p. 361]):

$$\begin{aligned} P(t, y, \theta) = 0 &\implies P(1-t, 1-y, \theta_3, \theta_2, \theta_1, \theta_4) = 0 \\ P(t, y, \theta) = 0 &\implies P\left(\frac{t}{t-1}, \frac{t-y}{t-1}, \theta_2, \theta_1, \theta_3, \theta_4\right) = 0. \end{aligned}$$

We remark that  $W_a(D_4^-)$  already contains transformations permuting  $\theta$  by the standard Klein four group (mapping  $\theta$  to  $(\theta_3, \theta_4, \theta_1, \theta_2)$  etc.), and so we already obtain all permutations of  $\theta$  just by adding the above two symmetries.<sup>(2)</sup> To obtain the desired  $K_4$  extension we refine the possible translations by adding the further symmetry (denoted  $x^2$  in [30]):

$$P(t, y, \theta) = 0 \implies P(1/t, 1/y, \theta_4 - 1, \theta_2, \theta_3, \theta_1 + 1) = 0.$$

Combined with  $x^1, x^3$  this generates an  $S_4$  which may be thought of as permuting the set of values of  $\theta_1, \theta_2, \theta_3, \theta_4 - 1$ . (Note that, modulo the permutations of  $\theta$ , we now have translations of the form  $\theta \mapsto (\theta_1 + 1, \theta_2, \theta_3, \theta_4 - 1)$ , generating  $Q(D_4)$ .)

**Remark 4.** — One can also just extend by the  $K_4$  and get an intermediate group, often called the extended Weyl group  $W'_a(D_4^-) = W(D_4^-) \rtimes P((D_4^-)^\vee)$  which is normal in  $W_a(F_4)$  and is the maximal subgroup that does not change the time  $t$  in the above action on  $P_{VI}$ . The quotient group  $S_3$  should thus be thought of as the automorphisms of  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ .

Our aim in the rest of this section is to explain how these symmetries arise naturally from symmetries of the Schlesinger equations. The immediate symmetries are:

- (twisted) Schlesinger transformations,
- negating the  $\theta_i$  independently, and
- arbitrary permutations of the  $\theta_i$ .

In more detail, the Schlesinger transformations (see [24]) are certain rational gauge transformations which shift the eigenvalues of the residues by integers. Applying such a transformation and then twisting by a logarithmic connection on the trivial line bundle (to return the system to  $\mathfrak{sl}_2$ ) is a symmetry of the Schlesinger equations. (This procedure of “twisting” clearly commutes with the flows of the Schlesinger equations: in concrete terms it simply amounts to adding an expression of the form  $\sum_1^3 c_i/(z-a_i)$ , for constant scalars  $c_i$ , to the Fuchsian system (1). Recall  $(a_1, a_2, a_3) = (0, t, 1)$ .)

<sup>(2)</sup>For example,  $R_5 r_1 r_3 R_5 r_2 r_4$  produces the permutation written, where  $r_i$  is the Okamoto transformation negating  $\theta_i$  —i.e.,  $r_i = R_i$  for  $i = 1, 2, 3$  and  $r_4 = R_5(R_1 R_2 R_3)R_5(R_1 R_2 R_3)R_5$ .

Secondly, the eigenvalues of the residues are only determined by the abstract Fuchsian system up to sign (i.e., one chooses an order of the eigenvalues of each residue to define  $\theta_i$ , and these choices can be swapped).

Finally, if we permute the labels  $a_1, \dots, a_4$  of the singularities of the Fuchsian system arbitrarily and then perform the (unique) automorphism of the sphere mapping  $a_1, a_3, a_4$  to  $0, 1, \infty$  respectively, we obtain another isomonodromic family of systems, which can be conjugated to give another Schlesinger solution.

As an example, consider the case of negating  $\theta_4$ . Suppose we have a solution of the Schlesinger equations  $\mathbf{A}(t)$  for a given choice of  $\theta$  and have normalised  $A_4$  as required in Theorem -1 (this is where the sign choice is used). If we conjugate  $\mathbf{A}$  by the permutation matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  we again get a solution of the Schlesinger equations, and by Theorem -1 this yields a solution to  $P_{VI}$  with parameters  $(\theta_1, \theta_2, \theta_3, -\theta_4)$ . This gives the corresponding Okamoto transformation in terms of Schlesinger symmetries. (It is a good, if unenlightening, exercise to compute the explicit formula—in effect computing the position of the zero of the bottom-left entry of (4) in terms of  $x, y$ —and check it agrees with the action of the corresponding word in the given generators of  $W_a(D_4^-)$ , although logically this verification is unnecessary since a) This is a symmetry of  $P_{VI}$  and b) Okamoto found all symmetries, and they are determined by their action on  $\{\theta\}$ .)

However, one easily sees that the group generated by these immediate symmetries does not contain the transformation  $R_5$  of Theorem 0. To obtain this symmetry we will recall (from [12]) how  $P_{VI}$  also governs the isomonodromic deformations of certain rank three Fuchsian systems and show that  $R_5$  arises from symmetries of the corresponding Schlesinger equations (indeed it arises simply from the choice of ordering of the eigenvalues at infinity). (Note that Noumi–Yamada [29] have also obtained this symmetry from an isomonodromy viewpoint, but only in terms of an *irregular* (non-Fuchsian)  $8 \times 8$  system whose isomonodromy deformations, in a generalised sense, are governed by  $P_{VI}$ . Note also that Arinkin and Lysenko ([2, Corollary 2]) give a nice explicit description of  $R_5$  as an isomorphism of the abstract varieties underlying the (compactified) moduli spaces of linear connections.)

To this end, let  $V = \mathbb{C}^3$  be a three-dimensional complex vector space and suppose  $B_1, B_2, B_3 \in \text{End}(V)$  are rank one matrices. Let  $\lambda_i = \text{Tr}(B_i)$  and suppose that  $B_1 + B_2 + B_3$  is diagonalisable with eigenvalues  $\mu_1, \mu_2, \mu_3$ , so that taking the trace implies

$$(7) \quad \sum_1^3 \lambda_i = \sum_1^3 \mu_i.$$

Consider connections of the form

$$(8) \quad \nabla = d - \widehat{B}dz, \quad \widehat{B}(z) = \frac{B_1}{z} + \frac{B_2}{z-t} + \frac{B_3}{z-1}.$$

The fact is that the isomonodromic deformations of such connections are also governed by  $P_{VI}$  (one might expect such a thing since the corresponding moduli spaces

are again two-dimensional). One proof of this ([11]) is to show directly that the corresponding Schlesinger equations are equivalent to those arising in the original  $2 \times 2$  case (this may be done easily by writing out the isomonodromy connections explicitly in terms of the coordinates on the spaces of residues given by the invariant functions, and comparing the resulting nonlinear differential equations).

The second proof of this result directly gives the function that solves  $P_{VI}$ ; First conjugate  $B_1, B_2, B_3$  by a single element of  $GL_3(\mathbb{C})$  such that

$$B_1 + B_2 + B_3 = \text{diag}(\mu_1, \mu_2, \mu_3).$$

(Note this uses the choice of ordering of eigenvalues of  $B_1 + B_2 + B_3$ .) Consider the polynomial defined to be the  $(2, 3)$  matrix entry of

$$(9) \quad z(z-1)(z-t)\widehat{B}(z).$$

By construction, this is a linear polynomial, so has a unique zero on the complex plane. Define  $y = y_{23}$  to be the position of this zero.

**Theorem 1** ([12, p. 201]). — *If we vary  $t$  and evolve  $\widehat{B}$  according to Schlesinger's equations then  $y(t)$  satisfies the  $P_{VI}$  equation with parameters*

$$(10) \quad \theta_1 = \lambda_1 - \mu_1, \quad \theta_2 = \lambda_2 - \mu_1, \quad \theta_3 = \lambda_3 - \mu_1, \quad \theta_4 = \mu_3 - \mu_2.$$

The proof given in [12] uses an extra symmetry of the corresponding Schlesinger equations ([12, Proposition 16]) to pass to the  $2 \times 2$  case. Note that [12] also gives the explicit relation between the  $2 \times 2$  and  $3 \times 3$  linear monodromy data, not just the relation between the Fuchsian systems.

**Remark 5.** — Apparently, ([16]), this procedure of [12] is essentially N. Katz's middle-convolution functor [26] in this context. For us it originated by considering the effect of performing the Fourier–Laplace transformation, twisting by a flat line bundle  $\lambda dw/w$  and transforming back (reading [3] carefully to see what happens to the connections and their monodromy). It is amusing that the middle-convolution functor first arose through the  $l$ -adic Fourier transform, essentially in this way it seems, and was then translated back into the complex analytic world, rather than having been previously worked out directly.

If we now conjugate  $\widehat{B}(z)$  by an arbitrary  $3 \times 3$  permutation matrix (i.e., a matrix which is zero except for precisely one 1 in each row and column), we obtain another solution of the Schlesinger equations, but with the  $\mu_i$  permuted accordingly. The happy fact that this  $S_3$  transitively permutes the six off-diagonal entries yields:

**Corollary.** — *Let  $(i, j, k)$  be some permutation of  $(1, 2, 3)$ . Then, the position  $y_{jk}$  of the zero of the  $(j, k)$  matrix entry of (9) satisfies  $P_{VI}$  with parameters*

$$(11) \quad \theta_1 = \lambda_1 - \mu_i, \quad \theta_2 = \lambda_2 - \mu_i, \quad \theta_3 = \lambda_3 - \mu_i, \quad \theta_4 = \mu_k - \mu_j.$$

*Proof.* — Conjugate by the corresponding permutation matrix and apply Theorem 1.  $\square$

For example, the permutation swapping  $\mu_2$  and  $\mu_3$  thus amounts to negating  $\theta_4$  (indeed one may view the original  $2 \times 2$  picture as embedded in this  $3 \times 3$  picture as the bottom-right  $2 \times 2$  submatrices, at least after twisting by a logarithmic connection on a line bundle to make  $A_1, A_2, A_3$  rank one matrices).

More interestingly, let us compute the action on the  $\theta$  parameters of the permutation swapping  $\mu_1$  and  $\mu_3$ :

$$\begin{aligned}\theta &= (\lambda_1 - \mu_1, \lambda_2 - \mu_1, \lambda_3 - \mu_1, \mu_3 - \mu_2), \\ \theta' &= (\lambda_1 - \mu_3, \lambda_2 - \mu_3, \lambda_3 - \mu_3, \mu_1 - \mu_2).\end{aligned}$$

Thus  $\theta'_i = \theta_i - \delta$  with  $\delta = \mu_3 - \mu_1$ . However, using the relation (7) we find

$$\sum_1^4 \theta_i = \sum_1^3 \lambda_i - 3\mu_1 + \mu_3 - \mu_2 = 2(\mu_3 - \mu_1)$$

so that  $\delta = \sum_1^4 \theta_i / 2$  as required for  $R_5$ . This leads to:

**Theorem 2** ([12, p. 202]). — *The permutation swapping  $\mu_1$  and  $\mu_3$  yields the Okamoto transformation  $R_5$ . In other words if  $y = y_{23}$  and  $\delta = \sum_1^4 \theta_i / 2$  and*

$$2x = \frac{(t-1)y' - \theta_1}{y} + \frac{y' - 1 - \theta_2}{y-t} - \frac{ty' + \theta_3}{y-1}$$

then

$$y_{21} = y + \frac{\delta}{x}.$$

**Remark 6.** — Of course, if one had a suitable parameterisation of the space of such  $3 \times 3$  linear connections (8) in terms of  $x$  and  $y$ , this could be proved by a direct computation. Such a parameterisation may be obtained as follows (lifted from the  $2 \times 2$  case in [24] using [12, Prop. 16]). (In particular, this shows how one might have obtained the transformation formula of Theorem 0 directly.) Fix  $\lambda_i, \mu_i$  for  $i = 1, 2, 3$  such that  $\sum \lambda_i = \sum \mu_i$ . We wish to write down the matrix entries of  $B_1, B_2, B_3$  as rational functions of  $x, y, t, \lambda_i, \mu_i$ . The usual  $2 \times 2$  parameterisation of Jimbo–Miwa [24] will appear in the bottom-right corner if  $\mu_1 = 0$ . First define  $\theta_i$  as in Theorem 1. Then, define  $z_i, u_i$  for  $i = 1, 2, 3$  as the unique solution to the 6 equations:

$$\begin{aligned}y &= tu_1 z_1, & x &= \sum z_i / (y - a_i), & \sum z_i &= \mu_1 - \mu_3, \\ \sum u_i z_i &= 0, & \sum w_i &= 0, & \sum (t - a_i) u_i z_i &= 1,\end{aligned}$$

where  $w_i = (z_i + \theta_i) / u_i$  and  $(a_1, a_2, a_3) = (0, t, 1)$  (cf. [24] and [8, Appendix A]). Now, define  $c_1, c_2, c_3$  as the solution to the 3 linear equations:

$$\sum c_i z_i = 0, \quad \sum c_i w_i = 0, \quad \sum (t - a_i) c_i z_i = 1.$$

The determinant of the corresponding  $3 \times 3$  matrix is generically nonzero so this yields explicit formulae for the  $c_i$  (using, for example, the formula for the inverse of a  $3 \times 3$  matrix) —we will not write them since they are somewhat clumsy and easily derived from the above equations.<sup>(3)</sup> Using  $z_i, u_i, w_i, c_i$  we construct forms  $\beta_i$  and vectors  $f_i$  for  $i = 1, 2, 3$  by setting

$$\beta_i = (0, w_i, -z_i) \in V^*, \quad f_i = \begin{pmatrix} c_i \\ u_i \\ 1 \end{pmatrix} \in V.$$

(The meaning of the above 9 equations is simply that if we set  $B_i^0 = f_i \otimes \beta_i \in \text{End}(V)$  and  $\tilde{B}^0 = z(z-1)(z-t)\hat{B}^0$  where  $\hat{B}^0 = \sum B_i^0/(z-a_i)$  then

$$\sum B_i^0 = \text{diag}(\mu_1, \mu_2, \mu_3) - \mu_1, \quad -\hat{B}_{33}^0|_{z=y} = x, \quad \tilde{B}_{23}^0 = z - y$$

and the coefficient of  $z$  in the top-right entry  $\tilde{B}_{13}^0$  is also 1.)

The  $f_i$  are in general linearly independent and we can define the dual basis  $\hat{f}_i \in V^*$ , with  $\hat{f}_i(f_j) = \delta_{ij}$ , explicitly. The desired matrices are then

$$B_i = f_i \otimes (\beta_i + \mu_1 \hat{f}_i) \in \text{End}(V).$$

Clearly,  $B_i$  is a rank-one matrix and one may check that  $\text{Tr}(B_i) = \lambda_i$  and that  $\sum B_i = \text{diag}(\mu_1, \mu_2, \mu_3)$ . Moreover, generically, any such triple of rank-one matrices is conjugate to the triple  $B_1, B_2, B_3$  up to overall conjugation by the diagonal torus, for some values of  $x$  and  $y$ . Now, if we define  $y_{ij}$  to be the value of  $z$  for which the  $i, j$  matrix entry of  $\tilde{B} := z(z-1)(z-t)\hat{B}$  vanishes, where  $\hat{B} = \sum_1^3 B_i/(z-a_i)$  then one may check explicitly (e.g., using Maple) that  $y_{23} = y$  and  $y_{21} = y + (\mu_3 - \mu_1)/x$  as required. Also  $x$  may be defined in general, as a function on the space of such connections, by the prescription:

$$x = \frac{\mu_1 - \mu_3}{\mu_3} \hat{B}_{33}|_{z=y}$$

which may be checked to hold in the above parameterisation, and specialises to the usual definition of  $x$  in the  $2 \times 2$  case when  $\mu_1 = 0$ . Moreover, one may check  $x$  is preserved under  $R_5$  and this agrees with the fact that one also has

$$x = \frac{\mu_3 - \mu_1}{\mu_1} \hat{B}_{11}|_{z=y+\delta/x}$$

in the above parameterisation. We should emphasise that this parameterisation is such that if  $y$  solves  $P_{\text{VI}}$  (with parameters  $\theta$ ) and  $x$  is defined by (6) then the family of connections (8) is isomonodromic as  $t$  varies. Indeed one may obtain a solution

<sup>(3)</sup>For the reader's convenience a text file with some Maple code to verify the assertions of this remark (and some others in this article) is available at [www.dma.ens.fr/~boalch/files/sps.mpl](http://www.dma.ens.fr/~boalch/files/sps.mpl) (or alternatively with the source file of [arxiv:math.AG/0503043](https://arxiv.org/abs/math/0503043)).

to Schlesinger's equations by also doing two quadratures as follows. (This amounts to varying the systems appropriately under the adjoint action of the diagonal torus, which clearly only conjugates the monodromy.) By construction, the above parameterisation is transverse to the torus orbits. We will parameterise the torus orbits by replacing  $B_i$  above by  $hB_ih^{-1}$  where  $h = \text{diag}(l, k, 1)$  for parameters  $l, k \in \mathbb{C}^*$ . One then finds the new residues  $B_i$  solve Schlesinger's equations provided also

$$(12) \quad \frac{d}{dt} \log k = \frac{\theta_4 - 1}{t(t-1)}(y-t)$$

(as in [24, p. 445]) and

$$(13) \quad \frac{d}{dt} \log l = \frac{\delta - 1}{t(t-1)} \left( y - t - \frac{\delta - \theta_4}{p} \right)$$

where  $p = x + \sum_1^3 \theta_i/(y - a_i)$ . As a consistency check one can observe that the equations (12) and (13) are exchanged by the transformation swapping  $\mu_1$  and  $\mu_2$ . Indeed the corresponding Okamoto transformation  $(R_1R_2R_3)R_5(R_1R_2R_3)$  maps  $y$  to  $y - \frac{\delta - \theta_4}{p}$  and changes  $\theta_4$  into  $\delta$ .

**Remark 7.** — The parameterisation of the  $3 \times 3$  Fuchsian systems given in the previous remark is tailored so that one can see how the Okamoto transformation  $R_5$  arises and see the relation to Schlesinger's equations (i.e., one may do the two quadratures to obtain a Schlesinger solution). However, when written out explicitly, the matrix entries are complicated rational functions of  $x, y, t, \lambda_i, \mu_i$  (the  $2 \times 2$  case in [24] is already quite complicated). If one is simply interested in writing down an isomonodromic family of Fuchsian system (starting from a  $P_{VI}$  solution  $y$ ) then one may conjugate the above family of Fuchsian systems into a simpler form, as follows. First, if we write each  $B_i$  of the previous remark with respect to the basis  $\{f_i\}$ , then  $B_i$  will only have non-zero matrix entries in the  $i$ th row. Then, one can further conjugate by the diagonal torus to obtain the following, simpler, explicit matrices:

$$(14) \quad B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \lambda_2 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \lambda_3 \end{pmatrix}$$

where

$$\begin{aligned} b_{12} &= \lambda_1 - \mu_3 y + (\mu_1 - xy)(y-1), & b_{32} &= (\mu_2 - \lambda_2 - b_{12})/t, \\ b_{13} &= \lambda_1 t - \mu_3 y + (\mu_1 - xy)(y-t), & b_{23} &= (\mu_2 - \lambda_3)t - b_{13}, \\ b_{21} &= \lambda_2 + \frac{\mu_3(y-t) - \mu_1(y-1) + x(y-t)(y-1)}{t-1}, & b_{31} &= (\mu_2 - \lambda_1 - b_{21})/t. \end{aligned}$$

Thus if  $y(t)$  solves  $P_{VI}$  (with parameters  $\theta$  as in (10)) and we define  $x(t)$  via (6) and construct the matrices  $B_i$  from the above formulae, then the family of Fuchsian systems

$$(15) \quad \frac{d}{dz} - \left( \frac{B_1}{z} + \frac{B_2}{z-t} + \frac{B_3}{z-1} \right)$$

will be isomonodromic as  $t$  varies, since it is conjugate to a Schlesinger solution. This seems to be the simplest way to write down explicit isomonodromic families of rank three Fuchsian systems from  $P_{VI}$  solutions (an example will be given in the following section).

### 3. Special solutions

Another application of the  $3 \times 3$  Fuchsian representation of  $P_{VI}$  is that it allows us to see new finite-branching solutions to  $P_{VI}$ . The basic idea is that, due to (2), if a Fuchsian system has finite linear monodromy group then the solution to the isomonodromy equations, controlling its deformations, will only have a finite number of branches. For example, this idea was used in the  $2 \times 2$  context by Hitchin [20, 21] to find some explicit solutions with dihedral, tetrahedral and octahedral linear monodromy groups. (Also there are 5 solutions in [17, 18, 27] equivalent to solutions with icosahedral linear monodromy groups.)

One can also try to use the same idea in the  $3 \times 3$  context. The first question to ask is: what are the possible finite monodromy groups of rank 3 connections of the form (8)? Well (at least if  $\lambda_i \notin \mathbb{Z}$ ), the local monodromies around  $0, t, 1$  will be conjugate to the exponentials of the residues, which will be matrices of the form “identity + rank one matrix”, i.e., they will be pseudo-reflections. Moreover, the finite groups generated by such pseudo-reflections, often called complex reflection groups, have been classified by Shephard and Todd [32]. Looking at their list we immediately see that we get a richer class of finite groups than the finite subgroups of  $SL_2(\mathbb{C})$ , and so expect to get new  $P_{VI}$  solutions.

For example, the smallest non-real exceptional complex reflection group is the Klein reflection group of order 336 (which is a two-fold cover of Klein’s simple group of holomorphic automorphisms of Klein’s quartic curve). This leads to:

**Theorem 3 ([12]).** — *The rational functions*

$$y = -\frac{(5s^2 - 8s + 5)(7s^2 - 7s + 4)}{s(s-2)(s+1)(2s-1)(4s^2 - 7s + 7)}, \quad t = \frac{(7s^2 - 7s + 4)^2}{s^3(4s^2 - 7s + 7)^2},$$

constitute a genus zero solution to  $P_{VI}$  with 7 branches and parameters  $\theta = (2, 2, 2, 4)/7$ . It governs isomonodromic deformations of a rank 3 Fuchsian connection of the form (8) with linear monodromy group isomorphic to the Klein reflection group and parameters  $\lambda_i = 1/2$ ,  $(\mu_1, \mu_2, \mu_3) = (3, 5, 13)/14$ . Moreover, this solution

is not equivalent to (or a simple deformation of) any solution with finite  $2 \times 2$  linear monodromy group.

As an example application of the formulae of remark 7 it is now easy to write down the corresponding isomonodromic family of rank three Fuchsian systems having monodromy equal to the Klein complex reflection group (we have conjugated the resulting system slightly to make it easier to write). The result is that for any  $s$  such that  $t(s) \neq 0, 1, \infty$  the system (15), with  $t = t(s)$  as in Theorem 3, has monodromy equal to the Klein reflection group, generated by reflections, where the residues  $B_i$  are given by (14) with each  $\lambda_i = 1/2$  and

$$\begin{aligned} b_{12} &= \frac{14s^3 - 21s^2 + 24s - 22}{21s(4s^2 - 7s + 7)}, & b_{13} &= \frac{22s^3 - 24s^2 + 21s - 14}{21(7s^2 - 7s + 4)}, \\ b_{21} &= \frac{14s^3 - 21s^2 + 24s + 5}{21(s-1)(4s^2 - s + 4)}, & b_{23} &= \frac{22s^3 - 42s^2 + 39s - 5}{21(7s^2 - 7s + 4)}, \\ b_{31} &= \frac{14 - 21s + 24s^2 + 5s^3}{21(s-1)(4s^2 - s + 4)}, & b_{32} &= \frac{22 - 42s + 39s^2 - 5s^3}{21s(4s^2 - 7s + 7)}. \end{aligned}$$

Observe that

$$t = \frac{(7s^2 - 7s + 4)^2}{s^3(4s^2 - 7s + 7)^2} = 1 - \frac{(4s^2 - s + 4)^2(s-1)^3}{s^3(4s^2 - 7s + 7)^2}$$

so that the matrix entries of the the residues  $B_i$  are all nonsingular whenever  $t(s) \neq 0, 1, \infty$ . (Up to conjugation, at the value  $s = 5/4$  this system equals that of [12, Corollary 31] although there is a typographical error just before ([12, p. 200]) in that the values of  $b_{23}b_{32} = \text{Tr}(B_2B_3)$  and  $b_{13}b_{31} = \text{Tr}(B_1B_3)$  have been swapped.)

Unfortunately, most of the other three-dimensional complex reflection groups do not seem to lead to new solutions of  $P_{VI}$ . However, the largest exceptional complex reflection group does give new solutions. In this case the group is the Valentiner reflection group of order 2160 (which is a 6-fold cover of the group  $A_6$  of even permutations of six letters). Now, one finds there are three inequivalent solutions that arise, all of genus one. (Choosing the linear monodromy representation amounts to choosing a triple of generating reflections, and in this case there are three inequivalent triples that can be chosen.)

**Theorem 4 ([13]).** — *There are three inequivalent triples of reflections generating the Valentiner complex reflection group. The  $P_{VI}$  solutions governing the isomonodromic deformations of the corresponding Fuchsian systems are all of genus one. They have 15, 15, 24 branches and parameters*

$$(\mu_1, \mu_2, \mu_3) = (5, 11, 29)/30, \quad (5, 17, 23)/30, \quad (2, 5, 11)/12,$$

respectively (with all  $\lambda_i = 1/2$ ). The explicit solutions appear in [13].



Somewhat surprisingly when pushed down to the equivalent  $2 \times 2$  perspective these solutions all correspond to Fuchsian systems with linear monodromy generating the binary icosahedral group in  $SU_2$ , and they are not equivalent to any of the 5 solutions already mentioned. (The 3 icosahedral solutions of Dubrovin and Mazzocco [17, 18], with 10, 10, 18 branches respectively do fit into this framework and correspond to the three inequivalent choices of generating reflections of the icosahedral reflection group, cf. also [12, pp. 181-183].)

This led to the question of seeing what other such ‘icosahedral solutions’ might occur (e.g., is the 24 branch solution the largest?). The classification was carried out in [13]. (Another motivation was to find other interesting examples on which to apply the machinery of [23, 12] to construct explicit solutions.) At first glance one finds there is a huge number of such linear representations; one is basically counting the number of conjugacy classes of triples of generators of the binary icosahedral group, and an old formula of Hall [19] says there are 26688. However, this is drastically reduced if we agree to identify solutions if they are related by Okamoto’s affine  $F_4$  action (since after all there is a simple algebraic procedure to relate any two equivalent solutions, using the formulae for the Okamoto transformations).

**Theorem 5 (see [13]).** — *There are exactly 52 equivalence classes of solutions to  $P_{VI}$  having linear monodromy group equal to the binary icosahedral group.*

- *The possible genera are: 0, 1, 2, 3, 7, and the largest solution has 72 branches.*
- *The first 10 classes correspond to the ten icosahedral entries on Schwarz’s list of algebraic solutions to the hypergeometric equation,*
- *The next 9 solutions have less than 5 branches and are simple deformations of known (dihedral, tetrahedral or octahedral) solutions,*

*The remaining 33 solutions are all now known explicitly, namely there are:*

- *The 5 already mentioned of Dubrovin, Mazzocco and Kitaev in [17, 18, 27],*
- *The 20 in [13] including the three Valentiner solutions, and*
- *The 8 in [6], constructed out of previous solutions via quadratic transformations.*

In particular, all of the icosahedral solutions with more than 24 branches (and in particular all the icosahedral solutions with genus greater than one) were obtained from earlier solutions using quadratic transformations, so in this sense the 24 branch Valentiner solution is the largest ‘independent’ icosahedral solution (it was certainly the hardest to construct).

The main idea in the classification was to sandwich the equivalence classes between two other, more easily computed, equivalence relations (geometric and parametric equivalence), which in this case turned out to coincide. A key step was to understand the relation between the linear monodromy data of Okamoto-equivalent solutions, for which the geometric description in Theorem 2 of the transformation  $R_5$  was very useful (see also [22]).

Examining the list of icosahedral solutions carefully it turns out that there is one solution which is “generic” in the sense that its parameters lie on none of the reflection hyperplanes of the  $F_4$  or  $D_4$  affine Weyl groups. This is closely related to the fact that the icosahedral rotation group  $A_5$  has four non-trivial conjugacy classes: one can choose a triple of pairwise non-conjugate elements generating  $A_5$  whose product is in the fourth non-trivial class. Viewing this triple as a representation of the fundamental group of a four-punctured sphere and choosing a lift to  $\mathrm{SL}_2(\mathbb{C})$  arbitrarily, gives the monodromy data of a Fuchsian system with such generic parameters.

**Corollary ([13]).** — *There is an explicit algebraic solution to the sixth Painlevé equation whose parameters lie on none of the reflecting hyperplanes of Okamoto’s affine  $F_4$  (or  $D_4$ ) action.*

This contrasts, for example, with the Riccati solutions whose parameters always lie on an affine  $D_4^-$  hyperplane (and needless to say no other explicit generic solutions are currently known).

One can also carry out the analogous classification for the tetrahedral and octahedral groups, and this led to five new octahedral solutions. In more detail:

**Theorem 6 (see [8]).** — *There are exactly 6 (resp. 13) equivalence classes of solutions to  $P_{VI}$  having linear monodromy group equal to the binary tetrahedral (resp. octahedral) group.*

- *The first two solutions of each type correspond to the two entries of the same type on Schwarz’s list of algebraic solutions to the hypergeometric equation,*
- *The next solutions (with less than 5 branches) were previously found by Hitchin [20, 21] and Dubrovin [17] (up to equivalence/simple deformation),*
- *A six-branch genus zero tetrahedral solution and two genus zero octahedral solutions (with 6 and 8 branches resp.) were found by Andreev and Kitaev [1, 27],*
- *All the solutions have genus zero except for one 12 branch octahedral solution of genus one. The largest octahedral solution has 16 branches and is currently the largest known genus zero solution.*

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